A remark on the covariance matrix of fractional Brownian motion

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Abstract

Let $X^H(t)$ be a fractional Brownian motion with index H $(0 < H \le 1/2)$, and let $D_n(t_0, t_1, \dots t_n)$ $(0 \le t_0 < t_1 < \cdots < t_n)$ denote the correlation matrix of $\{X^H(t_{k+1}) - X^H(t_k) : k = 1, \dots, n-1\}$. In this paper the asymptotic behaviour of $(1/n) \log \det D_n$ as n tends to ∞ is studied.

1. INTRODUCTION

Let $X^H(t)$ be a fractional Brownian motion with index H $(0 < H \le 1/2)$. That is, X^H is a real-valued centered Gaussian process such that

$$E[X^H(t)X^H(s)] = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}, \quad s, t \ge 0, \tag{1.1}$$

or equivalently,

$$X^H(0) = 0, \quad E[(X^H(t) - X^H(s))^2] = |t-s|^{2H}, \quad s, t \ge 0. \tag{1.2}$$

If $H = 1/2$, then X^H is the ordinary Brownian motion.

Definition 1.1. Let $0 \le t_0 < t_1 < \cdots < t_n$. We denote by $C_n(t_0, t_1, \dots, t_n)$ the covariance matrix of

$$(X^H(t_1) - X^H(t_0), X^H(t_2) - X^H(t_1), \dots, X^H(t_n) - X^H(t_{n-1})), \tag{1.3}$$

and by $D_n(t_0, t_1, \dots, t_n)$ the covariance matrix of the standardized increments, i.e.,

$$D_n(t_0, t_1, \dots, t_n) = \Big(\mathrm{Cov}\Big(\frac{X^H(t_i) - X^H(t_{i-1})}{(t_i - t_{i-1})^H}, \frac{X^H(t_j) - X^H(t_{j-1})}{(t_j - t_{j-1})^H}\Big)\Big)_{i,j}.$$

In particular, if $H = 1/2$, then we have

$$\det C_n(t_0, t_1, \dots, t_n) = (t_1 - t_0) \times \cdots \times (t_n - t_{n-1})$$

and

$$\det D_n(t_0, t_1, \ldots, t_n) = 1.$$

In the general case where $0 < H \leq 1/2$ we have the following inequality.

$$\left(\frac{1}{2}\right)^n \leq \frac{\det C_n(t_0, t_1, \ldots, t_n)}{(t_1 - t_0)^{2H}(t_2 - t_1)^{2H} \ldots (t_n - t_{n-1})^{2H}} \leq 1, \tag{1.4}$$

or equivalently,

$$\left(\frac{1}{2}\right)^n \leq \det D_n(t_0, t_1, \ldots, t_n) \leq 1. \tag{1.5}$$

Here the lower bound of (1.4) is due to Csörgő, et al.([1]), and the upper one is an easy consequence of the fact that for any positive definite matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, it holds that $\det A \leq a_{11}a_{22} \ldots a_{nn}$. Notice that (1.5) can be rewritten as follows:

$$-\log 2 \leq \frac{1}{n} \log \det D_n(t_0, t_1, \ldots, t_n) \leq 0.$$

The aim of the present paper is to show that the following limit exists in some sense and that the limit is in fact a constant which depends only on H.

$$\lim_{n \to \infty} \frac{1}{n} \log \det D_n(t_0, t_1, \ldots, t_n). \tag{1.6}$$

This problem was motivated in connection with evaluating the tail probabilities of the local times at the origin, though we shall not go into details here and refer to [3].

Theorem 1.1. *Assume that* $0 < H \leq 1/2$. *Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be independent random variables which are uniformly distributed on the interval* $[0, 1]$, *and let* $(\xi_1^{(n)}, \ldots, \xi_n^{(n)})$ *be its order statistic. Then there exists a constant* c_H $(-\log 2 \leq c_H \leq 0)$ *such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \det D_n(0, \xi_1^{(n)}, \ldots, \xi_n^{(n)}) = c_H \tag{1.7}$$

and

$$\lim_{n \to \infty} \left(\frac{1}{n} \log \det C_n(0, \xi_1^{(n)}, \ldots, \xi_n^{(n)}) + 2H \log n\right) = c_H \tag{1.8}$$

in probability.

The proof will be given in the next section.

The authors believe that the condition $0 < H \leq 1/2$ may be replaced by $0 < H < 1$ with $-\infty < c_H \leq 0$.

2. Proof of Theorem 1.1

We shall prove Theorem 1.1 by reducing it to the next theorem. However, since the idea of the proof remains valid for more general Gaussian processes, we shall not confine ourselves to fractional Brownian motions but consider a centered Gaussian process $Y(t)$ which has stationary increments with $Y(0) = 0$. We denote by $\varphi(t)$ the incremental covariance function: i.e., $E[(Y(t) - Y(s))^2] = \varphi(t-s)(0 \le s \le t)$, and let $C_n(t_0, t_1, \ldots, t_n)(0 \le t_0 < t_1 < \cdots < t_n)$ denote the covariance matrix of $(Y(t_1) - Y(t_0), Y(t_2) - Y(t_1), \ldots, Y(t_n) - Y(t_{n-1}))$ as in Section 1.

Theorem 2.1. *Assume that $\varphi(t)$ is concave and suppose*

$$\int_0^\infty e^{-t} |\log \varphi(t)| dt < \infty, \tag{2.1}$$

and let $0 = T_0 < T_1 < \cdots < T_n < \ldots$ be Poisson arrival times with intensity 1. *Then there exists a constant c $(-\infty < c < \infty)$ satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \log \det C_n(0, T_1, \ldots, T_n) = c, \qquad a.s. \tag{2.2}$$

We postpone the proof of Theorem 2.1, and we first see how Theorem 1.1 can be reduced to Theorem 2.1. So we consider the case of $Y(t) = X^H(t)$, and hence $\varphi(t) = t^{2H}$ $(0 < H \le 1/2)$. Since X^H is a self-similar process, we have

$$\det D_n(0, ct_1, \ldots, ct_n) = \det D_n(0, t_1, \ldots, t_n), \tag{2.3}$$

for every $c > 0$. Since $\{\xi_1^{(n)}, \ldots, \xi_n^{(n)}\}$ and $\{\frac{T_1}{T_{n+1}}, \ldots, \frac{T_n}{T_{n+1}}\}$ are equally distributed (see e.g. page 221 of Gut ([2])), (2.3) implies

$$\det D_n(0, \xi_1^{(n)}, \ldots, \xi_n^{(n)}) \overset{\mathcal{L}}{\sim} \det D_n(0, T_1, \ldots, T_n). \tag{2.4}$$

On the other hand, we have

$$\log \det D_n(0, T_1, \ldots, T_n) = \log \det C_n(0, T_1, \ldots, T_n) - \sum_{k=1}^n \log \varphi(T_k - T_{k-1}).$$

Since $\{\log \varphi(T_k - T_{k-1})\}_k$ are i.i.d. random variables, keeping the law of large numbers in mind we have from Theorem 2.1 that

$$\lim_{n\to\infty} \frac{1}{n} \log \det D_n(0, T_1, \ldots, T_n) = c - \int_0^\infty e^{-t} \log \varphi(t)\, dt, \tag{2.5}$$

where the right side of (2.5) is finite by the assumption (2.1). Combining (2.4) and (2.5), we see that the first half of Theorem 1.1 can be reduced to Theorem 2.1. The latter half can also be shown in a similar way by using

$$\det C_n(0, \xi_1^{(n)}, \ldots, \xi_n^{(n)}) \overset{\mathcal{L}}{\sim} T_{n+1}^{-2Hn} \det C_n(0, T_1, \ldots, T_n) \tag{2.6}$$

and $T_n/n \to 1$.

We now return to the proof of Theorem 2.1. The proof is based on the following subadditive ergodic theorem of Kingman ([5]).

Theorem A ([5]) . *Let* $\{X_{m,n}\}_{0\leq m<n}$ *be a family of random variables satisfying*

(S_1) $X_{m,n} \leq X_{m,k} + X_{k,n}, \quad 0 \leq m < k < n$,

(S_2) $\{X_{m+1,n+1}\} \overset{\mathcal{L}}{\sim} \{X_{m,n}\}$,

(S_3) $\inf\limits_{n\in\mathbb{N}} \dfrac{E[X_{0,n}]}{n} > -\infty$.

Then the finite limit

$$\zeta = \lim_{n\to\infty} \frac{X_{0,n}}{n} \tag{2.7}$$

exists with probability one and in mean.

Now put

$$X_{m,n} \equiv \log\det C_{n-m}(T_m, T_{m+1}, \ldots, T_n), \text{ for } 0 \leq m < n. \tag{2.8}$$

Then by an elementary property of positive definite matrices (see Proposition in Appendix), we have

$$\det C_{n-m}(t_m, \ldots, t_n) \leq \det C_{k-m}(t_m, \ldots, t_k) \cdot \det C_{n-k}(t_k, t_{k+1}, \ldots, t_n) \tag{2.9}$$

which implies

$$X_{m,n} \leq X_{m,k} + X_{k,n}, \qquad \text{for } 0 \leq m < k < n. \tag{2.10}$$

Thus we see that (S_1) is satisfied. Next, since Y has stationary increments, it holds that

$$\det C_{n-m}(t_m, t_{m+1}, \ldots, t_n) = \det C_{n-m}(0, t_{m+1} - t_m, \ldots, t_n - t_m). \tag{2.11}$$

Combining (2.8) and (2.11), we have

$$X_{m,n} = \log\det C_{n-m}(0, T_{m+1} - T_m, \ldots, T_n - T_m), \tag{2.12}$$

and hence it is easy to see that (S_2) is satisfied. From (1.4), we have

$$\begin{aligned} \frac{E[X_{0,n}]}{n} &= \frac{1}{n} E[\log\det C_n(0, T_1, \ldots, T_n)] \\ &\geq \frac{1}{n} E[-n\log 2 + \log\varphi(T_1) + \log\varphi(T_2 - T_1) + \cdots + \log\varphi(T_n - T_{n-1})] \\ &= -\log 2 + \int_0^\infty e^{-t}\log\varphi(t)dt. \end{aligned} \tag{2.13}$$

This combined with the assumption (2.1) implies

$$\inf_{n\in\mathbb{N}} \frac{E[X_{0,n}]}{n} > -\infty. \tag{2.14}$$

Thus, the condition (S_3) is satisfied. Therefore, Theorem A is applicable and we see that, under the assumptions of Theorem 2.1, there exists a real-valued random variable ζ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \log\det C_n(0, T_1, \dots, T_n) = \zeta \quad \text{a.s.} \tag{2.15}$$

For the proof of Theorem 2.1 it remains to show that the limiting random variable ζ is non-random. To this end we prepare

Lemma 2.1. *Let $C_n(t_0, \dots, t_n)$ $(0 \le t_0 < t_1 < \dots < t_n)$ be as in Theorem 2.1 and let $1 \le k < n$. Then,*
(i)

$$\begin{aligned}\det C_n(t_0, t_1, \dots, t_n) \\ \le \varphi(t_1 - t_0) \dots \varphi(t_k - t_{k-1}) \det C_{n-k}(t_k, t_{k+1}, \dots, t_n),\end{aligned} \tag{2.16}$$

(ii)

$$\begin{aligned}\det C_n(t_0, t_1, \dots, t_n) \\ \ge \left(\frac{1}{2}\right)^k \varphi(t_1 - t_0) \dots \varphi(t_k - t_{k-1}) \det C_{n-k}(t_k, t_{k+1}, \dots, t_n).\end{aligned} \tag{2.17}$$

Proof. (i) is an easy consequence of Proposition in Appendix. For the proof of (ii) let $X_1 = X(t_n) - X(t_{n-1}), \dots, X_n = X(t_1) - X(t_0)$ and let $V_n = (v_{ij})$ where $v_{ij} = E[X_i X_j]$. Notice that $\det V_n = \det C_n(t_0, t_1, \dots, t_n)$. Since φ is concave, all of the off diagonal elements of V_n are negative, and so we can check directly to see that $\sum_{j\neq k} |v_{jk}| \le (1/2) v_{kk}$ for every $k = 1, \dots, n$. Thus, V_n is a 'matrix with dominant principal diagonal'. Therefore, by applying Marcus' result ([6], see also [7]), we get the inequality of (ii). □

We are now ready to prove Theorem 2.1. Applying Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{n}\{-k\log 2 + \log\varphi(t_1 - t_0) + \dots + \log\varphi(t_k - t_{k-1})\} \\ &\le \frac{1}{n}\log\det C_n(t_0, t_1, \dots, t_n) - \frac{1}{n}\log\det C_{n-k}(t_k, t_{k+1}, \dots, t_n) \\ &\le \frac{1}{n}\{\log\varphi(t_1 - t_0) + \dots + \log\varphi(t_k - t_{k-1})\}.\end{aligned} \tag{2.18}$$

Hence, for every fixed $k \geq 1$,

$$\text{(2.19)} \quad \lim_{n\to\infty} \frac{1}{n} \log\det C_n(0, T_1, \ldots, T_n) = \lim_{n\to\infty} \frac{1}{n} \log\det C_{n-k}(T_k, T_{k+1}, \ldots, T_n).$$

Combining this with (2.15), we have

$$\text{(2.20)} \qquad \zeta \in \cap_k \mathfrak{B}(T_{k+1} - T_k, T_{k+2} - T_{k+1}, \ldots).$$

Therefore, applying Kolmogorov's 0-1 law, we see that ζ is in fact constant, which completes the proof of Theorem 2.1.

APPENDIX.

The following fact may be well-known, but we give the proof for the convenience of the reader.

Proposition. *Let $Q = (q_{ij})$ be an $n \times n$ symmetric positive definite matrix such that*

$$Q = \begin{pmatrix} A & C \\ {}^tC & B \end{pmatrix},$$

where A is a $k \times k$ matrix and B is an $(n-k)\times(n-k)$ matrix for some k $(1 \leq k < n)$. Then,

$$\det Q \leq \det A \cdot \det B.$$

Proof. We first remark that in the special case where $k = n - 1$ the result is well-known and it holds that $\det Q = (\det A)(q_{nn} - {}^tCAC) \leq (\det A) q_{nn} = \det A \cdot \det B$. (Notice that by mathematical induction it also holds that $\det Q \leq q_{11} \ldots q_{nn}$ as we mentioned before.) This idea is also applicable to the general case where $k \leq n - 1$ but we shall give another proof here. Since both matrices A and B are positive definite, we can diagonalize them by orthogonal matrices U and V, respectively, i.e.,

$${}^tUAU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}, \quad \text{and } {}^tVBV = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_{n-k} \end{pmatrix}.$$

Consider the $n \times n$ matrix

$$R = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Then

$$
{}^tRQR = \begin{pmatrix} \lambda_1 & & 0 & & & \\ & \ddots & & & * & \\ 0 & & \lambda_k & & & \\ & & & \mu_1 & & 0 \\ & * & & & \ddots & \\ & & & 0 & & \mu_{n-k} \end{pmatrix},
$$

and thus

$$
\det Q = \det {}^tRQR \leq \lambda_1 \ldots \lambda_k \cdot \mu_1 \ldots \mu_{n-k} = \det A \cdot \det B.
$$

□

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