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Pell Equation. I. Systematic classification of the solutions of the Pell equation

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Abstract Let the smallest non-trivial solution of Pell equation, $x^2 - D\,y^2 = 1$, be denoted by (x_1, y_1). The Pell equations for D were systematically classified into several types with respect to the form of the polynomial relations (PR's) among (D, x_1, y_1). The key strategies for this analysis are the value of y_1 and form of the continued fraction expansion of $\sqrt{D}$. Among the solutions of Pell equation with D below 100 only four D's were found to have no other D below ten thousand connected through a PR. All the PR's were shown to be derived from a pair of the "master equations". The results obtained in this paper show an effectiveness of the proposed strategies for the systematic analysis of the chaotic behavior of the solutions of Pell equation.

1. Introduction

In this series of papers Eqns. (1.1) and (1.2) will simply be called, respectively, Pell and Llep.

$$x^2 - D\,y^2 = 1 \tag{1.1}$$

$$x^2 - D\,y^2 = -1 \tag{1.2}$$

where only non-negative integer solutions (x, y) are to be sought for square-free D.[1-11] Pell has an infinite number of solutions for any D besides the trivial solution (x=1, y=0), whereas Llep has solutions only for a limited number of D. The smallest pairs of non-trivial solutions of Pell and Llep (if ever) will be denoted, respectively, as (x_1, y_1) and (r_1, s_1).

Several algorithms for solving both equations have been known, and the solutions of the Pell-like equation,

$$x^2 - D\,y^2 = N, \tag{1.3}$$

are involved in these processes, where N is either a positive or negative integer. Here let us call (1.3) G-Pell, meaning a generalized Pell's equation, or Pell-N. The continued fraction expansion (CFE) of the square root of any D, $\sqrt{D}$, is known to be periodic, and has an important role in solving (1.1–1.3).[6,10] Once (x_1, y_1) and/or (r_1, s_1) are obtained for a given D, all the larger solutions can be derived by standard

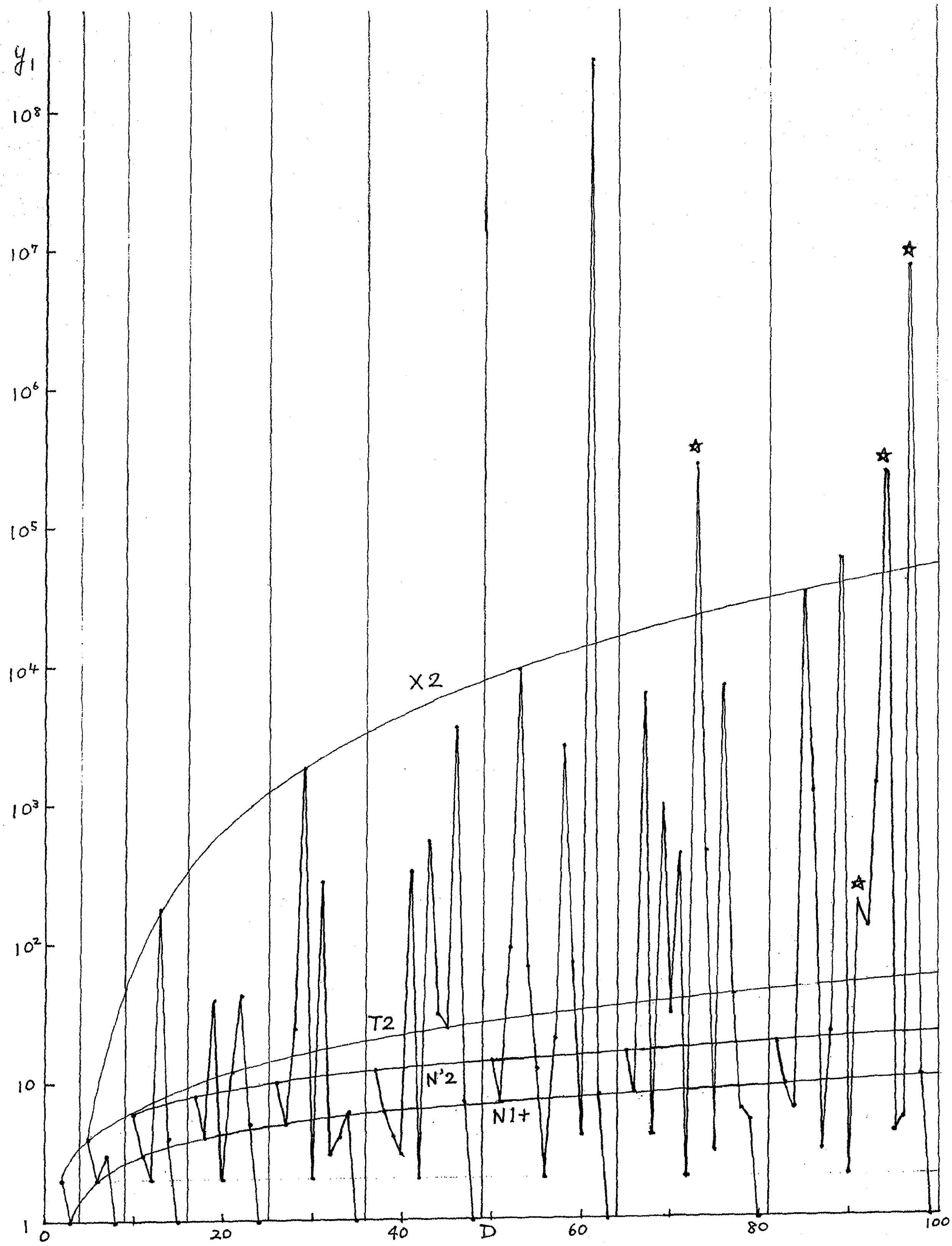

Fig. 1. Chaotic behavior of y_1 value against D of Pell equation. All the points except for the four stars were found to belong to some polynomial group as shown by a few smooth curves.

recipes. Although the essence of the charm of Pell and Llep lies in the chaotic behavior of their solutions as shown in Fig. 1, where logarithms of the values of y_1's are plotted against smaller D, no systematic survey for the polynomial relation (PR) among the solutions of Pell and Llep seems to have been reported.[10] Namely, there have sporadically been documented the PR's only for very small y_1's and s_1's or for special groups of D's.[1, 7, 8, 10, 12–15]

The purpose of the present series of paper is to find global mathematical structure of the solutions of the families of Pell and Llep by scrutinizing the lists of (x_1, y_1)'s, (r_1, s_1)'s, and CFE of $\sqrt{D}$. In this paper the solutions of Pell and Llep for D's below 100 were found to belong to a group or groups characterized by some PR, except for the four cases (D=73, 91, 94, and 97). All these PR's obtained here were found to be derived from a pair of the "master equations". Although the existence of more global master equations is anticipated, the method developed and results obtained here are believed to have paved an artery for global understanding of the mathematical structure of the solutions of Pell and Llep.

2. Analysis and results

2.1. Observation and setting up of strategy

By a standard recipe CFE's of $\sqrt{D}$ for D<10,000 were computed and sorted according to the length p of their periods. The values of (x_1, y_1) or (r_1, s_1) can be obtained through the internet[16] if necessary. There are many different groups of D's with their common y_1's contrary to the case with x_1. Thus we first propose

(Strategy 1) Find such a group of D's whose y_1's are in common or related by a relatively simple polynomial.

The simplest PR has long been recognized[12,13] as

$$n^2 - (n^2 - 1)\cdot 1^2 = 1, \tag{2.1}$$

which can be transformed into

$$(m + 1)^2 - (m^2 + 2\,m) = 1. \tag{2.2}$$

This expression is a special case of

$$(n\,a^2 \pm 1)^2 - (n^2\,a^2 \pm 2\,n)\,a^2 = 1 \tag{2.3}$$

reported by Speckmann[12] and Ricalde.[13]

Here we propose to name (2.3) as the master equation which derives a number of PR's of Pell.

(Master Equation) $$(k^2\,m \pm 1)^2 - k^2\,(k^2\,m^2 \pm 2\,m) = 1. \tag{M±}$$

The CFE's of $\sqrt{D}$ of C1 group (Table 2) with $m \geq 2$ are found to be expressed by a common formula with p=2.

$$\sqrt{m^2+2m} = m + \frac{1}{1+}\,\frac{1}{2m+}\,\frac{1}{1+}\,\frac{1}{2m+}\,\frac{1}{1+}\cdots$$
$$= [m;\overline{1,2m}] \quad (m \geq 1) \tag{2.4}$$

Also for N1+ group (Table 3) which is derived by putting k=m and m=1 into M+ as

$$(m^2 \pm 1)^2 - m^2\,(m^2 \pm 2) = 1, \tag{2.5}$$

their CFE is given by a simple form as

$$\sqrt{m^2+2} = [m;\overline{m,2m}] \qquad (m \geq 1). \tag{2.6}$$

Then we propose

(Strategy 2) Select those D's whose CFE's have as similar forms as possible.

Further it is helpful for us to propose

(Strategy 3) Take a plot of (D, y) or $(\log D, y)$ for the candidates to the members of the same group selected by Strategies 1 and 2 to check if those points are fit into a smooth curve.

After consecutive numbering with m to the set of (D, x, y) 's which are thought to form a group, we proceed to

(Strategy 4) Try to formulate a PR satisfying these (D, x, y) 's in a form as

$$[x_1(m)]^2 - D(m)\,[y_1(m)]^2 = 1. \tag{2.7}$$

2.2. C- and N-types

By applying the above strategies to our list of (D, x, y) 's, a number of new PR's were obtained. They were roughly classified into several types among which the simplest C-type with constant y_1=c is straightforwardly obtained as in Table 2. However, there are two subtypes which can be discriminated by the form of $D(m)$ as

C-type: $$D(m) = k^2\,m^2 \pm b\,m \tag{2.8}$$

C'-type: $$D(m) = k^2\,m^2 \pm b\,m + c. \tag{2.9}$$

For the former C-type two general expressions can be obtained with respect to odd and even k as given in Table 2. Since the values of (x_1, y_1) and p for C'-type are generally larger than those for C-type, the procedure for determining the PR of C'-type is a little more involved than C-type but still feasible. Although a general expression has not yet been obtained for C'-type, all their PR's are shown to be derived from either of the master equations M+ and M– as given in Table 2.

In this experimental analysis we have tentatively set $D(m)$ to be quadratic as in (2.8) and (2.9). Then the degree of polynomial expression of $x_1(m)$ is determined by that of $y_1(m)$, and accordingly the type of PR is classified into C, N, S, T, Q, and X as shown in Table 3.

As the difference in the behavior between C- and C'-types, similar discrimination is observed between the two types with linear $y(m)$ as in

N-type: $y(m) = a\,m$ (2.10)

and S-type: $y(m) = a\,m + b.$ (2.11)

The obtained results for the former are given in Table 4, some of which have already been reported by Speckmann[12)] and Ricalde[13)] but not systematically. The N-type groups are further subdivided into N and N' with a=1 and >1, respectively. The general expressions for them are given in Table 4.

As evident in Table 1, almost two thirds of D's below 100 are grouped into at least C- and/or N-types. Note that the largest y_1 value for them is as small as 66 for D=54.

2.3. Step up from Llep to Pell

In general the procedure for obtaining (r_1, s_1) of Llep is easier than the case with Pell of the same D. It is known that Llep has solutions only for D of odd p, and such D is expressed by the sum of a pair of square numbers. However, this is not a necessary but a sufficient condition. Anyway those numbers which are expressed by the sum of a pair of square numbers are printed in bold in Table 1.

By applying the relation between (x_1, y_1) and (r_1, s_1) to the PR of Llep for D with larger (x_1, y_1) the desired RP of Pell with the same D can be obtained. Given a PR of Llep for a group of D as in the following form,

$$A^2 - D\,B^2 = -1, \tag{2.12}$$

where A, B, and D are polynomials in terms of an integer variable m or a set of variables. In this paper, however, only the former case is assumed. Then square (2.12) followed by some manipulation the following equation can be obtained,

$$(A^2 + D\,B^2)^2 - D\,(2\,A\,B)^2 = 1, \tag{2.13}$$

which is the PR of Pell corresponding to the given Llep. Note that if (A, B) are (x_1, y_1), $(A^2 + D\,B^2, 2\,A\,B)$ give (x_2, y_2). Anyway one can use this discussion as

(Strategy 5) From the set of (A, B, D) for a PR of Llep one can obtain the PR of the corresponding Pell as $(A^2 + D\,B^2)^2 - D\,(2\,A\,B)^2 = 1$.

2.4. S-Type

In Table 1 one can see three D's, i.e., 13, 41, and 74, whose s_1 is 5. By searching larger D's two groups of Llep solutions with s_1=5 were found as in Table 5, from which it is not so difficult to find a pair of PR's for the groups S5± as given in Table 6.

By following similar procedures a variety of PR's of the groups of S-type were found as assembled in Table 6, which also shows that all the PR's obtained can be derived either from the pair of master equations M±. Now almost 90 per cent of D's in Table 1 belongs to any one group of C-, N-, or S-type.

As already seen in Fig. 1 the transition region of the value of y_1 from "small" to "large" seems to be around ten or twenty. Then the y_1 values less than a few scores are plotted against D smaller than 160 and their grouping was shown by the smooth curves as in Fig. 2. Although the C-type groups are not explicitly assigned to avoid confusion in the figure, it is to be noted that all the points are shown to belong to at least a group of either C-, N-, or S-type, except for (D=135, y_1=21), which belongs to T9 (See Table 7). Since the function $y_1(m)$ for N- and S-type groups is linear and $D(m)$ is assumed to be quadratic (See Table 3), all their curves are parabolic. However, from the bottom left to the top half of Fig. 2 a straight line can be seen connecting the several terminal points of C-, N-, or S-type. This is T2 group as will be explained in the next section.

Another important feature in Fig. 2 is the group of vertical parabola-like curves which are overlapping with each other and continuously growing up systematically. Some curves are densely populated by points, i.e., solutions of Pell, while some are not, possibly due to some in-phase and out-of-phase behavior caused by the crossing of horizontal parabolic curves of C- and N-types. Although these curves were drawn tentatively without any rigorous analysis and by throwing out larger y_1 values, existence of some unknown and hidden mathematical structure is anticipated.

2.5. T-, Q-, and X-Types

From the straight line in Fig. 2 one can select out those D's as 5, 21, 45, and 77, which eventually form the sequence $(2m+1)^2-4$. Then it is quite easy to derive the group T2 as given in Table 7. This group was already found by Ramasamy[15] but in a different form. Namely, he proved the following PR,

$$(4t^3+18t^2+24t+9)^2-(4t^2+12t+5)(2t^2+6t+4)^2 = 1,$$

which can be transformed into a more elegant expression by putting $t=m-1$ as

$$[(4m-2)(m+1)^2+1]^2-[(2m+1)^2-4][2m(m+1)]^2= 1,$$

showing that it can be derived from M+ by putting $k=m+1$ and $m=4m-2$.

This group was classified into T-type as its $x_1(m)$ is cubic, and two more T-type groups were found as in Table 7. On the other hand, only one quartic Q-type group was found in the present analysis.

The highest degree of $x_1(m)$ polynomial discovered so far is six for the two groups, X2 and X10 as

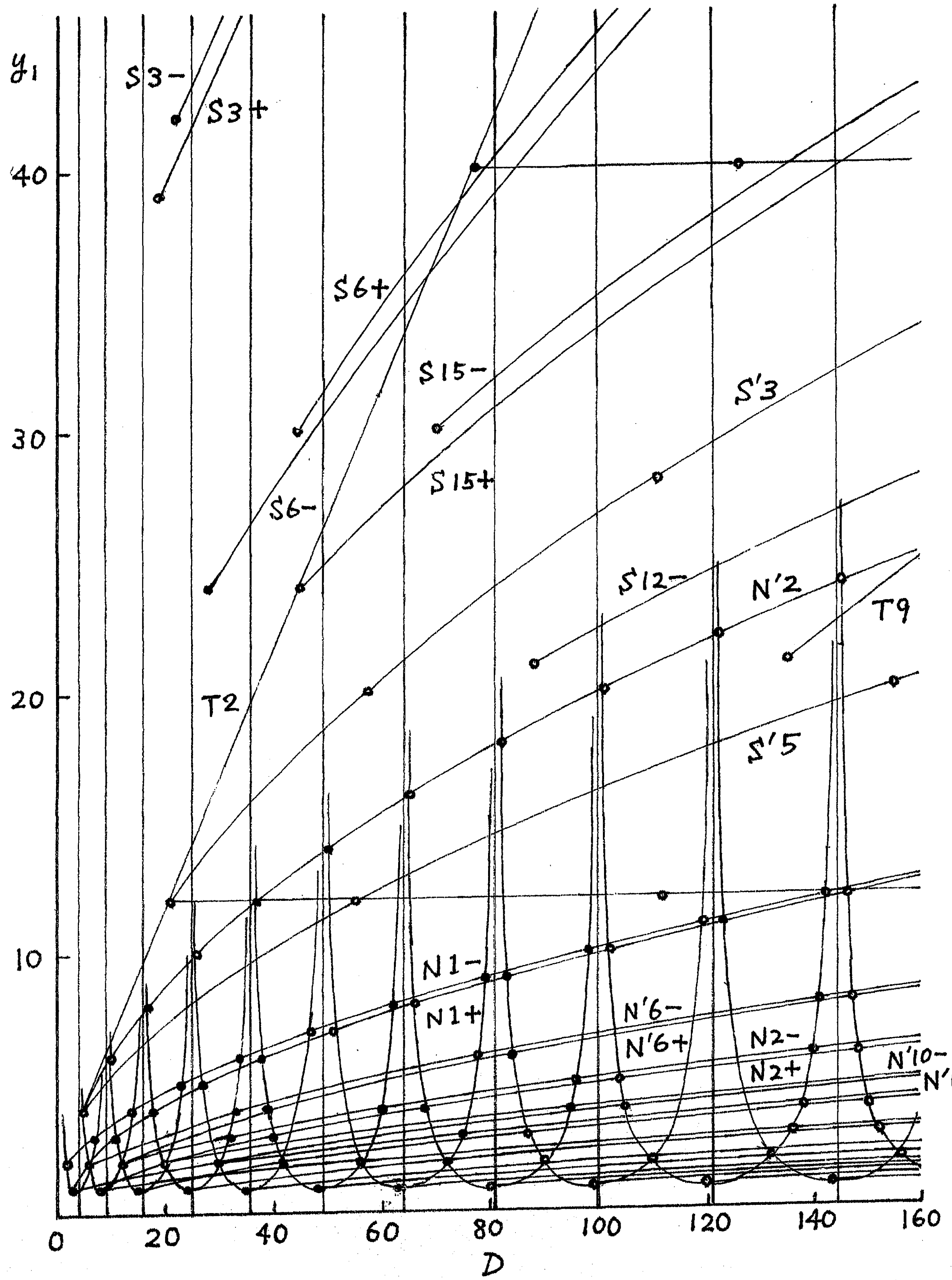

Fig. 2 y_1–D plot for smaller y_1 values. Note that the curve T2 in Fig. 1 is a straight line here.

given in Table 7. Their PR's have very peculiar forms.

There are only six D's below 100 whose p is 5. They are 13, 29, 53, 74, 85, and 89. Although 13 and 74 are already shown to form S5– group, 13, 29, 53, and 85, which can be expressed as $(2m+1)^2+4$ and whose CFE have similar forms, were found to form another group. It is not difficult to derive the PR of their Llep as,

$$\{2[m^3+(m+1)^3]\}^2-[(2m+1)^2+4][m^2+(m+1)^2]^2=-1.$$

Then by applying Strategy 5 to this formula the PR of Pell as shown in Table 7 was obtained.

In as early as 1901 Ricalde[13] reported the result of the same PR of Pell for X2 group without any discussion but erroneously. Namely, in his paper $x_1(m)$ is given as $8[m^3+(m+1)^3]+1$ with the square sign to the square bracket missing. Later in 1912 Whitford[1] introduced the Ricalde's PR just in this erroneous form. To the present authors' awareness this result has never been cited by any other authors. Although Ricalde must have derived the correct PR, the present authors believe that the correct expression is first published in this paper.

It can be shown that the PR's of both X2 and X10 are derived from M+, and close resemblance between their forms is observed. It is further interesting to know that D=61, which has the largest (x_1, y_1) below D<100 as large as ten digit numbers, has its youngest elder brother, D=317, with two-digit larger (x_1, y_1).

3. Problems to be studied

Although we have checked the CFE's of D's below ten thousand, four D's (73, 91, 94, 98) below one hundred have no other D connected through any PR. It is an open question if it is possible or not to find any group for them by expanding the upper limit of exploration. An affirmative anticipation is based on the observation that for S53+ group the next larger member of D=89 is as large as 3898.

Ramasamy[15] obtained a PR with three parameters as

$$(mt^3+nt^2+1)^2-t^2\,(m^2t^4+2mnt^3+n^2t^2+2mt+2n)=1.$$

However, the merit of this sophisticated master equation is not yet clarified.

Anyway the results obtained in this paper have shown an effectiveness of the proposed strategies for the systematic analysis of the chaotic behavior of the solutions of Pell and Llep. However, mathematical meaning of the interesting group of parabola-like curves in Fig. 2 need to be clarified.

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Table 1. Solutions and classification of Pell and Llep of D<100.

D	x_1	y_1	p	r_1	s_1	Group			
2	3	2	1	1	1	C2	N1–	N'2	S3–
3	2	1	2			C1	N1+	N3–	N'2–
5	9	4	1	2	1	C4+	N'2	N'6–	S'5
						T2	X2	X10+	
6	5	2	2			C2	N1+	N'6–	C'20+
7	8	3	4			C3–	N1–		
8	3	1	2			C1	N2+	N4–	S12–
10	19	6	1	3	1	C6+	N'2		
11	10	3	2			C3+	N1+	S11	
12	7	2	2			C2	N2–	N'6+	
13	649	180	5	18	5	S5–	T6–	X2	
14	15	4	4			C4–	N1–		
15	4	1	2			C1	N3+	N5–	
17	33	8	1	4	1	C8+	N'2		
18	17	4	2			C4+	N1+	S'9	
19	170	39	6			C'39–	S3+		
20	9	2	2			C2	N2+	N'10–	
21	55	12	6			C'12–	S'3	T2	
22	197	42	6			S3–			
23	24	5	4			C5–	N1–		
24	5	1	2			C1	N4+	N6–	
26	51	10	1	5	1	C10+	N'2		
27	26	5	2			C5+	N1+		
28	127	24	4			S6–	C'24+		
29	9801	1820	5	70	13	S13+	X2		
30	11	2	2			C2	N3–	N'10+	
31	1520	273	8			S7–			
32	17	3	4			C3–	N2–		
33	23	4	4			C4–	N'6–		
34	35	6	4			C6–	N1–		

35	6	1	2			C1	N5+	N7–
37	73	12	1	6	1	C12+	N’2	
38	37	6	2			C6+	N1+	
39	25	4	2			C4+	N’6+	
40	19	3	2			C3+	N2+	
41	2049	320	3	32	5	S5+		
42	13	2	2			C2	N3+	N’14–
43	3482	531	10			S9–		
44	199	30	8			S6+	C’30+	
45	161	24	6			C’24–	S15+	T2
46	24335	3588	12			S23–		
47	48	7	4			C7–	N1–	
48	7	1	2			C1	N6+	N8–
50	99	14	1	7	1	C14+	N’2	
51	50	7	2			C7+	N1+	
52	649	90	6			S’10–		
53	66249	9100	5	182	25	S53–	X2	
54	485	66	6			C’66–	S3+	
55	89	12	4			C’12+	S’5	
56	15	2	2			C2	N4–	N’14+
57	151	20	6			S’3	C’20–	
58	19603	2574	7	99	13	S13–		
59	530	69	6			S3–		
60	31	4	4			C4–	N2–	
61	1766319049	226153980	11	29718	3805	X10–		
62	63	8	4			C8–	N1–	
63	8	1	2			C1	N7+	N9–
65	129	16	1	8	1	C16+	N’2	
66	65	8	2			C8+	N1+	
67	48842	5967	10			S27–		
68	33	4	2			C4+	N2+	
69	7775	936	8			T6+		
70	251	30	6			C’30–	S15–	
71	3480	413	8			S7+		

72	17	2	2			C2	N4+	N'18–
73	2281249	267000	7	1068	125	☆		
74	3699	430	5	43	5	S5–		
75	26	3	8			C3–	N3–	
76	57799	6630	12			Q12		
77	351	40	6			T2	Q2	
78	53	6	4			C6–	N'6–	
79	80	9	4			C9–	N1–	
80	9	1	2			C1	N8+	N10–
82	163	18	1	9	1	C18+	N'2	
83	82	9	2			C9+	N1+	
84	55	6	2			C6+	N'6+	
85	285769	30996	5	378	41	X2		
86	10405	1122	10			S11–		
87	28	3	2			C3+	N3+	
88	197	21	6			C'21–	S12–	
89	500001	53000	5	500	53	S53–		
90	19	2	2			C2	N5–	N'18+
91	1574	165	8			☆		
92	1151	120	8			S10–		
93	12151	1260	10			T6–		
94	2143295	221064	16			☆		
95	39	4	4			C4–	N'10–	
96	49	5	4			C5–	N2–	
97	62809633	6377352	11	5604	569	☆		
98	99	10	4			C10–	N1–	
99	10	1	2			C1	N9+	N11–

$x_1^2 - D y_1^2 = 1$, $r_1^2 - D s_1^2 = -1$, p: length of periodic continued fraction of $\sqrt{D}$.

Those D's which are expressed by the sum of a pair of square numbers are printed in boldface.

☆: "Lonely star" which does not have any other D below 10,000 connected by a polynomial expression.

Table 2. Polynomial relations of C-type Pell. Italic means the 2nd solution.

C1[12,13] $(m+1)^2-(m^2+2m)\cdot 1^2=1$ $(y=1)$ [M+: k=1]

m	1	2	3	4	5	6	7	8	9
D	3	8	15	24	35	48	63	80	99
x	2	3	4	5	6	7	8	9	10

$p=2$, $\sqrt{D}=[m; 1, 2m]$ $(m\geq 2)$

C2[12] $(2m+1)^2-(m^2+m)\cdot 2^2=1$ $(y=2)$ [M+: k=1, m=2m]

m	1	2	3	4	5	6	7	8	9	10
D	2	6	12	20	30	42	56	72	90	110
x	3	5	7	9	11	13	15	17	19	21

$p=2$, $\sqrt{D}=[m; 2, 2m]$ $(m\geq 2)$

C3± $(9m\pm 1)^2-(9m^2\pm 2m)\cdot 3^2=1$ $(y=3)$ [M±: k=3]

	m	1	2	3	4	5		1	2	3	4	5
(+)	D	11	40	87	152	235	(−)	7	32	75	136	215
	x	10	19	28	37	46		8	17	26	35	44

$p=2$, $\sqrt{D}=[3m; 3, 6m]$ $p=4$, $\sqrt{D}=[3m-1; 1, 1, 1, 6m-2]$ $(m\geq 1)$

C4±[15] $(8m\pm 1)^2-(4m^2\pm m)\cdot 4^2=1$ $(y=4)$ [M±: k=2, m=2m]

	m	1	2	3	4	5		*1*	2	3	4	5
(+)	D	5	18	39	68	105	(−)	*3*	14	33	60	95
	x	9	17	25	33	41		*7*	15	23	31	39

$p=2$, $\sqrt{D}=[2m; 4, 4m]$ $p=4$, $\sqrt{D}=[2m-1; 1, 2, 1, 4m-2]$ $(m\geq 2)$

C5± $(25m\pm 1)^2-(25m^2\pm 2m)\cdot 5^2=1$ $(y=5)$ [M±: k=5]

	m	1	2	3	4	5		1	2	3	4	5
(+)	D	27	104	231	408	635	(−)	23	96	219	392	615
	x	26	51	76	101	126		24	49	74	99	124

$p=2$, $\sqrt{D}=[5m; 5, 10m]$ $p=4$, $\sqrt{D}=[5m-1; 1, 3, 1, 10m-2]$ $(m\geq 1)$

C6± $(18m\pm 1)^2-(9m^2\pm m)\cdot 6^2=1$ $(y=6)$ [M±: k=3, m=2m]

	m	1	2	3	4	5		*1*	2	3	4	5
(+)	D	10	38	84	148	230	(−)	*8*	34	78	140	220
	x	19	37	55	73	91		*17*	35	53	71	89

$p=2$, $\sqrt{D}=[3m; 6, 6m]$ $p=4$, $\sqrt{D}=[3m-1; 1, 4, 1, 6m-2]$ $(m\geq 2)$

C7± $(49m\pm1)^2-(49m^2\pm2m)\cdot7^2=1$ $(y=7)$ [M±: k=7]

	m	1	2	3	4		1	2	3	4
(+)	D	51	200	447	792	(−)	47	192	435	776
	x	50	99	148	197		48	97	146	195

$p=2$, $\sqrt{D}=[7m;\,7,\,14m]$ $p=4$, $\sqrt{D}=[7m-1;\,1,5,1,14m-2]$ $(m\geq1)$

C8± $(32m\pm1)^2-(16m^2\pm m)\cdot8^2=1$ $(y=8)$ [M±: k=4, m=2m]

	m	1	2	3	4		*1*	2	3	4
(+)	D	17	66	147	260	(−)	*15*	62	141	252
	x	33	65	97	129		*31*	63	95	127

$p=2$, $\sqrt{D}=[4m;\,8,\,8m]$ $p=4$, $\sqrt{D}=[4m-1;\,1,6,1,8m-2]$ $(m\geq2)$

C9± $(81m\pm1)^2-(81m^2\pm2m)\cdot9^2=1$ $(y=9)$ [M±: k=9]

	m	1	2	3	4		1	2	3	4
(+)	D	83	328	735	1304	(−)	79	320	723	1288
	x	82	163	244	325		80	161	242	323

$p=2$, $\sqrt{D}=[9m;\,9,\,18m]$ $p=4$, $\sqrt{D}=[9m-1;\,1,7,1,18m-2]$ $(m\geq1)$

C10± $(50m\pm1)^2-(25m^2\pm m)\cdot10^2=1$ $(y=10)$ [M±: k=5, m=2m]

	m	1	2	3	4		*1*	2	3	4
(+)	D	26	102	228	404	(−)	*24*	98	222	396
	x	51	101	151	201		*49*	99	149	199

$p=2$, $\sqrt{D}=[5m;\,10,\,10m]$ $p=4$, $\sqrt{D}=[5m-1;\,1,8,1,10m-2]$ $(m\geq2)$

		C12+	C14+	C16+	C18+	C20+
$m=1$	D	37	50	65	82	101
	x	73	99	129	163	201
$m=2$	D	146	198	258	326	402
	x	145	197	257	325	401

C$(2j+1)$± [M±: k=$2j+1$]

$$[(2j+1)^2m\pm1]^2-[(2j+1)^2m^2\pm2m]\cdot(2j+1)^2=1 \quad (j\geq0)$$

+: $p=2$, $\sqrt{D}=[(2j+1)\,m;\,(2j+1),\,(4j+2)\,m]$

−: $p=4$, $\sqrt{D}=[(2j+1)\,m-1;\,1,\,(2j-1),\,1,\,(4j+2)\,m-2]$ $(m\geq1)$

C$(2j)$± [M±: k=j, m=2m]

$$[2j^2m\pm1]^2-[j^2m^2\pm m]\cdot(2j)^2=1 \quad (j\geq1)$$

+: $p=2$, $\sqrt{D}=[j\,m;\,2j,\,2j\,m]$

−: $p=4$, $\sqrt{D}=[j\,m-1;\,1,2j-2,1,2j\,m-2]$ $(m\geq 2)$

C'12± $(72m\pm 17)^2-(36m^2\pm 17m+2)\cdot 12^2=1$ $(y=12)$ [M−: k=3, m=8m±2]

	m	1	2	3	4		1	2	3	4
(+)	D	55	180	377	646	(−)	21	112	275	510
	x	89	161	233	305		55	127	199	271

(+) $p=4$, $\sqrt{D}=[6m+1;2,2,2,12m+2]$ $(m\geq 1)$

(−) $p=6$, $\sqrt{D}=[6m-2;1,1,2,1,1,12m-4]$ $(m\geq 1)$

C'20± $(200m\pm 49)^2-(100m^2\pm 49m+6)\cdot 20^2=1$ $(y=20)$ [M−: k=5, m=8m±2]

	m	0	1	2	3		1	2	3	4
(+)	D	*6*	155	504	1802	(−)	57	308	759	1410
	x	*49*	249	449	649		151	351	551	751

(+) $p=4$, $\sqrt{D}=[10m+2;2,4,2,20m+4]$ $(m\geq 1)$

(−) $p=6$, $\sqrt{D}=[10m-3;1,1,4,1,1,20m-6]$ $(m\geq 1)$

C'21± $(441m\pm 244)^2-(441m^2\pm 488m+135)\cdot 21^2=1$ $(y=21)$ [M+: k=3, m=49m±27]

	m	0	1	2	3		1	2	3	4
(+)	D	135	1064	2875	5568	(−)	88	923	2640	5239
	x	244	685	1126	1567		197	638	1079	1520

(+) $p=8$, $\sqrt{D}=[21m+11;1,1,1,1,1,1,1,42m+22]$ $(m\geq 0)$

(−) $p=6$, $\sqrt{D}=[21m-12;2,1,1,1,2,42m-24]$ $(m\geq 1)$

C'24± $(288m\pm 127)^2-(144m^2\pm 127m+28)\cdot 24^2=1$ $(y=24)$ [M+: k=3, m=32m±14]

	m	0	1	2	3		1	2	3	4
(+)	D	28	299	858	1705	(−)	45	350	943	1824
	x	127	415	703	991		161	449	737	1025

(+) $p=4$, $\sqrt{D}=[12m+5;3,2,3,24m+10]$ $(m\geq 0)$

(−) $p=6$, $\sqrt{D}=[21m-6;1,2,2,2,1,24m-12]$ $(m\geq 1)$

C'30± $(450m\pm 199)^2-(225m^2\pm 199m+44)\cdot 30^2=1$ $(y=30)$ [M−: k=5, m=18m±8]

	m	0	1	2	3		1	2	3	4
(+)	D	44	468	1342	2666	(−)	70	546	1472	2848
	x	199	649	1099	1549		251	701	1151	1601

(+) $p=8$, $\sqrt{D}=[15m+6;1,1,1,2,1,1,1,30m+12]$ $(m\geq 0)$

(−) $p=6$, $\sqrt{D}=[15m-7;2,1,2,1,2,30m-14]$ $(m\geq 1)$

C'39± $(1521m \pm 1351)^2 - (1521m^2 \pm 2702m + 1200) \cdot 39^2 = 1$ $(y = 39)$ [M+: k=3, m=169m±150]

	m	0	1	2		1	2	3
(+)	D	1200	5423	12688	(−)	19	1880	6783
	x	1351	2872	4393		170	1691	3212

(+) $p = 8$, $\sqrt{D} = [39m + 34; 1, 1, 1, 3, 1, 1, 1, 78m + 68]$ $(m \geq 0)$

(−) $p = 6$, $\sqrt{D} = [39m - 35; 2, 1, 3, 1, 2, 78m - 70]$ $(m \geq 1)$

C'66± $(2178m \pm 1693)^2 - (1089m^2 \pm 1693m + 658) \cdot 66^2 = 1$ $(y = 66)$ [M+: k=3, m=242m±188]

	m	0	1	2		1	2	3
(+)	D	658	3440	8400	(−)	54	1628	5380
	x	1693	3871	6049		485	2663	4841

(+) $p = 8$, $\sqrt{D} = [33m + 25; 1, 1, 1, 6, 1, 1, 1, 66m + 50]$ $(m \geq 0)$

(−) $p = 6$, $\sqrt{D} = [33m - 26; 2, 1, 6, 1, 2, 66m - 52]$ $(m \geq 1)$

Table 3. Types of the solutions of Pell according to the degree of PR.

$[x_1(m)]^2 - D(m)\,[y_1(m)]^2 = 1$		
$y_1(m)$	$x_1(m)$	Type
m^0	m^1	C
m^1	m^2	N and S
m^2	m^3	T
m^3	m^4	Q
$> m^4$	$> m^5$	X

Tasble 5. Two groups of D's whose s_1 is 5. Their CFE's are also shown.

D	r_1	$\sqrt{D}$	D	r_1	$\sqrt{D}$
13	18	[3; 1, 1, 1, 1, 6]	41	32	[6; 2, 2, 12]
74	43	[8; 1, 1, 1, 1,16]	130	57	[11; 2, 2, 22]
185	68	[13; 1, 1, 1, 1, 26]	269	82	[16; 2, 2, 32]
346	93	[18; 1, 1, 1, 1, 36]	458	107	[21; 2, 2, 42]

Table 4. Polynomial relations of N-type Pell.

	N1+[12]		N1−[12]	
m, y_1	D	x_1	D	x_1
1	3	2	−1	0
2	6	5	2	3
3	11	10	7	8
4	18	17	14	15
5	27	26	23	24
6	38	37	34	35
7	51	50	47	48
8	66	65	62	63
9	83	82	79	80
10	102	101	98	99

N1±: $(m^2 \pm 1)^2 - (m^2 \pm 2) \cdot m^2 = 1$

(+): $p=2$, $\sqrt{D}=[m; m, 2m]$ $(m \geq 1)$

(−): $p=4$, $\sqrt{D}=[m-1; 1, m-2, 1, 2m-2]$ $(m \geq 3)$

[M±: $k=m$, $m=1$] → N1±

N2+		N2−[12]	
D	x_1	D	x_1
8	3	0	—
20	9	12	7
40	19	32	17
68	33	60	31
104	51	96	49
148	73	140	71
200	99	192	97

N2±: $(2m^2 \pm 1)^2 - (4m^2 \pm 4) \cdot m^2 = 1$

(+): $p=2$, $\sqrt{D}=[2m; m, 4m]$ $(m \geq 1)$

(−): $p=4$, $\sqrt{D}=[2m-1; 1, m-2, 1, 4m-2]$ $(m \geq 3)$

[M±: $k=m$, $m=2$] → N2±

	N3+		N3−		N4+		N4−	
m, y_1	D	x_1	D	x_1	D	x_1	D	x_1
1	15	4	3	2	24	5	8	3
2	42	13	30	11	72	17	56	15
3	87	28	75	26	152	37	136	35
4	150	49	138	47	264	65	248	63

N3±: $(3m^2 \pm 1)^2 - (9m^2 \pm 6) \cdot m^2 = 1$

(+): $p=2$, $\sqrt{D}=[3m; m, 6m]$ $(m \geq 1)$

(−): $p=4$, $\sqrt{D}=[3m-1; 1, m-2, 1, 6m-2]$ $(m \geq 3)$

[M±: $k=m$, $m=3$] → N3±

N4±: $(4m^2 \pm 1)^2 - (16m^2 \pm 8) \cdot m^2 = 1$

(+): $p=2$, $\sqrt{D}=[4m; m, 8m]$ $(m \geq 1)$

(−): $p=4$, $\sqrt{D}=[4m-1; 1, m-2, 1, 8m-2]$ $(m \geq 3)$

[M±: $k=m$, $m=4$] → N4±

	N5+		N5−		N6+		N6−	
m, y_1	D	x_1	D	x_1	D	x_1	D	x_1
1	35	6	15	4	48	7	24	5
2	110	21	90	19	156	25	132	23
3	235	46	215	44	336	55	312	53

N5±: $(5m^2 \pm 1)^2 - (25m^2 \pm 10) \cdot m^2 = 1$

(+): $p=2$, $\sqrt{D}=[5m; m, 10m]$ $(m \geq 1)$

N6±: $(6m^2 \pm 1)^2 - (36m^2 \pm 12) \cdot m^2 = 1$

(+): $p=2$, $\sqrt{D}=[6m; m, 12m]$ $(m \geq 1)$

(–): p=4, $\sqrt{D}$=[5m–1; 1, m–2, 1, 10m–2] (m≥3) (–): p=4, $\sqrt{D}$=[6m–1; 1, m–2, 1, 12m–2] (m≥3)

[M±: k=m, m=5] → N5± [M±: k=m, m=6] → N6

	N7+		N7–		N8+		N8–	
m, y_1	D	x_1	D	x_1	D	x_1	D	x_1
1	63	8	35	6	80	9	48	7
2	210	29	182	27	272	33	240	31
3	455	64	427	62	592	73	560	71

N7±: $(7m^2 \pm 1)^2 - (49m^2 \pm 14) \cdot m^2 = 1$

(+): p=2, $\sqrt{D}$=[7m; m, 14m] (m≥1)

(–): p=4, $\sqrt{D}$=[7m–1; 1, m–2, 1, 14m–2] (m≥3)

[M±: k=m, m=7] → N7±

N8±: $(8m^2 \pm 1)^2 - (64m^2 \pm 16) \cdot m^2 = 1$

(+): p=2, $\sqrt{D}$=[8m; m, 16m] (m≥1)

(–): p=4, $\sqrt{D}$=[8m–1; 1, m–2, 1, 16m–2] (m≥3)

[M±: k=m, m=8] → N8

	N9+		N9–		N10+		N10–	
m, y_1	D	x_1	D	x_1	D	x_1	D	x_1
1	99	10	63	8	120	11	80	9
2	342	37	306	35	420	41	380	39

N9±: $(9m^2 \pm 1)^2 - (81m^2 \pm 18) \cdot m^2 = 1$

(+): p=2, $\sqrt{D}$=[9m; m, 18m] (m≥1)

(–): p=4, $\sqrt{D}$=[9m–1; 1, m–2, 1, 18m–2] (m≥3)

[M±: k=m, m=9] → N9±

N10±: $(10m^2 \pm 1)^2 - (100m^2 \pm 20) \cdot m^2 = 1$

(+): p=2, $\sqrt{D}$=[10m; m, 20m] (m≥1)

(–): p=4, $\sqrt{D}$=[10m–1; 1, m–2, 1, 20m–2] (m≥3)

[M±: k=m, m=10] → N10±

	N11+		N11–	
m, y_1	D	x_1	D	x_1
1	143	12	99	10
2	506	45	462	43
3	1111	100	1067	98

[M±: k=m, m=11] → N11±

N11±: $(11m^2 \pm 1)^2 - (121m^2 \pm 22) \cdot m^2 = 1$

(+): p=2, $\sqrt{D}$=[11m; m, 22m] (m≥1)

(–): p=4, $\sqrt{D}$=[11m–1; 1, m–2, 1, 22m–2] (m≥3)

N'2[12,13] [M+: k=m, m=2] → N'2

m	y_1	D	x_1	m	y_1	D	x_1
1	2	2	3	7	14	50	99
2	4	5	9	8	16	65	129
3	6	10	19	9	18	82	163
4	8	17	33	10	20	101	201
5	10	26	51	11	22	122	243
6	12	37	73	12	24	145	289

N'2: $(2m^2 + 1)^2 - (m^2 + 1)(2m)^2 = 1$

N$j\pm$ [M$\pm$: k=m, m=j] [M$\pm$: k=m, m=4j+2]

$(j\,m^2\pm1)^2-(j^2m^2\pm2j)\,m^2=1 \quad (j\geq1)$

(+): p=2, $\sqrt{D}$=[$j\,m$; m, $2\,j\,m$] ($m\geq1$)

(−): p=4, $\sqrt{D}$=[$j\,m$−1; 1, m−2, 1, $2\,j\,m$−2] ($m\geq1$)

N'(4j+2)$\pm$ [M$\pm$: k=m, m=4j+2]

$[(4j+2)\,m^2\pm1]^2-[(2j+1)^2m^2\pm(2j+1)]\,(2m)^2=1 \quad (j\geq1)$

(+): p=2, $\sqrt{D}$=[(2j+1) m; 2m, (4j+2)m] ($m\geq1$)

(−): p=4, $\sqrt{D}$=[(2j+1) m−1; 1, 2m−2, 1, (4j+2)m] ($m\geq1$)

		N'6+		N'6−		N'10+		N'10−	
m	y_1	D	x_1	D	x_1	D	x_1	D	x_1
1	2	12	7	6	5	30	11	20	9
2	4	39	25	33	23	105	41	95	39
3	6	84	55	78	53	230	91	220	89
4	8	147	97	141	95	405	161	395	159
5	10	228	151	222	149	630	251	620	249

N'6$\pm$: $(6m^2\pm1)^2-(9m^2\pm3)\,(2m)^2=1$

(+): p=2, $\sqrt{D}$=[3m; 2m, 6m] ($m\geq1$)

(−): p=4, $\sqrt{D}$=[3m−1; 1, 2m−2, 1, 6m−2]

[M$\pm$: k=m, m=6] → N'6$\pm$ ($m\geq2$)

N'10$\pm$: $(10m^2\pm1)^2-(25m^2\pm5)\,(2m)^2=1$

(+): p=2, $\sqrt{D}$=[5m; 2m, 10m] ($m\geq1$)

(−): p=4, $\sqrt{D}$=[5m−1; 1, 2m−2, 1, 10m−2]

[M$\pm$: k=m, m=10] → N'10$\pm$ ($m\geq2$)

		N'14+		N'14−		N'18+		N'18−	
m	y_1	D	x_1	D	x_1	D	x_1	D	x_1
1	2	56	15	42	13	90	19	72	17
2	4	203	57	189	55	333	73	315	71
3	6	448	127	434	125	738	163	720	161
4	8	791	225	777	223	1305	289	1287	287

N'14$\pm$: $(14m^2\pm1)^2-(49m^2\pm7)\,(2m)^2=1$

(+): p=2, $\sqrt{D}$=[7m; 2m, 14m] ($m\geq1$)

(−): p=4, $\sqrt{D}$=[7m−1; 1, 2m−2, 1, 14m−2]

[M$\pm$: k=m, m=14] → N'14$\pm$ ($m\geq2$)

N'18$\pm$: $(18m^2\pm1)^2-(81m^2\pm9)\,(2m)^2=1$

(+): p=2, $\sqrt{D}$=[9m; 2m, 18m] ($m\geq1$)

(−): p=4, $\sqrt{D}$=[9m−1; 1, 2m−2, 1, 18m−2]

[M$\pm$: k=m, m=18] → N'18$\pm$ ($m\geq2$)

Table 6. Polynomial relations of S-type Pell. Italic means the 2nd solution.

S3± $[(9m\pm4)^2+1]^2-(9m^2\pm8m+2)\,[3(9m\pm4)]^2=1$ [M+: $k=9m\pm4$, $m=1$]

		$m=1$	2	3	4	5
(+)	D	19	54	107	178	267
	x	170	485	962	1601	2402
	y	39	66	93	120	147
	$p=6$, $\sqrt{D}=[3m+1;\,2,1,3m,1,2,6m+2]$ $(m\geq1)$					
(−)	D	3	22	59	114	187
	x	*26*	197	530	1025	1682
	y	*15*	42	69	96	123
	$p=6$, $\sqrt{D}=[3m-2;\,1,2,3m-2,2,1,6m-4]$ $(m\geq1)$					

S'3 $[6(2m+1)^2+1]^2-(9m^2+9m+3)\,[4(2m+1)]^2=1$ [M+: $k=2m+1$, $m=6$]

D	21	57	111	183	273
x	55	151	295	487	727
y	12	20	28	36	44

$p=6$, $\sqrt{D}=[3m+1;\,1,1,2m,1,1,6m+2]$ $(m\geq1)$

S5± $[2(25m\pm7)^2+1]^2-(25m^2\pm14m+2)[10(25m\pm7)]^2=1$ [M+: $k=25m\pm7$, $m=2$]

(+)	D	41	130	269	458	697
	x	2049	6499	13449	22899	34849
	y	320	570	820	1070	1320
	$p=3$, $\sqrt{D}=[5m+1;\,2,2,10m+2]$ $(m\geq1)$					
(−)	D	13	74	185	346	557
	x	649	3699	9249	17299	27849
	y	180	430	680	930	1180
	$p=5$, $\sqrt{D}=[5m-2;\,1,1,1,1,10m-4]$ $(m\geq1)$					

S'5 $[10(2m+1)^2-1]^2-[5(5m^2+5m+1)][4(2m+1)]^2=1$ [M−: $k=2m+1$, $m=10$]

	$m=0$	1	2	3	4
D	5	55	155	305	505
x	9	89	249	489	809
y	4	12	20	28	36

$p=4$, $\sqrt{D}=[5m+2;\,2,2m,2,10m+4]$ $(m\geq1)$

S6± $[2(9m\pm1)^2-1]^2-(36m^2\pm8m)\,[3(9m\pm1)]^2=1$ [M–: k=9m±1, m=2]

		$m=1$	2	3	4	5
(+)	D	44	160	348	608	940
	x	199	721	1567	2737	4231
	y	30	57	84	111	138

p=8 $\sqrt{D}=[6m;\,1,1,1,3m-1,1,1,1,12m]$ ($m\geq1$)

(–)	D	28	128	300	544	860
	x	127	577	1351	2449	3871
	y	24	51	78	105	132

p=4 $\sqrt{D}=[6m-1;\,3,3m-1,3,12m-2]$ ($m\geq1$)

S7± $[(49m\pm10)^2-1]^2-(49m^2\pm20m+2)\,[7(49m\pm10)]^2=1$ [M–: k=49m±10, m=1]

(+)	D	71	238	503	866	1327
	x	3480	11663	24648	42435	65024
	y	413	756	1099	1442	1785

p=8 $\sqrt{D}=[7m+1;\,2,2,1,7m,1,2,2,14m+2]$ ($m\geq1$)

(–)	D	31	158	383	706	1127
	x	1520	7743	18768	34595	55224
	y	273	616	959	1302	1645

p=8 $\sqrt{D}=[7m-2;\,1,1,3,7m-2,3,1,1,14m-4]$ ($m\geq1$)

S9± $[(81m\pm22)^2+1]^2-(81m^2\pm44m+6)\,[9(81m\pm22)]^2=1$ [M+: k=81m±22, m=1]

(+)	D	131	418	867	1478	2251
	x	10610	33857	70226	119717	182330
	y	927	1656	2385	3114	3843

p=6 $\sqrt{D}=[9m+2;\,2,4,9m+2,4\ 2,18m+4]$ ($m\geq1$)

(–)	D	43	242	603	1126	1811
	x	3482	19601	48842	91205	146690
	y	531	1260	1989	2718	3447

p=10 $\sqrt{D}=[9m-3;\,1,1,3,1,9m-4,1,3,1,1,18m-6]$ ($m\geq1$)

S'9 $[18(2m+1)^2-1]^2-[9(9m^2+9m+2)]\,[4(2m+1)]^2=1$ [M–: k=2m+1, m=18]

	$m=0$	1	2	3	4
D	18	180	504	990	1638
x	17	161	449	881	1457
y	4	12	20	28	36

p=4 $\sqrt{D}=[9m+4;\,2,2m,2,18m+8]$ ($m\geq1$)

S10± $[2(25m\pm1)^2-1]^2-(100m^2\pm8m)\,[5(25m\pm1)]^2=1$ [M–: k=25m±1, m=2]

		$m=1$	2	3	4	5
(+)	D	108	416	924	1632	2540
	x	1352	5201	11551	20401	31751
	y	130	255	380	505	630
	p=8	$\sqrt{D}$=[10m; 2, 1, 1, 5m–1, 1, 1, 2, 20m]				(m≥1)
(–)	D	92	384	876	1568	2460
	x	1151	4801	10951	19601	30751
	y	120	245	370	495	620
	p=8	$\sqrt{D}$=[10m–1; 1, 1, 2, 5m–1, 2, 1, 1, 20m–2]				(m≥1)

S'10± $[2(25m\pm7)^2+1]^2-(100m^2\pm56m+8)\,[5(25m\pm7)]^2=1$ [M+: k=25m±7, m=2]

(+)	D	164	520	1076	1832	2788
	x	2049	6499	13449	22899	34849
	y	160	285	410	535	660
	p=6	$\sqrt{D}$=[10m+2; 1, 4, 5m+1, 4, 1, 20m+4]				(m≥1)
(–)	D	52	296	740	1384	2228
	x	649	3699	9249	17299	27849
	y	90	215	340	465	590
	p=6	$\sqrt{D}$=[10m–3; 4, 1, 5m–3, 1, 4, 20m–6]				(m≥1)

S11± $[(121m\pm19)^2+1]^2-(121m^2\pm38m+3)[11(121m\pm19)]^2=1$ [M+: k=121m±19, m=1]

(+)	D	162	563	1206	2091	3218
	x	19601	68122	145925	253010	389377
	y	1540	2871	4202	5533	6864
	p=10	$\sqrt{D}$=[11m+1; 1, 2, 1, 2, 11m+1, 2, 1, 2, 1, 22m+2]				(m≥1)
(–)	D	86	411	978	1787	2838
	x	10405	49730	118337	216226	343397
	y	1122	2453	3784	5115	6446
	p=10	$\sqrt{D}$=[11m–2; 3, 1, 1, 1, 11m–3, 1, 1, 1, 3, 22m–4]				(m≥1)

S12± $[4(9m\pm2)^2+1]^2-[8(18m^2\pm8m+1)][3(9m\pm2)]^2=1$ [M+: k=9m±2, m=4]

(+)	D	216	712	1496	2568	3928
	x	485	1601	3365	5777	8837
	y	33	60	87	114	141
	p=6	$\sqrt{D}$=[12m+2; 1, 2, 3m, 2, 1, 24m+4]				(m≥1)

		$m=0$	1	2	3	4
(–)	D	8	88	456	1112	2056
	x	17	197	1025	2501	4625
	y	6	21	48	75	102

$p=6$ $\sqrt{D}=[12m-3; 2, 1, 3m-2, 1, 2, 24m-6]$ $(m\geq 1)$

S13± $[2(169m\pm 70)^2+1]^2-(169m^2\pm 140m+29)[26(169m\pm 70)]^2=1$ [M+: $k=169m\pm 70$, $m=2$]

		$m=1$	2	3	4	5
(+)	D	29	338	985	1970	3293
	x	9801	114243	332929	665859	1113033
	y	1820	6214	10608	15002	19396

$p=5$ $\sqrt{D}=[13m+5; 2, 1, 1, 2, 26m+10]$ $(m\geq 1)$

(–)	D	58	425	1130	2173	3554
	x	19603	143649	381939	734473	1201251
	y	2574	6968	11362	15756	20150

$p=7$ $\sqrt{D}=[13m-6; 1, 1, 1, 1, 1, 1, 26m-12]$ $(m\geq 1)$

S15± $[10(9m\pm 4)^2+1]^2-[5(45m^2\pm 40m+9)][6(9m\pm 4)]^2=1$ [M+: $k=9m\pm 4$, $m=1$]

		$m=0$	1	2	3	4
(+)	D	45	470	1345	2670	4445
	x	161	1691	4841	9611	16001
	y	24	78	132	186	240

$p=6$ $\sqrt{D}=[15m+6; 1, 2, 6m+2, 2, 1, 30m+12]$ $(m\geq 0)$

		$m=1$	2	3	4	5
(–)	D	70	545	1470	2845	4670
	x	251	1961	5291	10241	16811
	y	30	84	138	192	246

$p=6$ $\sqrt{D}=[15m-7; 2, 1, 6m-4, 1, 2, 30m-14]$ $(m\geq 1)$

S23± $[(529m\pm 373)^2-1]^2-(529m^2\pm 746m+263)[23(529m\pm 373)]^2=1$ [M–: $k=529m\pm 373$, $m=1$]

		$m=0$	1	2	3	4
(+)	D	263	1538	3871	7262	11711
	x	139128	813603	2047760	3841599	6195120
	y	8579	20746	32913	45080	57247

$p=12$ $\sqrt{D}=[23m+16; 4, 1, 1, 1, 1, 23m+15, 1, 1, 1, 1, 4, 46m+32]$ $(m\geq 0)$

		$m=1$	2	3	4	5
(–)	D	46	887	2786	5743	9758
	x	24335	469224	1473795	3038048	5161983
	y	3588	15755	27922	40089	52256

$p=12$ $\sqrt{D}=[23m-17;1,3,1,1,2,23m-17,2,1,1,3,1,46m-34]$ $(m\geq1)$

S27± $[(729m\pm508)^2+1]^2-(729m^2\pm1016m+354)[27(729m\pm508)]^2=1$ [M+: $k=729m\pm508$, $m=1$]

		$m=0$	1	2	3
(+)	D	354	2099	5302	9963
	x	258065	1530170	3865157	7263026
	y	13716	33399	53082	72765

$p=10$ $\sqrt{D}=[27m+18;1,4,2,2,27m+18,2,2,4,1,54m+36]$ $(m\geq0)$

		$m=1$	2	3	4
(–)	D	67	1238	3867	7954
	x	48842	902501	2819042	5798465
	y	5967	25650	45333	65016

$p=10$ $\sqrt{D}=[27m-19;5,2,1,1,27m-20,1,1,2,5,54m-38]$ $(m\geq1)$

S53 $[2(2809m+500)^2+1]^2-(2809m^2+1000m+89)[106(2809m+500)]^2=1$ [M+: $k=2809m+500$, $m=2$]

	$m=0$	1	2
D	89	3898	13325
x	500001	21898963	74859849
y	53000	350754	648508

$p=5$ $\sqrt{D}=[53m+9;2,3,3,2,106m+18]$ $(m\geq0)$

Table 7. Polynomial relations of T-, Q-, and X-type Pell. Italic means the 2nd solution.

T2 $[(4m-2)(m+1)^2+1]^2-(4m^2+4m-3)[2m(m+1)]^2=1$ [M+: $k=m+1$, $m=4m-2$]

	$m=1$	2	3	4	5
D	5	21	45	77	117
x	9	55	161	351	649
y	4	12	24	40	60

$p=6$, $\sqrt{D}=[2m;1,m-1,2,m-1,1,4m]$ $(m\geq 2)$

T6± $[(18m\pm 7)(162m^2\pm 126m+23)]^2-[(2m\pm 1)(18m\pm 5)][18(3m\pm 1)(9m\pm 4)]^2=1$ [M±: $k=27m\pm 9$, $m=4m\pm 2$]

(+)

D	69	205	413	693	1045
x	7775	39689	113399	246401	456191
y	936	2772	5580	9360	14112

$p=8$, $\sqrt{D}=[6m+2;3,3m,1,4,1,3m,3,12m+4]$ $(m\geq 1)$

(−)

D	13	93	245	469	765
x	649	12151	51841	137215	285769
y	180	1260	3312	6336	10332

$p=10$, $\sqrt{D}=[6m-3;1,1,1,3m-2,6,3m-2,1,1,1,12m-6]$ $(m\geq 2)$

T9 $[27m(2m+1)^2+1]^2-[27m(3m+2)][(2m+1)(6m+1)]^2=1$ [M+: $k=2m+1$, $m=27m$]

D	135	432	891	1512	2295
x	244	1351	3970	8749	16336
y	21	65	133	225	341

$p=8$, $\sqrt{D}=[9m+2;1,2m+1,1,1,1,2m+1,1,18m+4]$ $(m\geq 1)$

Q12 $[18(18m-5)^2(36m^2-20m+3)+1]^2-[4(36m^2-20m+3)][6(18m-5)(162m^2-90m+13)]^2=1$

[M+: $k=18m-5$, $m=18(36m^2-20m+3)$]

	$m=$ 1	2	3	4
D	76	428	1068	1996
x	57799	1850887	11539207	40320199
y	6630	89466	353094	902490

$p=12$, $\sqrt{D}=[12m-4;1,2,3m-2,1,5,6m-2,5,1,3m-2,2,1,24m-8]$ $(m\geq 1)$

X2 $\{8[m^3+(m+1)^3]^2+1\}^2-[(2m+1)^2+4]\{4[m^3+(m+1)^3][m^2+(m+1)^2]\}^2=1$ [M+: $k=m^3+(m+1)^3$, m=8]

$m=$	0	1	2	3	4	5
D	*5*	13	29	53	85	125
x	*9*	649	9801	66249	285769	930249
y	*4*	180	1820	9100	30996	83204

p=5, $\sqrt{D}=[2m+1; m, 1, 1, m, 4m+2]$ $(m\geq 1)$

X10± $\{8[(25m\pm5)^3+(25m\pm6)^3]^2+1\}^2-(100m^2\pm44m+5)$

$\times\{20[(25m\pm5)^3+(25m\pm6)^3][(25m\pm5)^2+(25m\pm6)^2]\}^2=1$ [M+: $k=2[(25m\pm5)^3+(25m\pm6)^3]$, m=2]

$\{2[(25m\pm5)^3+(25m\pm6)^3]\}^2-(100m^2\pm44m+5)-\{5[(25m\pm5)^2+(25m\pm6)^2]\}^2=-1$

	m	D	r	s	x	y
(–)	1	61	29718	3805	1766319049	226153980
	2	317	352618	19805	248678907849	13967198980
	3	773	1343018	48305	3607394696649	129748968980
	4	1429	3375918	89305	22793644685449	602972713980
	p=11, $\sqrt{D}=[10m-3; 1, 4, 5m-2, 1, 2, 2, 1, 5m-2, 4, 1, 20m-6]$ $(m\geq 1)$					
(+)	0	*5*	*682*	*305*	*930249*	*416020*
	1	149	113582	9305	25801741449	2113761020
	2	493	683982	30805	935662752649	42140131020
	3	1037	2086882	64805	8710152963849	270480776020

p=9, $\sqrt{D}=[10m+2; 4, 1, 5m, 3, 3, 5m, 1, 4, 20m+4]$ $(m\geq 1)$