Continuant, caterpillar, and topological index *Z*. III. Graph-theoretical algorithm for and interpretation of solving linear Diophantine equations.

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Abstract The conventional algorithm for solving the linear Diophantine equation in two variables is greatly improved graph-theoretically by using the *Z*-caterpillars, namely, by substituting all the relevant series of integers with the caterpillar graphs whose topological indices represent those integers. By this graph-theoretical analysis, the mathematical structure of the linear Diophantine equation and its relation with the Euclid's algorithm, continued fraction, and Euler's continuant are clarified.

1. Introduction

During its long history of more than two thousand years, several algorithms have been proposed[1-3] for solving the linear Diophantine equations in two variables ($lx \pm my = \pm 1$) seemingly leaving no room for another efficient one. The one using the continued fractions has been shown to be most efficient and popular. However, very recently the present author discovered that the topological index Z proposed by him in 1971[4,5] has a key role in bridging between various concepts in elementary algebra and geometry.[6,7] In this paper a novel algorithm is proposed for solving this equation by using the Z-index applied to the relevant caterpillar graphs. Its close relationship with Euler's continuant[8,9] for solving continued fraction and Euclid algorithm for calculating the GCD of a pair of integers is also discussed.

2. Graph-theoretical features of continued fraction

2.1. Definitions of continued fraction, continuant, and caterpillar graph

[Def. 1] A positive rational number, $Q_N = p/q$ (>1), can be expressed by a finite simple continued fraction as

$$Q = \frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_{N-1} + \cfrac{1}{a_N}}}}} = [a_0; a_1, a_2, \cdots, a_{N-1}, a_N], \tag{2.1}$$

with all positive partial quotients, $\{a_n\}$. If p and q are prime with each other, or $(p, q)=1$, it can be shown that $a_N>1$.

[Def. 2] Euler defined the continuant polynomial, or continuant, recursively as follows:[8,9)]

$$\mathrm{K}_0() = 1,$$

$$\mathrm{K}_1(x_1) = x_1,$$

$$\mathrm{K}_2(x_1, x_2) = x_1 x_2 + 1,$$

$$\mathrm{K}_N(x_1, x_2, \cdots, x_N) = x_N \mathrm{K}_{N-1}(x_1, x_2, \cdots, x_{N-1}) + \mathrm{K}_{N-2}(x_1, x_2, \cdots, x_{N-2}) . \tag{2.2}$$

Here all the N parameters $\{x_n\}$ are positive integers. Three well-known properties of the continuant are given here as theorems without proof.

[Theorem 1] Reversible character.

$$\mathrm{K}_N(x_1, x_2, \cdots, x_N) = \mathrm{K}_N(x_N, \cdots, x_2, x_1) . \tag{2.3}$$

[Theorem 2] Recursive relation (cf. (2.2)).

$$\mathrm{K}_N(x_1, x_2, \cdots, x_N) = x_1 \mathrm{K}_{N-1}(x_2, x_3, \cdots, x_N) + \mathrm{K}_{N-2}(x_3, x_4, \cdots, x_N) . \tag{2.4}$$

[Theorem 3] Tridiagonal determinantal expression.

$$\mathrm{K}_N(x_1, x_2, \cdots, x_N) = \begin{vmatrix} x_1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & x_2 & 1 & 0 & \cdots & 0 \\ 0 & -1 & x_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x_{N-1} & 1 \\ 0 & 0 & \cdots & 0 & -1 & x_N \end{vmatrix} . \tag{2.5}$$

A graph is a mathematical object composed of vertices and edges.[10)] In this paper only those tree graphs (without a cycle) with no directed and multiple edges are concerned. A path graph, S_N, is constructed by consecutively joining N vertices with N–1 edges, and a star graph, $\mathrm{K}_{1,N}$, is constructed from a central vertex and N (≥ 0) edges of unit length emanating from it. Now suppose a path graph S_N and prepare the set of N stars, $X_N=\{x_1, x_2, \cdots, x_N\}$, where natural number x_n denotes K_{1,x_n-1}. The set X_N has N x_n terms and each term x_n may take an arbitrary natural number. Then mount each element of X_N onto each vertex of S_N one by one either from left or right to another end. Let us call the resultant graph a caterpillar $\mathrm{C}_N(x_1, x_2, \cdots, x_N)$ composed of $|V|=\Sigma x_n$ vertices.[11,12)]

In order to characterize a graph the topological index (TopIx) Z was proposed by the present author as the total matchings plus one.[4,5,13,14)] The TopIx of path graph S_N is the Fibonacci number, f_N, which is defined recursively as

$$f_N = f_{N-1} + f_{N-2}, \tag{2.6}$$

with

$$f_0 = f_1 = 1, \tag{2.7}$$

while

$$Z(K_{1,N-1}) = N. \tag{2.8}$$

TopIx, $Z_N(x_1, x_2,\cdots, x_N)$, of caterpillar graph, $C_N(x_1, x_2,\cdots, x_N)$, can be calculated recursively by using the TopIces of the component star graphs just in the same procedure as the definition of the continuant, $K_N(x_1, x_2, \cdots, x_N)$. Thus the following theorem has been proved by the present author:[15)]

[Theorem 4] Continuant is identical to Z-caterpillar, *i.e.*, the TopIx of caterpillar $C_N(x_1, x_2,\cdots, x_N)$.

$$Z_N(x_1, x_2,\cdots, x_N) = K_N(x_1, x_2,\cdots, x_N). \tag{2.9}$$

Many problems in elementary number theory have been shown to be solved quite easily by the aid of continuant.[8,9)] However, discussion using the visualized Z-caterpillars has not only simplified the calculation but also revealed geometrical or graph-theoretical meaning and interpretation to a number of concepts and quantities involved in these problems. Advantage of the Z-caterpillar over continuant comes from the fact that continuant is a special case of TopIx for only caterpillar graphs. Now for treating Z-caterpillars two theorems will be introduced as powerful tools and algorithms for further discussion.

[Theorem 5] (Additive rule)

$$Z_{m+n}(x_1, x_2,\cdots, x_n)= Z_m(x_1,\cdots, x_m)\, Z_n(x_{m+1},\cdots, x_{m+n}) +Z_{m-1}(x_1,\cdots, x_{m-1})\, Z_{n-1}(x_{m+2},\cdots, x_{m+n}) \tag{2.10}$$

$$= \begin{vmatrix} Z_m(x_1,\cdots,x_m) & Z_{m-1}(x_1,\cdots,x_{m-1}) \\ -Z_{n-1}(x_{m+2},\cdots,x_{m+n}) & Z_n(x_{m+1},\cdots,x_{m+n}) \end{vmatrix} . \tag{2.11}$$

[Theorem 6] (Cassini's identity)[9)]

$$Z_n(x_1,\cdots, x_n)\, Z_{n-2}(x_2, \cdots, x_{n-1}) - Z_{n-1}(x_1,\cdots, x_{n-1})\, Z_{n-1}(x_2, \cdots, x_n) = (-1)^n. \tag{2.12}$$

Instead of giving proofs for these two theorems here, Fig. 1 will be given to demonstrate their visual understanding. The proof of Theorem 6 will be given in Appendix.

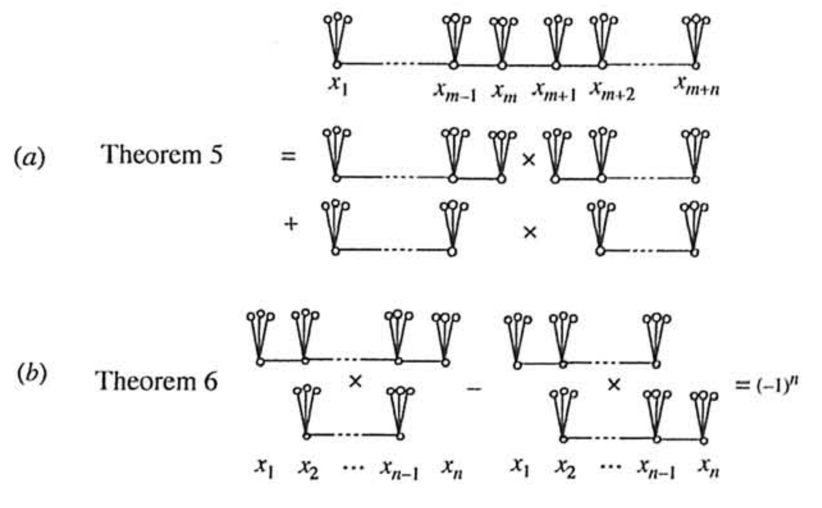


Fig. 1 (*a*) Theorem 5 and (*b*) Theorem 6 for caterpillar graphs.

2.2 Graph-theoretical meaning of continued fraction and related quantities

The standard recipe[16,17] for obtaining the set of quotients for a given pair of coprime integers p and q is exemplified below by taking p=1027 and q=712:

$$\frac{p}{q}=\frac{1027}{712}=\mathbf{1}+\frac{315}{712},\ \frac{712}{315}=\mathbf{2}+\frac{82}{315},\ \frac{315}{82}=\mathbf{3}+\frac{69}{82},\ \frac{82}{69}=\mathbf{1}+\frac{13}{69},\ \frac{69}{13}=\mathbf{5}+\frac{4}{13},\ \frac{13}{4}=\mathbf{3}+\frac{1}{4},\ \frac{4}{1}=\mathbf{4}+0.$$

The obtained set of the integers (1, 2, 3, 1, 5, 3, 4) in bold face above constitutes the quotient set $(a_0, a_1,\cdots, a_N)$ giving the corresponding continued fraction as

$$\frac{1027}{712}=1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{3+\cfrac{1}{4}}}}}}=[1;2,3,1,5,3,4].$$

The convergents of a continued fraction are successively expressed as follows:

$$p_0/q_0=[a_0],\quad p_1/q_1=[a_0;a_1],\quad \cdots,\quad p_j/q_j=[a_0;a_1,\cdots,a_j],\quad \cdots,\quad p_N/q_N=[a_0;a_1,\cdots,a_N]=p/q. \tag{2.13}$$

In the above example we have

$$\frac{p_0}{q_0}=\frac{1}{1},\quad \frac{p_1}{q_1}=1+\frac{1}{2}=\frac{3}{2}=1.5,\quad \frac{p_2}{q_2}=1+\cfrac{1}{2+\cfrac{1}{3}}=\frac{10}{7}=1.428\cdots,\quad \frac{p_3}{q_3}=1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1}}}=\frac{13}{9}=1.444\cdots,$$

$$\frac{p_4}{q_4}=1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5}}}}=\frac{75}{52}=1.4423\cdots,\quad \frac{p_5}{q_5}=1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{3}}}}}=\frac{238}{165}=1.44242\cdots,\quad \text{and finally}\quad \frac{p_6}{q_6}=\frac{1027}{712}=1.4424157\cdots.$$

According to Euler the above convergents can be expressed by the ratios of continuants as follows:[8,9] Given the set of $\{a_N\}$, the jth convergent ($0\le j\le N$) can be calculated by using the following relations:

$$p_j=\mathrm{K}_{j+1}(a_0,a_1,\cdots,a_j), \tag{2.14}$$

$$q_j=\mathrm{K}_j(a_1,a_2,\cdots,a_j), \tag{2.15}$$

to give

$$\frac{p_0}{q_0}=\frac{\mathrm{K}_1(1)}{\mathrm{K}_0()}=\frac{1}{1}\Rightarrow\frac{\mathrm{K}_2(1,2)}{\mathrm{K}_1(2)}=\frac{3}{2}\Rightarrow\frac{\mathrm{K}_3(1,2,3)}{\mathrm{K}_2(2,3)}=\frac{10}{7}\Rightarrow\frac{\mathrm{K}_4(1,2,3,1)}{\mathrm{K}_3(2,3,1)}=\frac{13}{9}$$

$$\Rightarrow\frac{\mathrm{K}_5(1,2,3,1,5)}{\mathrm{K}_4(2,3,1,5)}=\frac{75}{52}\Rightarrow\frac{\mathrm{K}_6(1,2,3,1,5,3)}{\mathrm{K}_5(2,3,1,5,3)}=\frac{238}{165}\Rightarrow\frac{\mathrm{K}_7(1,2,3,1,5,3,4)}{\mathrm{K}_6(2,3,1,5,3,4)}=\frac{1027}{712}=\frac{p_6}{q_6}=\frac{p}{q}$$

where the following recursive relations

$$p_j=a_j\,p_{j-1}+p_{j-2} \tag{2.16}$$

$$q_j=a_j\,q_{j-1}+q_{j-2} \tag{2.17}$$

hold with their initial conditions,

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \tag{2.18}$$

$$q_0 = 1, \quad q_1 = a_1. \tag{2.19}$$

However, one has to perform the above recursive calculation up to $p_N/q_N = p/q$ in one direction from the beginning to the end. On the other hand, as will be explained later one need not use this iterative calculation by treating the Z-caterpillars for this problem.

The Z-caterpillar graphs corresponding to the continuants, or the set of $\{p_j\}$ and $\{q_j\}$ of (2.14) and (2.15), can be drawn systematically as the P_j and Q_j graphs in Fig. 2, where the relations

$$Z(P_j) = p_j \tag{2.20}$$

$$Z(Q_j) = q_j \tag{2.21}$$

are also shown. Thus the graph-theoretical meaning of the partial convergent, p_j/q_j, for any j is clearly demonstrated. All the graphs, P_j and Q_j, giving p_j and q_j are the subgraphs derived successively from the caterpillar constructed from the set of the quotients for p/q. Note, however, that the order of the quotients appearing in p_j and q_j is just the reverse of what we have obtained above. Then the arguments j's assigning the order of p_j and q_j are consistent to those for r_j which we are going to explain.

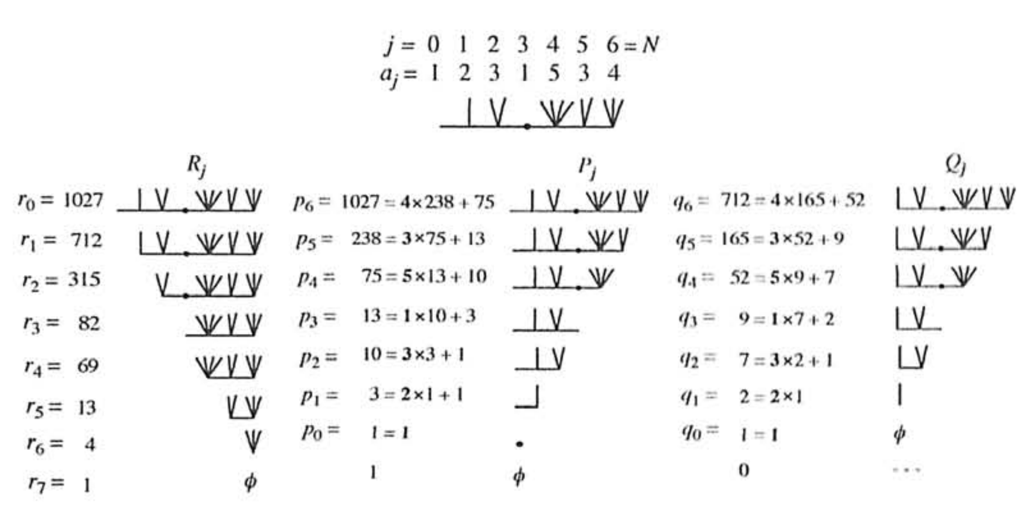


Fig. 2 The sets of P, Q, and R Z-caterpillars related to the rational number 1027/712.

Since the pair of integers, p and q, are prime with each other, their greatest common divisor (GCD) should be unity. This fact can be ascertained by executing the Euclidean algorithm for finding GCD=1, as is exemplified in Fig. 3.

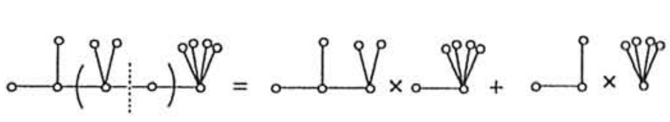
Except for r_0 $(=p)$ and r_1 $(=q)$, r_j's are the set of residue in this algorithm, and the last entry r_{N+1} is obtained to be the GCD of p and q. In this case we get $r_{N+1}=1$ with $N=6$, the length of the decimal part of the continued fraction of p/q. Note that when we obtained the set of quotients by the standard recipe, the numbers appearing alternately in the numerators and denominators form the set of r_j's, whose Z-caterpillars are drawn

	j	$r_j \quad a_j \quad r_{j+1} \quad r_{j+2}$	
$r_0 = a_0 r_1 + r_2$	0	$1027 = 1\times 712 + 315$	$r_0 = p,\ r_1 = q$
$r_1 = a_1 r_2 + r_3$	1	$712 = 2\times 315 + 82$	
$r_2 = a_2 r_3 + r_4$	2	$315 = 3\times 82 + 69$	
	3	$82 = 1\times 69 + 13$	
$\vdots$	4	$69 = 5\times 13 + 4$	
	5	$13 = 3\times 4 + 1$	
$r_N = a_N r_{N+1}$	6(N)	$4 = 4\times 1$	$r_N = a_N,\ r_{N+1} = 1$

Fig. 3 Euclidean algorithm for getting the GCD of (1027, 712) to be unity.

as R_j graphs in Fig. 3. They are another series of subgraphs derived from the original caterpillar by deletingfrom the head, contrary to the case of P_j graphs which are obtained by deleting from the tail.

Thus it is quite easy to draw the caterpillars, P_j, Q_j, and R_j, and also their Z-values without recourse to iterative calculation as continuant. An example is shown in Fig. 4 for P_4 graph by using Theorem 5.



$$= 10\times 6 + 3\times 5 = 75$$

Fig. 4 Calculation of TopIx for caterpillar P_4 by using Theorem 5.

With these graph-theoretical tools and algorithms we can solve linear Diophantine equation in two variables.

3 Solving linear Diophantine equation in two variables

3.1 D-Eqn is solved by Z-caterpillars

Consider the linear Diophantine equations of various forms,

$$lX \pm mY = \pm 1. \tag{3.1}$$

For a given pair of coprime natural numbers of l and m ($l > m$), one can obtain the quotient set ($a_0, a_1, \cdots, a_N$) either by continued fraction expansion or Euclidean algorithm. However, from now on let us use the set of $\{x_n\}$ for expressing Z-caterpillars instead of $\{a_n\}$ system. The transformation is straightforward just by changing the arguments $(0, 1, \cdots, N)$ to new $(1, 2, \cdots, N)$. Anyway one can draw the Z-caterpillars, G and Λ, corresponding to l and m as

$$\mathrm{G} = \mathrm{C}_N(x_1, x_2, \cdots, x_N), \qquad Z(\mathrm{G}) = l \tag{3.2}$$

and

$$\Lambda = \mathrm{C}_{N-1}(x_2, x_3, \cdots, x_N), \qquad Z(\Lambda) = m, \tag{3.3}$$

so as to fulfill the condition, $x_N > 1$. The Cassini's identity, or Theorem 6, has an important clue for solving (3.1). Then if one can prepare the following pair of subgraphs of G as,

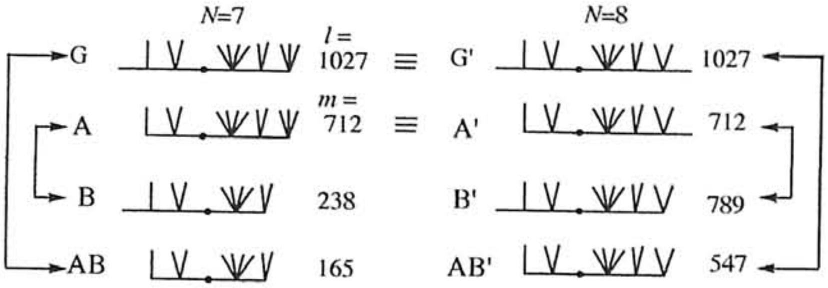
$$\mathrm{B} = \mathrm{C}_{N-1}(x_1, x_2, \cdots, x_{N-1}) \tag{3.4}$$

and

$$\mathrm{AB} = \mathrm{C}_{N-2}(x_2, x_3, \cdots, x_{N-1}), \tag{3.5}$$

these four caterpillars, (3.2)-(3.5), are qualified to apply Theorem 6 to get the following relation (See Fig. 5*a*),

$$Z(\mathrm{G})\, Z(\mathrm{AB}) - Z(\mathrm{A})\, Z(\mathrm{B}) = (-1)^N. \tag{3.6}$$



$Z(\mathrm{G})\, Z(\mathrm{AB}) - Z(\mathrm{A})\, Z(\mathrm{B})$
$= 1027 \times 165 - 712 \times 238$
$= -1 = (-1)^7$

$1027\,(165{+}712\,t) - 712\,(238{+}1027\,t) = -1$

$l\,X - m\,Y = -1$

(*a*)

$Z(\mathrm{G'})\, Z(\mathrm{AB'}) - Z(\mathrm{A'})\, Z(\mathrm{B'})$
$= 1027 \times 547 - 712 \times 789$
$= 1 = (-1)^8$

$1027\,(547{+}712\,t) - 712\,(789{+}1027\,t) = 1$

$l\,X - m\,Y = 1$

(*b*)

Fig. 5 Graph-theoretical solution of linear Diophantine equations.

Since $x_N > 1$, one can modify the tail part of G and A into G' and A', respectively, as

$$\mathrm{G'} = \mathrm{C}_{N+1}(x_1, x_2, \cdots, x_{N-1}, x_N - 1, 1) \tag{3.7}$$

and

$$\mathrm{A'} = \mathrm{C}_{N}(x_2, x_3, \cdots, x_{N-1}, x_N - 1, 1), \tag{3.8}$$

as exemplified in the upper part of Fig. 5*b*. Now both the lengths of these caterpillars are increased by one. Then by deleting each tail of G' and A' one gets another pair of caterpillars as

$$\mathrm{B'} = \mathrm{C}_{N}(x_1, x_2, \cdots, x_{N-1}, x_N - 1), \tag{3.9}$$

and

$$\mathrm{AB'} = \mathrm{C}_{N-1}(x_2, x_3, \cdots, x_{N-1}, x_N - 1). \tag{3.10}$$

Now these four caterpillars, (3.7)-(3.10), are also qualified to apply Theorem 6 to get the following relation as in Fig. 5*b*,

$$Z(\mathrm{G'})\, Z(\mathrm{AB'}) - Z(\mathrm{A'})\, Z(\mathrm{B'}) = (-1)^{N+1}. \tag{3.11}$$

The values of N in (3.11) and Fig. 5*b* are formally different, but the essential point is implied in Theorem 6.

The numerical calculations given in Fig. 5 tell us that the smallest solutions, X_0 and Y_0, for the linear Diophantine equations in two variables,

$$lX - mY = -1 \tag{3.12}$$

and
$$lX - mY = 1 \tag{3.13}$$

are, respectively, given in Figs. 5*a* and *b*.

Then by adding both sides of

$$l\,m\,t - m\,l\,t = 0$$

to both sides of (3.12) and (3.13) substituted by X_0 and Y_0 for X and Y, one gets

$$l\,(X_0 + m\,t) - m\,(Y_0 + l\,t) = -1 \tag{3.14}$$

and
$$l\,(X_0 + m\,t) - m\,(Y_0 + l\,t) = 1, \tag{3.15}$$

which are the general solutions of (3.12) and (3.13) by substituting t with any integer.

Quite similarly one can obtain the general solutions of

$$lX + mY = 1 \quad \text{and} \quad lX + mY = -1. \tag{3.16}$$

These algorithms are more easily understandable by following some concrete example as in Fig. 5 rather than by tracing formal discussion. However, in applying Theorem 6 careful caution is necessary for assigning the value of N, the length of the largest caterpillar involved.

Thus in the present analysis we could succeed in clarifying the whole mathematical structure of the linear Diophantine equation in two variables by using the Z-caterpillar graphs.

3.2 Solutions of D-Eqn are *Z*-Caterpillars

Finally we will show that all the solutions of linear Diophantine equations can be expressed by the Z-caterpillar graphs derived from that of G (3.2). Instead of exposing formal discussion let us use the example given in the above discussion. Namely, as given in Fig. 5 we have already obtained the general solutions of the pair of linear Diophantine equations in two variables of the form

$$1027\,X - 712\,Y = -1 \tag{3.17}$$

and
$$1027\,X - 712\,Y = 1 \tag{3.18}$$

as
$$X = 165 + 712\,t \quad \text{and} \quad Y = 238 + 1027\,t \tag{3.19}$$

and
$$X = 547 + 712\,t \quad \text{and} \quad Y = 789 + 1027\,t, \tag{3.20}$$

respectively.

In Fig. 6 are given these solutions, (3.20) (t=–4~3), of X and Y for (3.18) together with their Z-graphs, all of which are caterpillars and contain the skeleton of their respective solutions for t=–1 as subgraphs. Note that the set of solutions (3.19) for (3.17) are obtained just by changing the sign to the set of solutions (3.20). Then it is deduced that either in the case of (3.17) or (3.18) the three numbers appearing in (3.19) and (3.20), respectively, construct the absolute magnitudes of all the solutions for X and Y. The general forms of these solutions for the case of (3.18) are given in Fig. 6. Although the general forms for other Diophantine equations of the type (3.1) are not given here explicitly, they can easily be obtained in a similar fashion. The most important point is that the Z-graph G (3.2) representing l in (3.1) derives all the Z-graphs relevant to the process and solutions of the linear Diophantine equation in two variables.

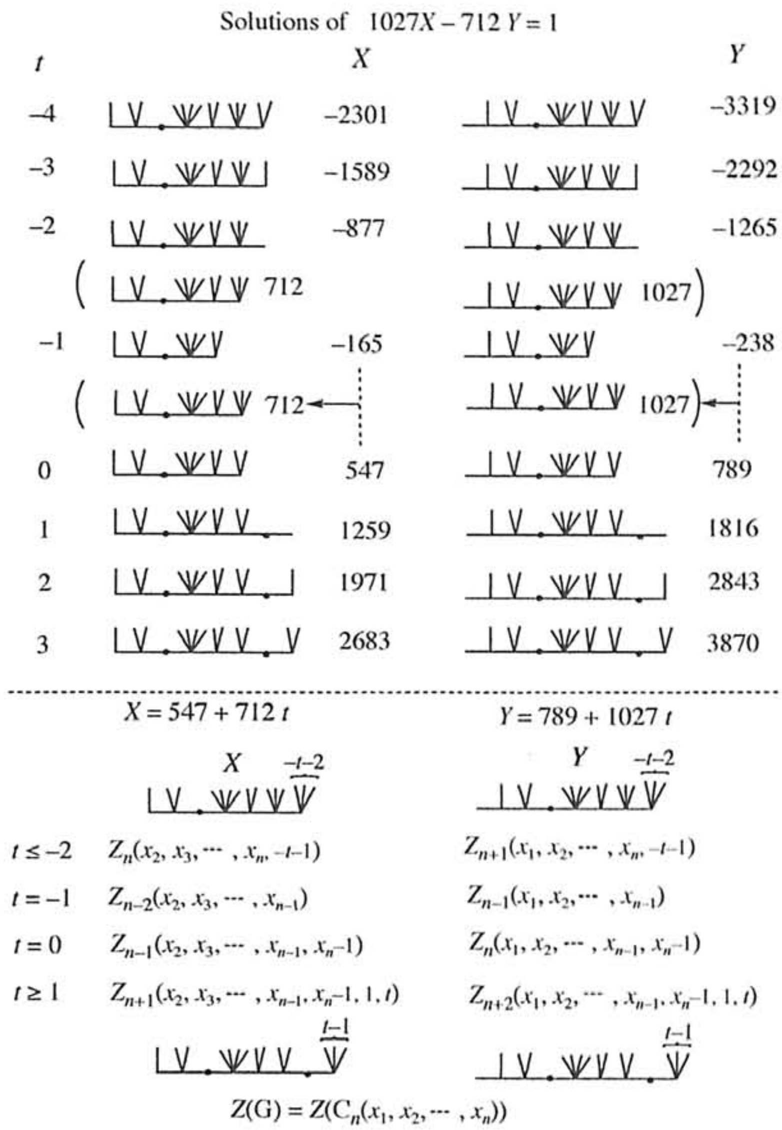


Fig. 6 Z-graphs of the solution of linear Diophantine equation (3.18).

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Appendix Proof of Theorem 6

Consider the square determinant D of order $2n-2$ as shown below

$$\mathrm{D} = \left|\begin{array}{ccc|c|c|ccc}
x_1 & 1 & & 0 & 0 & & & 0 \\
-1 & \ddots & \ddots & & & & \mathbf{0} & \\
 & \ddots & x_{n-1} & 1 & & & & \\
\hline
0 & & -1 & x_n & 1 & & & 0 \\
\hline
0 & & & -1 & x_{n-1} & 1 & & 0 \\
\hline
 & & & & -1 & x_{n-2} & \ddots & \\
 & \mathbf{0} & & & & \ddots & \ddots & 1 \\
0 & & & 0 & 0 & & -1 & x_2
\end{array}\right| , \qquad (\mathrm{A.1})$$

which is constructed by joining the two determinantal continuants of $\mathrm{K}_n(x_1, \cdots, x_n)$ and $\mathrm{K}_{n-2}(x_{n-1}, \cdots, x_2)$ with a pair of off-diagonal elements of 1 and -1. Note that the latter continuant is obtained from the former by deleting both the terminal rows and columns and reversing all the elements along the diagonal. This determinant D can be deemed as the Z-index of caterpillar $\mathrm{C}_{2n-2}(x_1,\cdots, x_{n-1}, x_n, x_{n-1}, x_{n-2},\cdots, x_2)$. Then one can apply Theorem 5 to D in two ways as follows:

$$\begin{aligned}
\mathrm{D} &= Z_n(x_1,\cdots, x_n)\, Z_{n-2}(x_{n-1}, \cdots, x_2) + Z_{n-1}(x_1,\cdots, x_{n-1})\, Z_{n-3}(x_{n-2}, \cdots, x_2) \\
&= Z_{n-1}(x_1,\cdots, x_{n-1})\, Z_{n-1}(x_n, \cdots, x_2) + Z_{n-2}(x_1,\cdots, x_{n-2})\, Z_{n-2}(x_{n-1}, \cdots, x_2).
\end{aligned} \qquad (\mathrm{A.2})$$

Then we have

$$\begin{aligned}
&Z_n(x_1,\cdots, x_n)\, Z_{n-2}(x_{n-1}, \cdots, x_2) - Z_{n-1}(x_1,\cdots, x_{n-1})\, Z_{n-1}(x_n, \cdots, x_2) \\
&\quad = -\left[Z_{n-1}(x_1,\cdots, x_{n-1})\, Z_{n-3}(x_{n-2}, \cdots, x_2) - Z_{n-2}(x_1,\cdots, x_{n-2})\, Z_{n-2}(x_{n-1}, \cdots, x_2)\right]
\end{aligned}$$

$$= (-1)^{n-2} \, [Z_2(x_1, x_2)\, Z_0(\phi) - Z_1(x_1)\, Z_1(x_2)], \tag{A.3}$$

where ϕ indicates the vacant graph.

Since the terms in the square brackets are calculated to be

$$(x_1x_2 + 1) \times 1 - x_1x_2 = 1,$$

after some change in the order of the elements we have

$$Z_n(x_1, \cdots, x_n)\, Z_{n-2}(x_2, \cdots, x_{n-1}) - Z_{n-1}(x_1, \cdots, x_{n-1})\, Z_{n-1}(x_2, \cdots, x_n) = (-1)^n. \quad \square \tag{A.4}$$