Pell Equation. II. Mathematical structure of the family of the solutions of the Pell equation

Haruo Hosoya

(Received September 21, 2006)

(Revised December 20, 2006)

Abstract Mathematical structure of the families of solutions of Pell equations $x^2 - D y^2 = 1$ (called Pell-1) and $x^2 - D y^2 = -1$ (Llep-1) are studied by using Cayley-Hamilton theorem. Besides discovery of several new recursive relations, it was found that the solutions (x_n, y_n) of Pell-1 are expressed by the Chebyshev polynomials of the first and second kinds, T_n and U_n, in terms of the smallest solutions (x_1, y_1). The solutions (t_n, u_n) of Pellep-1 which are the combination of Pell-1 and Llep-1 are expressed by using the conjugate Chebyshev polynomials. Similar results are obtained for the solutions of Pellep-4 through the modified Chebyshev polynomials and their conjugates. The solutions of Pellep-4 with several D values are found to form various interesting mathematical series of numbers, such as Fibonacci, Lucas, Pell numbers.

1. Introduction

Let us call Eqns. (1.1) and (1.2), respectively, Pell and Llep,[1)]

$$x^2 - D y^2 = 1 \qquad (1.1)$$

$$x^2 - D y^2 = -1, \qquad (1.2)$$

where only non-negative integer solutions (x, y) are to be sought for square-free D.[1-10)] Pell has an infinite number of solutions forming a family for any D besides the trivial solution (x_0=1, y_0=0), whereas Llep has solutions only for special values of D. A union of the solutions of Pell and Llep with the same D will be called here Pellep. It is known that mathematical structure of the Pell equation will be globally understandable by extending the problem to Pellep-N as

$$x^2 - D y^2 = \pm N. \qquad (1.3)$$

In algebraic number theory Pellep-4 is shown to be as important as Pellep-1.

The smallest pairs of non-trivial solutions of Pell and Llep (if ever) will be denoted, respectively, as (x_1, y_1) and (r_1, s_1). In the first paper of this series Pells and Lleps with smaller D (<100) were classified into several types with respect to the form of the polynomial relations (PR's) among (D, x_1, y_1). This is a kind of discussion along the equator. In this paper mathematical structure of the family of solutions of Pell and Pellep with the same D will be analyzed as a longitudinal discussion. It will be shown that the solutions of Pellep-1 and Pellep-4 can be generally expressed in terms of their smallest solutions through typical Chebyshev polynomials. Further, the family of solutions of Pell-4, Llep-4, and Pellep-4 with special *D's* are shown to be closely related to famous series of numbers, such as Fibonacci, Lucas, and Pell numbers. The results obtained here have contributed to global understanding of the mathematical structure of the solutions of the Pell equation to a certain extent.

2. Analysis of Pellep-1

2.1. Fundamental Pell-1, Llep-1, and Pellep-1

Due to its long history it is difficult to pin point the original credit for each theorem and technique for solving and relating the mathematics of Pell and Llep. Then in principle only recent contributions in this field will be credited in this paper.

First consider the case with D=2, *i.e.*, the original Pell and Llep as

$$x^2 - 2y^2 = 1 \tag{2.1}$$

$$x^2 - 2y^2 = -1. \tag{2.2}$$

The pair of equations (2.1) and (2.2) are nothing else but a formulation of the famous problem already posed by ancient Greek mathematicians as follows:

Although there is no square whose area is just twice of another square,
the areas of many squares differ by only one from twice another square,
such as $3^2 = 2 \times 2^2 + 1$, $7^2 = 2 \times 5^2 - 1$, *etc.*

The former identity corresponds to (x_1, y_1) of (2.1), while the latter (r_2, s_2) of (2.2). Although the identity $1^2 = 2 \times 1^2 - 1$ is not applied to this problem, it represents (r_1, s_1) of (2.2). From these observations one can write the following pair of identities:

$$(3+2\sqrt{2})(3-2\sqrt{2}) = 1 \qquad (1+\sqrt{2})(1-\sqrt{2}) = -1.$$

By taking their nth powers one gets

$$(3+2\sqrt{2})^n(3-2\sqrt{2})^n = 1 \tag{2.3}$$

and

$$(1+\sqrt{2})^n(1-\sqrt{2})^n = (-1)^n. \tag{2.4}$$

By using (2.3) it has been proved that all the solutions (x_n, y_n) of (2.1) can be obtained from the following equation:

$$x_n + \sqrt{2}\,y_n = (3+2\sqrt{2})^n = (1+\sqrt{2})^{2n} \quad (n \ge 0), \tag{2.5}$$

whereas for Llep its general solution is obtained by

$$r_n + \sqrt{2}s_n = (1+\sqrt{2})(3+2\sqrt{2})^{n-1} = (1+\sqrt{2})^{2n-1} \qquad (n \ge 1) \tag{2.6}$$

Now it is obvious that $(1+\sqrt{2})^n$ generates the union (t_n, u_n) of the solutions of (2.1) and (2.2), or the bigger family of Pellep, as expressed by the following equation,

$$t_n + \sqrt{2}u_n = (1+\sqrt{2})^n \quad (n \geq 0)\,. \tag{2.7}$$

In Table 1 the lower members of the solutions of all these Pell, Llep, and Pellep are given, where $r_0=-1$ and $s_0=1$

Table 1. Solutions of Pell, Llep, and Pellep families with $D=2$

n	Pell ($N=+1$) $(3+2\sqrt{2})^n$		Llep ($N=-1$) $(1+\sqrt{2})(3+2\sqrt{2})^{n-1}$		Pellep ($N=\pm 1$) $(1+\sqrt{2})^n$	
	x_n	y_n	r_n	s_n	t_n	u_n
0	1	0	−1	1	1	0
1	3	2	1	1	1	1
2	17	12	7	5	3	2
3	99	70	41	29	7	5
4	577	408	239	169	17	12
5	3363	2378	1393	985	41	29

are added so that formal discussion can be extended down to $n=0$ in (2.6) and other recursive relations. Note that (2.5)– (2.7) are also valid for negative n's.

From (2.5) we have

$$\begin{aligned} x_{n+1} + \sqrt{2}y_{n+1} &= (3+2\sqrt{2})(x_n + \sqrt{2}y_n) \\ &= (3x_n + 4y_n) + \sqrt{2}(2x_n + 3y_n)\,. \end{aligned} \tag{2.8}$$

Since x_n and y_n are all integers, the following pair of equations can be obtained:

$$x_{n+1} = 3\,x_n + 4\,y_n \tag{2.9}$$

$$y_{n+1} = 2\,x_n + 3\,y_n. \tag{2.10}$$

Once we get a pair of simultaneous recursive relations for

$$u_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{2.11}$$

in the form of

$$u_{n+1} = \mathbf{A}u_n \tag{2.12}$$

with 2×2 coefficient matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.13}$$

it is straightforward to solve them into a simpler recursive relation by using the following corollary of Cayley-Hamilton theorem. Namely, we have

(Corollary 1: Cayley-Hamilton)

The square of any 2×2 matrix **A** (2.13) is reduced to be

$$\mathbf{A}^2 = (tr\,\mathbf{A})\,\mathbf{A} - (det\,\mathbf{A})\,\mathbf{I}\,. \tag{2.14}$$

Proof: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$ □

Since

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

from (2.9) nd (2.10), we have

$$\begin{aligned} u_{n+2} &= \mathbf{A}^2 u_n = [(tr\,\mathbf{A})\,\mathbf{A} - (det\,\mathbf{A})\,\mathbf{I}]\,u_n \\ &= (a+d)\,u_{n+1} - (ad-bc)\,u_n \\ &= 6\,u_{n+1} - u_n. \end{aligned}$$

Quite the same result can be obtained for (r_n, s_n) in (2.6), and one gets the following recursive relations:

$$f_n = 6 f_{n-1} - f_{n-2} \qquad (f = r, s, x, y). \tag{2.15}$$

Then it is straightforward to get the general expressions for these variables as

$$x_n = (\alpha^n + \beta^n)/2 \tag{2.16}$$

$$y_n = (\alpha^n - \beta^n)/2\sqrt{2} \tag{2.17}$$

with

$$\alpha = (3 + 2\sqrt{2}) \quad \text{and} \quad \beta = (3 - 2\sqrt{2}), \tag{2.18}$$

which are the solutions of the characteristic polynomial

$$x^2 - 6x + 1 = 0 \tag{2.19}$$

for (2.15).

From (2.7) for (t_n, u_n) the recursive relation

$$f_n = 2 f_{n-1} + f_{n-2} \qquad (f = t, u) \tag{2.20}$$

and general formulas

$$t_n = (\rho^n + \sigma^n)/2 \tag{2.21}$$

$$u_n = (\rho^n - \sigma^n)/2\sqrt{2} \tag{2.22}$$

with

$$\rho = (1 + \sqrt{2}) \quad \text{and} \quad \sigma = (1 - \sqrt{2}) \tag{2.23}$$

are obtained, where (2.23) is the solution of the characteristic polynomial

$$x^2 - 2x - 1 = 0 \tag{2.24}$$

for (2.20).

Note that the series u_n have been known as Pell numbers,[8,10,11)] and y_n and s_n are their subgroups. This problem will be discussed in more detail later in this section. It is interesting to observe that (2.19) and (2.24) are formally related with each other in the following identity

$$(x^2 - 2x - 1)(x^2 + 2x - 1) = x^4 - 6x^2 + 1. \tag{2.25}$$

From the second factor of the left hand side of (2.25) the following recursive relation can be derived:

$$f_n = -2 f_{n-1} + f_{n-2} . \qquad (2.26)$$

If one selects 0 and 1 as the initial values for f_0 and f_1 and apply (2.26), a Pell number-like but sign-alternating series are obtained as: 0, 1, –2, 5, –12, 29, –70, ⋯.

See Table 1, and notice that all the ratios, x_n/y_n , r_n/s_n, and t_n/u_n converge to $\sqrt{2}$, which will be proved in the following discussion. It has been known that the square root of a positive number D can be approximated by the conversion limit of the series of numbers $\{c_n\}$ successively obtained by the following recursive relation

$$c_{n+1} = (c_n + D/c_n\,) \,/\, 2. \qquad (2.27)$$

If one starts with $c_0 = 1$ for D=2 in (2.27), a series of numbers rapidly converging to $\sqrt{2}$ are obtained as

3/2, 17/12, 577/408, 665887/470832, ⋯.

The fourth rational number has 12-digit accuracy to $\sqrt{2}$, and all the above entries are the 2^mth members of t_n/u_n, or the 2^{m-1}th member of x_n/y_n in Table 1. If one selects $7/5 = r_2/s_2$ for c_1, the $3 \cdot 2^{m-1}$th members of x_n/y_n will come out for $m \geq 1$. The reason why (2.27) generates larger solutions of Pell from a smaller one can be explained as follows. By putting $c_n = x_n \,/\, y_n$ into (2.27) one gets

$$x_{n+1}/\,y_{n+1} = (x_n \,/\, y_n + D\, y_n \,/\, x_n)/2 = (x_n^2 + D\, y_n^2) \,/\, (2\, x_n\, y_n). \qquad (2.28)$$

The numerator and denominator of the right-hand-side of (2.28) are shown to satisfy also the Pell with the same D as

$$(x_n^2 + D\, y_n^2)^2 - D\,(2\, x_n\, y_n)^2 = (x_n^2 - D\, y_n^2)^2 = 1. \qquad (2.29)$$

That is, $(x_{n+1},\, y_{n+1})$ obtained from (2.28) is not the next larger solution of $(x_n\ ,\ y_n)$ but a so-called "leapfrog" solution whose ratio is far more approaching to $\sqrt{D}$. Now go on to the discussion of the general Pellep-1 problem with an arbitrary D.

2.2. Properties of the solutions of Pellep-1 with an arbitrary D

Assume that Llep-1 is solvable for a given D. Then all the solutions of the corresponding Pell-1, Llep-1, and Pellep-1 can be obtained by the following set of equations like (2-5)–(2.7) for the case with D=2:

$$x_n + \sqrt{D}\, y_n = (x_1 + \sqrt{D} y_1)^n \quad (n \geq 0), \qquad (2.30)$$

$$r_n + \sqrt{D} s_n = (r_1 + \sqrt{D} s_1)\,(x_1 + \sqrt{D} y_1)^{n-1} \quad (n \geq 1), \qquad (2.31)$$

$$t_n + \sqrt{D} u_n = (r_1 + \sqrt{D} s_1)^n \quad (n \geq 0). \qquad (2.32)$$

Then one gets the following equations similar to (2.29):[4)]

$$(x_1^2 - D\, y_1^2)\,(x_n^2 - D\, y_n^2) = (x_1\, x_n + D\, y_1\, y_n)^2 - D\,(x_1\, y_n + y_1\, x_n)^2 = 1, \qquad (2.33)$$

$$(x_1^2 - D\, y_1^2)\,(r_n^2 - D\, s_n^2) = (x_1\, r_n + D\, y_1\, s_n)^2 - D\,(x_1\, s_n + y_1\, r_n)^2 = -1, \qquad (2.34)$$

$$(r_1^2 - D\, s_1^2)\,(t_n^2 - D\, u_n^2) = (r_1\, t_n + D\, s_1\, u_n)^2 - D\,(r_1\, u_n + s_1\, t_n)^2 = (-1)^{n+1}. \qquad (2.35)$$

Note here that $r_1 = t_1$ and $s_1 = u_1$ (See Table 1). From (2.33) the following set of recursive equations for the solutions of Pell-1 can be derived.

$$x_{n+1} = x_1 x_n + D y_1 y_n \tag{2.36}$$

$$y_{n+1} = y_1 x_n + x_1 y_n. \tag{2.37}$$

For Llep-1 one gets

$$r_{n+1} = x_1 r_n + D y_1 s_n \tag{2.38}$$

$$s_{n+1} = y_1 r_n + x_1 s_n. \tag{2.39}$$

Also for Pellep-1 one gets

$$t_{n+1} = r_1 t_n + D s_1 u_n \tag{2.40}$$

$$u_{n+1} = s_1 t_n + r_1 u_n. \tag{2.41}$$

By using the procedure in the preceding section the following recursive relations can be obtained:

$$f_n = 2 x_1 f_{n-1} - f_{n-2} \qquad (f = x, y, r, s \text{ of Pell-1 and Llep-1}) \tag{2.42}$$

$$f_n = 2 r_1 f_{n-1} + f_{n-2} \qquad (f = t, u \text{ of Pellep-1}). \tag{2.43}$$

In deriving (2.43) the relation $r_1^2 - D s_1^2 = -1$ is used. The results (2.15) and (2.20) obtained for D=2 are special cases of (2.42) and (2.43), respectively.

Thus if D is given and the smallest solutions (x_1, y_1) and (r_1, s_1) (if ever) are found, it is straightforward to obtain larger family members of the solutions to a desired extent by using (2.36)–(2.41) or (2.42)–(2.43). However, because of the recursive relations (2.42) and (2.43) all these solutions can be expressed by a more compact functional form with respect to (x_1, x_1) , (r_1, s_1), and (t_1, u_1).

In Table 2 are given the smaller family members of (x_n, y_n) and (t_n, u_n) expressed by the smallest solutions (x_1, y_1) and (t_1, u_1). While the absolute values of the coefficients of the corresponding terms are the same for (x_n, y_n) and (t_n, u_n), the former set of variables have a sign-alternating property contrary to all positive terms for the latter. On the other hand it is difficult to express (r_n, s_n) in terms of only (r_1, s_1) as inferred from the hybrid recursive relation of (2.38) and (2.39).

Table 2. Smaller members of (x_n, y_n) and (t_n, u_n) expressed by their smallest members

n	$x_n = X_n(x_1)$	$y_n = y_1 Y_n(x_1)$	$t_n = V_n(t_1)$	$u_n = u_1 W_n(t_1)$.
1	x_1	y_1	t_1	u_1
2	$2 x_1^2 - 1$	$y_1(2 x_1)$	$2 t_1^2 + 1$	$u_1(2 t_1)$
3	$4 x_1^3 - 3 x_1$	$y_1(4 x_1^2 - 1)$	$4 t_1^3 + 3 t_1^2$	$u_1(4 t_1^2 + 1)$
4	$8 x_1^4 - 8 x_1^2 + 1$	$y_1(8 x_1^3 - 4 x_1)$	$8 t_1^4 + 8 t_1^2 + 1$	$u_1(8 t_1^3 + 4 t_1)$

Then let us denote (x_n, y_n) and (t_n, u_n) in functional forms as

$$x_n = X_n(x_1)$$

$$y_n = y_1 Y_n(x_1) \tag{2.44}$$

and

$$t_n = V_n(t_1)$$
$$u_n = u_1 W_n(t_1). \tag{2.45}$$

The explicit expressions for lower members of these functions are already given in Table 2.

From (2.36) we have

$$X_n(x_1) = x_1 X_{n-1}(x_1) + D y_1^2 Y_{n-1}(x_1)$$
$$= x_1 X_{n-1}(x_1) + (x_1^2 - 1) Y_{n-1}(x_1), \tag{2.46}$$

while from (2.37)

$$Y_n(x_1) = X_{n-1}(x_1) + x_1 Y_{n-1}(x_1). \tag{2.47}$$

Again by using Corollary 1 to the pair of recursive equations (2.46) and (2.47), the following pair of recursive relations are obtained,

$$X_n(x_1) = 2x_1 X_{n-1}(x_1) - X_{n-2}(x_1) \tag{2.48}$$
$$Y_n(x_1) = 2x_1 Y_{n-1}(x_1) - Y_{n-2}(x_1). \tag{2.49}$$

This pair of polynomials, X_n and Y_n, respectively, are nothing else but the Chebyshev polynomials of the first and second kinds, T_n and U_n.[12-14] U_n is sometimes called Pell polynomial.[10,15,16] Although there exist a number of definitions for them, it is sufficient to ascertain their recursive relation,

$$f_n(x) = 2x f_{n-1}(x) - f_{n-2}(x) \tag{2.50}$$

and initial values for n=0 and 1 for proving the following identities:

$$X_n(x) = T_n(x)$$
$$Y_n(x) = U_{n-1}(x) \tag{2.51}$$

Then one may be allowed to state the following theorem without giving its proof explicitly.

(Theorem 1)

All the family members of the solutions of Pell-1 are generally expressed in terms of the Chebyshev polynomials as

$$x_n = T_n(x_1)$$
$$y_n = y_1 U_{n-1}(x_1). \tag{2.52}$$

□

For the sake of later discussion the polynomial expressions for T_n and U_n will be given here,[10)]

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} (x^2 - 1)^k \tag{2.53}$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} x^{n-2k} (x^2 - 1)^k \tag{2.54}$$

Their lower members are listed in Table 3.

Table 3. Chebyshev polynomials of the first and second kinds,

1st kind	2nd kind
$T_0(x) = 1$	$U_0(x) = 1$
$T_1(x) = x$	$U_1(x) = 2x$
$T_2(x) = 2x^2 - 1$	$U_2(x) = 4x^2 - 1$
$T_3(x) = 4x^3 - 3x$	$U_3(x) = 8x^3 - 4x$
$T_4(x) = 8x^4 - 8x^2 + 1$	$U_4(x) = 16x^4 - 12x^2 + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$	$U_5(x) = 32x^5 - 32x^3 + 6x$
$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$	$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$

$T^*_n(x)$, and $U^*_n(x)$ have exactly the same forms, respectively, as $T_n(x)$, and $U_n(x)$ but with all positive signs.

Next consider Pellep-1. Similarly to the case with Pell-1, from (2.40) and (2.45) we have

$$\begin{aligned} V_n(t_1) &= t_1 V_{n-1}(t_1) + D u_1^2 W_{n-1}(t_1) \\ &= t_1 V_{n-1}(t_1) + (t_1^2 + 1) W_{n-1}(t_1), \end{aligned} \tag{2.55}$$

while from (2.41) and (2.45)

$$W_n(t_1) = V_{n-1}(t_1) + t_1 W_{n-1}(t_1). \tag{2.56}$$

By applying Corollary 1 to the pair of recursive equations (2.55) and (2.56), the following recursive relations are obtained:

$$V_n(t_1) = 2t_1 V_{n-1}(t_1) + V_{n-2}(t_1) \tag{2.57}$$

$$W_n(t_1) = 2t_1 W_{n-1}(t_1) + W_{n-2}(t_1). \tag{2.58}$$

As shown in Table 2 all the terms of V_n and W_n have positive signs and their absolute values are, respectively, the same as those of X_n and Y_n, or of T_n and U_n. Let us here call V_n and W_n the pair of conjugate Chebyshev polynomials of the first and second kinds and denote them as T^*_n and U^*_n, respectively. By modifying (2.53) and (2.54) their polynomial expressions can be given as follows:

$$T^*_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} (x^2+1)^k \tag{2.59}$$

$$U^*_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} x^{n-2k} (x^2+1)^k \tag{2.60}$$

Then one may again state the following theorem:

(Theorem 2)

All the family members of the solutions of Pellep-1 are generally expressed in terms of the conjugate Chebyshev polynomials as

$$t_n = T^*{}_n(t_1)$$

$$u_n = u_1 U^*{}_{n-1}(t_1), \tag{2.61}$$

where t_1 and u_1 are the smallest solutions of (1.3) with N=1. □

As evident from Table 1, the solutions of Llep-1 and Pell-1 alternately appear in the solutions of Pellep-1. Then one gets the following Corollary:

(Corollary 2)

All the solutions of Pell-1 and Llep-1 can, respectively, be expressed by the conjugate Chebyshev polynomials as

$$x_n = T^*{}_{2n}(t_1)$$

$$y_n = u_1 U^*{}_{2n-1}(t_1), \tag{2.62}$$

and

$$r_n = T^*{}_{2n-1}(r_1)$$

$$s_n = s_1 U^*{}_{2n-2}(r_1). \tag{2.63}$$

□

Again notice that $r_1 = t_1$ and $s_1 = u_1$. Remark that x_n and y_n can be expressed by either (x_1, y_1), or (t_1, u_1). As an example Table 4 gives various ways for calculating the value of x_3 of Pell-1 with D=13, which are introduced in this paper.

Table 4. Worked out example for calculating x_3 of Pell-1 with D=13.

(2.36) $x_3 = x_1x_2 + D\,y_1\,y_2 = 649\times842401+13\times180\times233640$

(2.44) $= 2x_1x_2 - x_1 = 2\times649\times842401-649$

(2.55) $= T_3(x_1) = 4\times649^3-3\times649$

(2.67) $= T_6{}^*(t_1) = 32\times18^6+48\times18^4+18\times18^2+1$

$= 1093435849$

$x_2 = x_1{}^2 + D\,y_1{}^2 = 649^2+13\times180^2 = 842401$

$y_2 = 2x_1x_2 = 2\times649\times180 = 233640$

y_3 can also be calculated to be 303264540.

In the first paper of this series the solutions (x_1, y_1) and (r_1, s_1) (if ever) of the Pell-1 and Llep-1 with smaller D (<100) are given. All their family members can be obtained quite easily as in Table 4. The values of D were classified into several types with respect to the form of the polynomial relation connecting (D, x_1, y_1). As there have been found a number of interesting mathematical properties among the respective classes, discussion of the family members of each class will diverge. Selected topics will be introduced elsewhere.

Table 5. General expressions for giving the solutions of Pell-1, Llep-1, and Pellep-1 in terms of Chebyshev polynomials

Pell-1	Llep-1	Pellep-1
$x_n = T_n(x_1) \quad = T^*_{2n}(t_1)$	$r_n = T^*_{2n-1}(r_1)$	$t_n = T^*_n(t_1)$
$y_n = y_1 U_{n-1}(x_1) = u_1 U^*_{2n-1}(t_1)$	$s_n = s_1 U^*_{2n-2}(r_1)$	$u_n = u_1 U^*_{n-1}(t_1)$

Although the relation (2.52) has already been known for Pell-1,[9)] to the author's awareness general solutions of Llep-1 and Pellep-1 have not been reported yet. Then the new results in which Chebyshev and conjugate Chebyshev polynomials are involved will be summarized in Table 5.

3. Analysis of Pellep-N

3.1. Characteristic properties of Pellep-4 solutions

One can extend the above discussion to generalized Pell equations, or Pellep-N. However, according to our preliminary research neat general expressions for the solutions of Pellep-N seem to exist only for N=4. This is in concordance with the fact that Pellep-4 has a key role in algebraic number theory.[3,5,6)]

First we give the list of (x_1, y_1) and (r_1, s_1) (if ever) for Pellep-4 with D<100 in Table 6. Let us compare them with the corresponding results for Pellep-1.[1)] As a general rule which is applied to almost two thirds of the cases, (x_1, y_1) and (r_1, s_1) (if ever) for Pellep-4 are just the double of those for Pellep-1. Those D values which do not apply to this rule is printed in bold in Table 6. It is a natural consequence that there always exist Lle-p-4 solutions for D with Lepp-1, but for several D's without Llep-1 new Llep-4 solutions (r_1, s_1) come out as printed also in bold. Note, however, that in this case (x_1, y_1) obey the general rule, *i.e.*, twice the values for Llep-1. These D values are also printed in bold. Other (x_1, y_1) and (r_1, s_1) (if ever) values of Pellep-4 for D printed in bold take rather smaller values than Pellep-1 solutions. For example, 10- and 9-digit (x_1, y_1) numbers for D=61 are lowered down to 4- and 3-digits for Pell-4. Here we are not going into more detailed discussion, but it is to be remarked that only in the cases with bold D's there appear odd numbers in the solution of Pellep-4.

By following the procedures used in Pellep-1 analysis general expressions for the solutions of Pellep-4 were obtained as summarized in Table 7, where modified types of Chebyshev polynomials of the first and second kinds, $C_n(x)$ and $S_n(x)$,[12-14)] are conveniently used together with their conjugate polynomials $C^*_n(x)$ and $S^*_n(x)$. $C_n(x)$ and $S_n(x)$ are related to $T_n(x)$ and $U_n(x)$ as

$$C_n(x) = 2\, T_n(x/2) \tag{3.1}$$

$$S_n(x) = U_n(x/2), \tag{3.2}$$

and can explicitly be expressed by

Table 6. List of (x_1, y_1) and (r_1, s_1) of Pellep-4

D	2	3	**5**	6	7	**8**	10	11	**12**	**13**	14	15	17	18
x_1	6	4	3	10	16	6	38	20	4	11	30	8	66	34
y_1	4	2	1	4	6	2	12	6	1	3	8	2	16	8
r_1	2		1			**2**	6			3			8	
s_1	2		1			**1**	2			1			2	

D	19	**20**	**21**	22	23	24	26	27	**28**	**29**	30	31	**32**	33
x_1	340	18	5	394	48	10	102	52	16	27	22	3040	6	46
y_1	78	4	1	84	10	2	20	10	3	5	4	546	1	8
r_1		**4**					10			5				
s_1		**1**					2			1				

D	34	35	37	38	39	**40**	41	42	43	**44**	**45**	46	47
x_1	70	12	146	74	50	38	4098	26	6964	20	7	48670	96
y_1	12	2	24	12	8	6	640	4	1062	3	1	7176	14
r_1			12			**6**	64						
s_1			2			**1**	10						

D	48	50	51	**52**	**53**	54	55	56	57	58	59	**60**	**61**
x_1	14	198	100	1298	51	970	178	30	302	39206	1060	8	1523
y_1	2	28	14	180	7	132	24	4	40	5148	138	1	195
r_1		14		**36**	7					198			39
s_1		2		**5**	1					26			5

D	62	63	65	66	67	**68**	**69**	70	71	72	73	74
x_1	126	16	258	130	97684	66	25	502	6960	34	4562498	7398
y_1	16	2	32	16	11934	8	3	60	826	4	534000	860
r_1			16			**8**					2136	86
s_1			2			**1**					250	10

D	75	**76**	77	78	79	80	82	83	84	**85**	86	87	88
x_1	52	340	9	106	160	18	326	164	110	83	20810	56	394
y_1	6	39	1	12	18	2	36	18	12	9	2244	6	42
r_1							18			9			
s_1							2			1			

D	89	90	91	**92**	**93**	94	95	**96**	97	98	99
x_1	1000002	38	3148	48	29	4286590	78	10	125619266	198	20
y_1	106000	4	330	5	3	442128	8	1	1275470	20	2
r_1	1000								11208		
s_1	106								1138		

See text for the explanation of D's and (r_1, s_1) values printed in bold.

Table 7. General expressions for giving the solutions of Pell-4, Llep-4, and Pellep-4 in terms of modified Chebyshev polynomials

Pell-4	Llep-4	Pellep-4
$x_n = C_n(x_1) \quad = C^*_{2n}(t_1)$	$r_n = C^*_{2n-1}(r_1)$	$t_n = C^*_n(t_1)$
$y_n = y_1 S_{n-1}(x_1) = u_1 S^*_{2n-1}(t_1)$	$s_n = s_1 S^*_{2n-2}(r_1)$	$u_n = u_1 S^*_{n-1}(t_1)$

Table 8. Modified Chebyshev polynomials of the first and second kinds,

1st kind	2nd kind
$C_0(x) = 2$	$S_0(x) = 1$
$C_1(x) = x$	$S_1(x) = x$
$C_2(x) = x^2 - 2$	$S_2(x) = x^2 - 1$
$C_3(x) = x^3 - 3x$	$S_3(x) = x^3 - 2x$
$C_4(x) = x^4 - 4x^2 + 2$	$S_4(x) = x^4 - 3x^2 + 1$
$C_5(x) = x^5 - 5x^3 + 5x$	$S_5(x) = x^5 - 4x^3 + 3x$
$C_6(x) = x^6 - 6x^4 + 9x^2 - 2$	$S_6(x) = x^6 - 5x^4 + 6x^2 - 1$

$C^*_n(x)$, and $S^*_n(x)$ have exactly the same forms, respectively, as $C_n(x)$, and $S_n(x)$ but with all positive signs.

$$C_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} \qquad (n>0) \tag{3.3}$$

$$S_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} \qquad (n\geq 0). \tag{3.4}$$

Lower members of C_n and S_n are given in Table 8. All the terms of C^*_n and S^*_n have positive signs and their absolute values are, respectively, the same as those of C_n and S_n, Their definitions can be given by

$$C^*_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} \qquad (n>0) \tag{3.5}$$

$$S^*_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} \qquad (n\geq 0). \tag{3.6}$$

The recursive relations for the solutions of Pellep-4 were also obtained as in Table 9, where they are compared with those for Pellep-1.

Table 9. Recursive relations of the solutions of Pellep-1 and Pellep-4

		Pell-N		Pellep-N
N=1	T, U	$f_n = 2x f_{n-1} - f_{n-2}$	T^*, U^*	$f_n = 2x f_{n-1} + f_{n-2}$
N=4	C, S	$f_n = x f_{n-1} - f_{n-2}$	C^*, S^*	$f_n = x f_{n-1} + f_{n-2}$

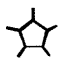
In these recursive relations x takes the smallest solution of Pell (x_1) or Llep (r_1). Then it is inferred from Table 6 that only in the case of Pellep-4 with D=5 Fibonacci and /or Lucas series may appear in the solution of Pellep-N, and it is actually the case. In Table 10 the solutions of Pellep-4 with D=5, 8, 13, and 20 (but not 21!) are shown together with the graphs whose topological indices Z's match these interesting series of numbers. This graph-theoretical interpretation of the Pell polynomial in terms of Z,[17,18] which has been proposed by the present author, will be the main topic of the next paper.

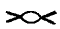
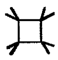
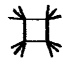
3.2. Other Pellep-N cases

Because it is still in a preliminary stage of study, only one point is to be mentioned here. For Pellep-9 most solutions are multiples of 3, but as in the case with D=7 new comers appear as follows:

$4^2 - 7\times 1^2 = 9$, $53^2 - 7\times 20^2 = 9$, *etc.* They will destroy the regular pattern like Pellep-1 and Pellep-4.

Table 10. Families of solutions of Pellep-4 with D=5, 8, 13, and 20. The graphs are those whose topological indices just correspond to x's and y's for each Pellep-4.

$D=5$	$f_n=f_{n-1}+f_{n-2}$		$D=8$	$f_n=2f_{n-1}+f_{n-2}$	
•	$1^2-5\times 1^2=-4$	ϕ	—	$2^2-8\times 1^2=-4$	ϕ
	$3^2-5\times 1^2=\ 4$	•		$6^2-8\times 2^2=\ 4$	|
	$4^2-5\times 2^2=-4$	—		$14^2-8\times 5^2=-4$	
	$7^2-5\times 3^2=\ 4$			$34^2-8\times 12^2=\ 4$	
	$11^2-5\times 5^2=-4$			$82^2-8\times 29^2=-4$	

$D=13$	$f_n=3f_{n-1}+f_{n-2}$		$D=20$	$f_n=4f_{n-1}+f_{n-2}$	
	$3^2-13\times 1^2=-4$	ϕ		$4^2-20\times 1^2=-4$	ϕ
	$11^2-13\times 3^2=\ 4$			$18^2-20\times 4^2=\ 4$	
	$36^2-13\times 10^2=-4$			$76^2-20\times 17^2=-4$	
	$119^2-13\times 33^2=\ 4$			$322^2-20\times 72^2=\ 4$	
	$393^2-13\times 109^2=-4$			$1364^2-20\times 305^2=-4$	

References

1) H. Hosoya and N. Asamoto, *Natural Sci. Rept. Ochanomizu Univ.*, **57** (2006), 57-83

2) E. E. Whitford, The Pell Equation, College of the City of New York, New York (1912).

3) T. Takagi, Elementary Number Theory (in Japanese), Kyoritsu, Tokyo (1931).

4) A. H. Beiler, Recreations in the Theory of Numbers. The Queen of Mathematics Entertainment, Dover, New York (1964).

5) G. Fujisaki, Y. Morita, Y. Yamamoto, Departure for Number Theory (in Japanese), Nihon-hyoronsha, Tokyo (1980).

6) H. H. Stark, An Introduction to Number Theory, MIT Press, Boston (1995).

7) J. H. Silverman, A Friendly Introduction to Number Theory, 2nd Ed., Pearson Educ., New York (2001).

8) D. M. Burton, Elementary Number Theory, 5th Ed., McGraw-Hill, Boston (2002).

9) E. J. Barbeau, Pell's Equation, Springer, New York (2002). Extensive bibliographies on Pell equations are given.

10) E. W. Weisstein, CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, Boca Raton, FL, (2003)..

11) A. F. Horadam, *Fibonacci Quart.*, **9** (1971), 245-252.

12) L. A. Lyusternik, O. A. Shelvonenkis, and A. R. Yanpol'skii, Handbook for Computing Elementary Functions, Pergamon, Oxford (1965).

13) M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, New York (1984).

14) H. Hosoya, *Natural Sci. Rept. Ochanomizu Univ.*, **32** (1981) 127-138.

15) A. F. Horadam and J. M. Mahon, *Fibonacci Quart.*, **9** (1971) 245-252.

16) T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, New York (2001).

17) H. Hosoya, *Bull. Chem. Soc. Jpn.*, **44** (1971) 2332-2339.

18) H. Hosoya, *Fibonacci Quart.*, **11** (1973) 255-266.