LAYER HEAT POTENTIALS FOR A BOUNDED CYLINDER WITH FRACTAL LATERAL BOUNDARY

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ABSTRACT. Let Ω_D be a bounded cylinder with fractal lateral boundary S_D. We define a generalized layer heat potential Φf of a L^p-function f on S_D for Ω_D and investigate the boundary behavior of Φf. We show that if f belongs to a Besov space on S_D, depending on the fractal dimension of S_D, Φf has a similar boundary behavior to that of a double layer heat potential for a bounded cylinder with smooth lateral boundary.

1. Introduction

Let D be a bounded smooth domain in $\mathbf{R}^d$ and set

$$\Omega_D = D \times (0,T) \quad \text{and} \quad S_D = \partial D \times [0,T].$$

The double layer heat potential Φf of $f \in L^p(S_D)$ is defined by

$$\Phi f(X) = -\int_0^T \int_{\partial D} \langle \nabla_y W(X-Y), n_y \rangle f(Y) d\sigma(y) ds \tag{1.1}$$

for $X = (x,t) \in (R^d \setminus \partial D) \times \mathbf{R}$, where $\langle\ ,\ \rangle$ is the inner product in $\mathbf{R}^d$, n_y is the unit outer normal to ∂D, σ is the surface measure on ∂D and W is the fundamental solution for the heat operator, i.e.,

$$W(X) = W(x,t) = \begin{cases} \dfrac{\exp\left(-\frac{|x|^2}{4t}\right)}{(4\pi t)^{d/2}} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The double layer heat potential is important not only physically but also mathematically. For example, R. M. Brown proved that the solution to the initial-Dirichlet problem in a Lipschitz cylinder for the heat operator can be written by a double layer heat potential and the solution to the initial-Neumann problem in a Lipschitz cylinder for the heat operator is given by a single layer heat potential (cf. [B]).

If D is a bounded domain with fractal boundary, then n_y and the surface measure can not be defined. But if f is a C^1-function on $\mathbf{R}^{d+1}$ with compact support, then we see by the Green formula that for $X = (x,t) \in D \times \mathbf{R}$

$$\begin{aligned} \Phi f(X) = &\int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y f(Y), \nabla_y W(X-Y) \rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} f(Y) \Delta_y W(X-Y) dy \end{aligned} \tag{1.2}$$

and for $X = (x, t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$

$$\Phi f(x) = -\int_0^T ds \int_D \langle \nabla_y f(Y), \nabla_y W(X - Y)\rangle dy - \int_0^T ds \int_D f(Y)\Delta_y W(X - Y) dy. \tag{1.3}$$

So we see that, if a function f defined on S_D can be extended to be a function $\mathcal{E}(f)$ on $\mathbf{R}^d \times [0, T]$ such that for each $t \in [0, T]$ the function $x \mapsto \mathcal{E}(f)(x, t)$ is a C^1-function on $\mathbf{R}^d \setminus \partial D$ and, for each $x \in \mathbf{R}^d \setminus \partial D$ and each j ($j = 1, 2, \cdots, d$) the function $t \mapsto \frac{\partial \mathcal{E}(f)}{\partial x_j}(x, t)$ is measurable, then it is possible that the right-hand sides of (1.2) and (1.3) are defined.

In this paper we assume that D is a bounded domain in $\mathbf{R}^d$ ($d \geq 2$) and ∂D is a β-set satisfying $d - 1 \leq \beta < d$. Here, according to [JW] we say that a closed set F is a β-set if there exist a positive Radon measure μ on F and positive real numbers r_0, b_1, b_2 such that

$$b_1 r^\beta \leq \mu(B(z, r) \cap F) \leq b_2 r^\beta \tag{1.4}$$

for all $z \in F$ and all $r \leq r_0$, where $B(z, r)$ stands for the open ball in $\mathbf{R}^d$ with center z and radius r.

We note that, if D is a bounded Lipschitz domain, then ∂D is a $(d-1)$-set and the surface measure μ has the property (1.4) for $F = \partial D$ and $\beta = d - 1$. Furthermore if ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set such that the β-dimensional Hausdorff measure μ restricted to ∂D has the property (1.4) for $F = \partial D$ (cf. [H]).

In this paper we will define the double layer heat potentials for functions in a Besov space on the lateral boundary $S_D = \partial D \times [0, T]$ of a bounded domain $D \times (0, T)$ in case ∂D is a β-set.

To do so, we fix a positive Radon measure μ satisfying (1.4) and the product measure μ_T of μ and the 1-dimensional Lebesgue measure restricted to [0,T]. Let $p \geq 1$, $\alpha > 0$ and $\lambda > 0$. We denote by $L^p(\mu_T)$ the set of all L^p-functions defined on S_D with respect to μ_T and by $\Lambda_\alpha^p(\mu_T)$ the Banach space of all functions in $L^p(\mu_T)$ such that

$$\int_0^T dt \iint \frac{|f(x,t) - f(y,t)|^p}{|x - y|^{\beta + p\alpha}} d\mu(x) d\mu(y) < \infty$$

and by $\Lambda_{\alpha,\lambda}^p(\mu_T)$ the Banach space of all functions in $L^p(\mu_T)$ such that

$$\int_0^T dt \iint \frac{|f(x,t) - f(y,t)|^p}{|x - y|^{\beta + p\alpha}} d\mu(x) d\mu(y) + \int d\mu(x) \int_0^T \int_0^T \frac{|f(x,t) - f(x,s)|^p}{|t - s|^{1 + p\lambda}} dt ds < \infty.$$

For $f \in \Lambda_\alpha^p(\mu_T)$ and $f \in \Lambda_{\alpha,\lambda}^p(\mu_T)$ the norms are defined by

$$\|f\|_{p,\alpha} = \left(\int_0^T dt \int |f(x,t)|^p d\mu(x)\right)^{1/p} + \left(\int_0^T dt \iint \frac{|f(x,t) - f(y,t)|^p}{|x - y|^{\beta + p\alpha}} d\mu(x) d\mu(y)\right)^{1/p}$$

and

$$\|f\|_{p,\alpha,\lambda} = \left(\int_0^T dt \int |f(x,t)|^p d\mu(x)\right)^{1/p} + \left(\int_0^T dt \iint \frac{|f(x,t)-f(y,t)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y)\right)^{1/p} + \left(\int d\mu(x) \int_0^T \int_0^T \frac{|f(x,t)-f(x,s)|^p}{|t-s|^{1+p\lambda}} dtds\right)^{1/p},$$

respectively.

We will prove the following lemma in §2.

Lemma 1.1. *Let $d-1 \le \beta < d$ and F be a compact β-set in $\mathbf{R}^d$ satisfying (1.4) and $F \subset B(0,R/2)$. If $f \in L^p(\mu_T)$, then there exists a function $\mathcal{E}(f)$ defined on $\mathbf{R}^d \times [0,T]$ satisfying the following properties:*

(i) $\mathcal{E}(f)(\cdot,t)$ is a C^1-function on $\mathbf{R}^d \setminus F$ for each $t \in [0,T]$, and both of $\mathcal{E}(f)(x,\cdot)$ and $\left(\frac{\partial \mathcal{E}(f)}{\partial x_j}\right)(x,\cdot)$ $(j=1,\cdots,d)$ are measurable for each $x \in \mathbf{R}^d$ and for each $x \in \mathbf{R}^d \setminus F$, respectively,

(ii) $\mathcal{E}(f) = f$ on F and $\operatorname{supp} \mathcal{E}(f)(\cdot,t) \subset B(0,2R)$ for each $t \in [0,T]$.

Moreover, if $f \in \Lambda^p_\alpha(\mu_T)$, then the function $\mathcal{E}(f)$ also has the following properties:

(iii) if $p\alpha + d - \beta + p\epsilon > 0$, then

$$\int_0^T dt \int |\mathcal{E}(f)(y,t)|^p \operatorname{dist}(y,F)^{p\epsilon} dy \le c\|f\|_{p,\alpha},$$

(iv) if $p(\alpha-1) + d - \beta + p\epsilon > 0$, then

$$\int_0^T dt \int |\nabla \mathcal{E}(f)(y,t)|^p \operatorname{dist}(y,F)^{p\epsilon} dy \le c\|f\|_{p,\alpha}$$

where $\operatorname{dist}(y,F) = \inf_{z\in F} |y-z|$.

Moreover $f \mapsto \mathcal{E}(f)(x,t)$ is a bounded linear functional on $\Lambda^p_\alpha(\mu_T)$ for every $(x,t) \in (\mathbf{R}^d \setminus F) \times (0,T)$.

Using Lemma 1.1 we define

$$\Phi f(X) = \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \langle \nabla_y \mathcal{E}(f)(y,s), \nabla_y W(X-Y)\rangle dy + \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \mathcal{E}(f)(y,s)\Delta_y W(X-Y) dy \tag{1.5}$$

for $X=(x,t) \in D \times \mathbf{R}$ and

$$\Phi f(X) = -\int_0^T ds \int_D \langle \nabla_y \mathcal{E}(f)(y,s), \nabla_y W(X-Y)\rangle dy - \int_0^T ds \int_D \mathcal{E}(f)(y,s)\Delta_y W(X-Y) dy \tag{1.6}$$

for $X=(x,t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$.

Furthermore we also use the parabolic metric

$$\delta(X,Y) = \left(|x-y|^2 + |t-s|\right)^{1/2}$$

for $X=(x,t)$ and $Y=(y,s)$ and define, for $\tau>0$ and for each $Z\in S_D$, the nontangential approach regions by

$$\Gamma_\tau(Z) = \{X \in D\times(0,T) : \delta(X,Z) < (1+\tau)\delta(X;S_D)\}, \tag{1.7}$$

and

$$\Gamma_\tau^e(Z) = \{X \in (\mathbf{R}^d\setminus\overline{D})\times(0,T) : \delta(X,Z) < (1+\tau)\delta(X;S_D)\}, \tag{1.8}$$

where

$$\delta(X;A) = \inf_{Y\in A}\delta(X,Y)$$

for a subset A of $\mathbf{R}^{d+1}$.

Under these notations we will prove the following theorem in §7.

Theorem. *Assume that D is a bounded domain in $\mathbf{R}^d$ $(d\geq 2)$ such that ∂D is a β-set. Furthermore assume that $\Gamma_\tau(Z)\cap B(Z,r)\neq\emptyset$ and $\Gamma_\tau^e(Z)\cap B(Z,r)\neq\emptyset$ for every $r\leq\epsilon_0$ and for μ_T-a.e. $Z\in S_D$. If $p>1$, $0\leq\beta-(d-1)<\alpha<1$ and $f\in\Lambda^p_{\alpha,\alpha/2}(\mu_T)$, then the double heat layer potential Φf defined by (1.5) and (1.6) is caloric in $(\mathbf{R}^d\setminus\partial D)\times\mathbf{R}$ and*

$$\lim_{X\to Z, X\in\Gamma_\tau(Z)}\Phi f(X) = Kf(Z) + \frac{1}{2}f(Z)$$

and

$$\lim_{X\to Z, X\in\Gamma_\tau^e(Z)}\Phi f(X) = Kf(Z) - \frac{1}{2}f(Z)$$

at μ_T-almost every $Z\in S_D$, where K is a bounded operator from $\Lambda^p_{\alpha,\alpha/2}(\mu_T)$ to $L^p(\mu_T)$ defined by

$$\begin{aligned}Kf(Z) = &\frac{1}{2}\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}\langle\nabla\mathcal{E}(f)(y,s),\nabla_y W(Z-Y)\rangle dy\\ &+\frac{1}{2}\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}(\mathcal{E}(f)(Y)-f(Z))\Delta_y W(Z-Y)dy\\ &+\frac{f(Z)}{2}\int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}}W(Z-Y)dy\\ &-\frac{1}{2}\int_0^T ds\int_D\langle\nabla\mathcal{E}(f)(y,s),\nabla_y W(Z-Y)\rangle dy\\ &-\frac{1}{2}\int_0^T ds\int_D(\mathcal{E}(f)(Y)-f(Z))\Delta_y W(Z-Y)dy\\ &-\frac{f(Z)}{2}\int_{D\times\{0\}}W(Z-Y)dy.\end{aligned}\tag{1.9}$$

2. Extensions of boundary functions

In this section we assume that F is a compact β-set such that $d-1 \le \beta < d$ and $F \subset B(O, R/2)$. Fix a positive Radon measure μ on F satisfying (1.4). We may assume that (1.4) holds for every r satisfying $r \le 4R$. To extend $f \in L^p(\mu_T)$ to be a function on $\mathbf{R}^d$, we use the Whitney decomposition. More precisely, let G be an open set in $\mathbf{R}^d$. A cube Q is called a k-cube if it is of the form

$$[l_1 2^{-k}, (l_1+1)2^{-k}] \times \cdots \times [l_d 2^{-k}, (l_d+1)2^{-k}],$$

where $k, l_1, \cdots, l_d$ are integers. We denote by $\mathcal{W}_k(G)$ the family of all k-cubes in G and set $\mathcal{W}(G) = \cup_{k=-\infty}^{\infty} \mathcal{W}_k(G)$. The following theorem is well-known (cf. [S, Theorem 1 in Chapter 6]).

Theorem A. *Let G be a non-empty open set in $\mathbf{R}^d$. Then there exists a family $\mathcal{V}(G) = \{Q_j\}$ of cubes in $\mathcal{W}(G)$ having the following properties:*
(i) $\cup_j Q_j = G$,
(ii) $\mathrm{int}\, Q_j \cap \mathrm{int}\, Q_k = \emptyset$ $(j \ne k)$,
(iii) $\mathrm{diam}\, Q_j \le \mathrm{dist}(Q_j, \mathbf{R}^d \setminus G) \le 4 \mathrm{diam}\, Q_j$,
where $\mathrm{int}\, A$, $\mathrm{diam}\, A$ and $\mathrm{dist}(A, B)$ stand for the interior of A, the diameter of A and the distance between A and B, respectively.

Using this, we will extend $f \in L^p(\mu_T)$ to be a function on $\mathbf{R}^d \times [0, T]$.

Proof of Lemma 1.1. (i) and (ii). Let Q_0 denote the closed cube of unit length centered at the origin. Fix a positive real number η satisfying $\eta < 1/4$ and choose a C^∞-function ϕ on $\mathbf{R}^d$ such that

$$\phi = 1 \text{ on } Q_0, \quad \mathrm{supp}\, \phi \subset (1+\eta) Q_0, \quad 0 \le \phi \le 1. \tag{2.1}$$

We simply denote by $\mathcal{V} = \{Q_j\}$ the family $\mathcal{V}(\mathbf{R}^d \setminus F)$. Further let $q^{(j)}$, l_j be the center of Q_j and the common length of its sides, respectively. For each j pick a point $a^{(j)} \in F$ satisfying $\mathrm{dist}(Q_j, F) = \mathrm{dist}(Q_j, a^{(j)})$ and fix it. Set

$$t(x) = \sum_j \phi\Big(\frac{x - q^{(j)}}{l_j}\Big)$$

and

$$\phi_j^*(x) = \frac{\phi(x - q^{(j)}/l_j)}{t(x)}.$$

Let $p \ge 1$ and $f \in L^p(\mu_T)$. First let $X \in (\mathbf{R}^d \setminus F) \times [0, T]$. We define

$$\mathcal{E}_0(f)(x,t) = \sum_j \frac{1}{\mu(B(a^{(j)}, \eta l_j) \cap F)} \left(\int_{B(a^{(j)}, \eta l_j)} f(z,t) d\mu(z) \right) \phi_j^*(x)$$

if $\int |f(z,t)|^p d\mu(z) < \infty$ and 0 otherwise. Further define $\mathcal{E}_0(f)(X) = f(X)$ if $X \in F \times [0, T]$.
Moreover pick a C^∞ function ϕ_0 in $\mathbf{R}^d$ such that

$$\phi_0 = 1 \text{ on } \overline{B(0,R)}, \quad \mathrm{supp}\, \phi_0 \subset B(0, 2R), \quad 0 \le \phi_0 \le 1 \tag{2.2}$$

and define

$$\mathcal{E}(f)(x,t) = \mathcal{E}_0(f)(x,t) \phi_0(x).$$

Then it is easy to see that $\mathcal{E}(f)$ has the properties (i) and (ii). □

To show (iii) and (iv) in Lemma 1.1, we prepare the following lemma.

Lemma 2.1. *Let $b > 0$, $k \in \mathbf{Z}$ and $a^{(j)}$ be the point on F corresponding to each k-cubes $Q_j \in \mathcal{V}(\mathbf{R}^d \setminus F)$. Then each point $z \in F$ is contained in at most N numbers of $\{B(a^{(j)}, b/2^k) \cap F\}_j$, where N is a natural number independent of k.*

Proof. Set $l = 2^{-k}$. Suppose $z \in B(a^{(i)}, \eta l) \cap B(a^{(j)}, \eta l) \cap F$. Choose $x^{(i)} \in Q_i$ and $x^{(j)} \in Q_j$ satisfying $\operatorname{dist}(Q_i, F) = \operatorname{dist}(x^{(i)}, a^{(i)})$ and $\operatorname{dist}(Q_j, F) = \operatorname{dist}(x^{(j)}, a^{(j)})$, respectively. Then

$$\begin{aligned}|x^{(i)} - x^{(j)}| &\le |x^{(i)} - a^{(i)}| \\ &\quad + |a^{(i)} - z| + |z - a^{(j)}| + |a^{(j)} - x^{(j)}| \le (8\sqrt{d} + 2b)l,\end{aligned}$$

whence $\operatorname{dist}(Q_i, Q_j) \le (8\sqrt{d} + 2b)l$. Therefore we have the conclusion. □

We next prove Lemma 1.1, (iii) and (iv).

Proof of Lemma 1.1, (iii) and (iv). Let Q be a closed cube in $\mathcal{V}$ with side length l and a be the fixed boundary point such that $\operatorname{dist}(F, Q) = \operatorname{dist}(a, Q)$. Further let $w \in B(a, \eta l) \cap F$ and Q_j be a cube in $\mathcal{V}(\mathbf{R}^d \setminus F)$ satisfying $Q \cap Q_j^* \neq \emptyset$, where Q_j^* stands for the cube which has the same center as Q_j but is expanded by the factor $1 + \eta$. We note that $c_1 l \le l_j \le c_2 l$ for some constants c_1 and c_2 depending only on d.

Choose $y \in Q \cap Q_j^*$. Then for every $z \in B(a^{(j)}, \eta l_j)$ we have

$$|z - w| \le |z - a^{(j)}| + |a^{(j)} - x| + |y - a| + |a - w| \le c_3 l_j.$$

From this, the definition of $\mathcal{E}_0(f)$ and $\mathcal{E}_0(1)(y,t) = 1$ we deduce, for every $y \in Q$ and a.e. $t \in [0, T]$,

$$\begin{aligned}(2.3)\qquad |\nabla_y \mathcal{E}_0(f)(y,t)| &= |\nabla_y(\mathcal{E}_0(f - \mathcal{E}_0(f)(w,t))(y,t)| \\ &\le c_4 \sum \frac{\chi_{Q_j^*}(y)}{l_j^{\beta+1}} \int_{B(a^{(j)}, \eta l_j)} |f(z,t) - f(w,t)| d\mu(z) \\ &\le c_5 \sum l_j^{-\beta/q+\alpha-1} \chi_{Q_j^*}(y) \int_{B(a^{(j)}, \eta l_j)} \frac{|f(z,t) - f(w,t)|}{|z - w|^{\beta/p+\alpha}} d\mu(z) \\ &\le c_6 l^{\alpha-1} \left(\int \frac{|f(z,t) - f(w,t)|^p}{|z - w|^{\beta+\alpha p}} d\mu(z) \right)^{1/p}.\end{aligned}$$

Similarly, since

$$\begin{aligned}|\mathcal{E}_0(f)(y,t) - f(w,t)| &\\ \le c_7 \sum \frac{\chi_{Q_j^*}(y)}{l_j^{\beta}} &\int_{B(a^{(j)}, \eta l_j)} |f(z,t) - f(w,t)| d\mu(z) \\ \le c_8 l^{\alpha} &\left(\int \frac{|f(z,t) - f(w,t)|^p}{|z - w|^{\beta+\alpha p}} d\mu(z) \right)^{1/p},\end{aligned}$$

we have

$$(2.4)\qquad |\mathcal{E}_0(f)(y,t)| \le c_9 l^{\alpha} \left(\int \frac{|f(z,t) - f(w,t)|^p}{|z - w|^{\beta+\alpha p}} d\mu(z) \right)^{1/p} + |f(w,t)|.$$

Combining this with (1.4), we obtain

$$\begin{aligned}&\int_Q |\nabla \mathcal{E}(f)(y,t)|^p \operatorname{dist}(y, F)^{p\epsilon} dy \\ &\le c_{10} \int_Q (|\mathcal{E}_0(f)(y,t)|^p + |\nabla \mathcal{E}_0(f)(y,t)|^p) \operatorname{dist}(y, F)^{p\epsilon} dy \\ &\le c_{11} l^{p(\alpha-1)+d-\beta+p\epsilon} \int_{B(a,\eta l)} d\mu(w) \left(\int \frac{|f(z,t) - f(w,t)|^p}{|z - w|^{\beta+\alpha p}} d\mu(z) + |f(w,t)|^p \right).\end{aligned}$$

On account of Lemma 2.1 we have

$$\sum_{Q\in\mathcal{V}\cap\mathcal{W}_k(\mathbf{R}^d\setminus F)}\int_0^T dt\int_Q|\nabla\mathcal{E}(f)(y,t)|^p\mathrm{dist}(y,F)^{p\epsilon}dy$$
$$\leq c_{12}(2^{-k})^{p(\alpha-1)+d-\beta+p\epsilon}\|f\|_{p,\alpha}^p$$

Since $p(\alpha-1)+d-\beta+p\epsilon>0$ and supp $\mathcal{E}(f)\subset B(0,2R)\times[0,T]$, we have (iv). Similarly we can show (iii) by using (2.4). □

3. Properties of W

In this section we recall and study properties of the function W.

Lemma 3.1. (i) $W(X)\leq c\delta(X,0)^{-d}$,
(ii) $|\nabla_x W(X)|\leq c\delta(X,0)^{-d-1}$ *if* $X\neq 0$,
(iii) $|\frac{\partial^2}{\partial x_i\partial x_j}W(X)|\leq c\delta(X,0)^{-d-2}$, $|\frac{\partial}{\partial t}W(X)|\leq c\delta(X,0)^{-d-2}$ *if* $X\neq 0$,
(iv) $|\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}W(X)|\leq c\delta(X,0)^{-d-3}$, $|\frac{\partial^2}{\partial x_j\partial t}W(X)|\leq c\delta(X,0)^{-d-3}$ *if* $X\neq 0$,
(v) $|W(X-Y)-W(Z-Y)|\leq c\delta(X,Z)^\epsilon\{\delta(X,Y)^{-d-\epsilon}+\delta(Z,Y)^{-d-\epsilon}\}$
if $0\leq\epsilon\leq 1$ *and* $X\neq Y$, $Z\neq Y$,
(vi) $|\nabla_y W(X-Y)-\nabla_y W(Z-Y)|\leq c\delta(X,Z)^\epsilon\{\delta(X,Y)^{-d-1-\epsilon}+\delta(Z,Y)^{-d-1-\epsilon}\}$
if $0\leq\epsilon\leq 1$ *and* $X\neq Y$, $Z\neq Y$.

Proof. The assertions (i), (ii), (iii) and (iv) are well known (cf. [**B**, p.5]). The assertions (v) and (vi) will be shown by the same method as in the proof of Lemma 2.3 in [**W2**]. □

Let D_0 be a bounded piecewise smooth domain in $\mathbf{R}^d$ and u, v be smooth functions on $\overline{D_0}\times[0,\rho]$. Using the divergence theorem, we obtain

$$\int_0^\rho\int_{D_0}(uL^*v-vLu)\,dxdt \tag{3.1}$$
$$=\int_0^\rho dt\int_{\partial D_0}\langle u\nabla_x v-v\nabla_x u,n_x\rangle d\sigma(x)-\int_{D\times\{t=0\}}uvdx+\int_{D\times\{t=\rho\}}uvdx,$$

where

$$L=\Delta-\frac{\partial}{\partial t}\quad\text{and}\quad L^*=\Delta+\frac{\partial}{\partial t}.$$

If $Lu=L^*v=0$ in $D_0\times(0,\rho)$, then (3.1) implies

$$\int_0^\rho dt\int_{\partial D_0}\langle u\nabla_x v-v\nabla_x u,n_x\rangle d\sigma(x)-\int_{D_0\times\{t=0\}}uvdx+\int_{D_0\times\{t=\rho\}}uvdx=0. \tag{3.2}$$

Let $X=(x,t)$ $(0\leq t\leq T)$ be an exterior point of $D_0\times(0,T)$. Then, setting $u=1$ and $v(Y)=W(X-Y)$ and noting $W(X-Y)=0$ for $Y=(y,T)$, we deduce from (3.2)

$$\int_0^T ds\int_{\partial D_0}\langle\nabla_y W(X-Y),n_y\rangle d\sigma(y)-\int_{D_0\times\{s=0\}}W(X-Y)dy=0. \tag{3.3}$$

Hereafter we assume that D is a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set satisfying $\overline{D}\subset B(O,R/2)$.

Let us use the Whitney decompositions to approximate D and $\mathbf{R}^d\setminus\overline{D}$. More precisely let $\mathcal{V}(D)$ be the Whitney decomposition of D and define

$$A_n=\cup_{k=k_0}^n\cup_{Q\in\mathcal{V}_k(D)}Q,$$

where $\mathcal{V}_k(D) = \mathcal{V}(D) \cap \mathcal{W}_k(D)$ and k_0 is the smallest integer k such that $\mathcal{V}_k(D) \neq \emptyset$.

Similarly we also define

$$B_n = \left(\cup_{k=-\infty}^{n} \cup_{Q\in\mathcal{V}_k(\mathbf{R}^d\setminus\overline{D})} Q\right).$$

Then we have the following lemma.

Lemma 3.2. *Set*

$$g_n(X) = \int_0^T ds \int_{A_n} \Delta_y W(X-Y)dy \quad \textit{and} \quad h_n(X) = \int_0^T ds \int_{B_n} \Delta_y W(X-Y)dy.$$

Then $\lim_{n\to\infty} g_n(X)$ *and* $\lim_{n\to\infty} h_n(X)$ *exist on* $\mathbf{R}^d \times [0,T]$ *and for* $X=(x,t)\in \mathbf{R}^d\times(0,T]$

$$\lim_{n\to\infty} g_n(X) = \int_{D\times\{0\}} W(X-Y)dy - \chi_D(x)$$

and

$$\lim_{n\to\infty} h_n(X) = \int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}} W(X-Y)dy - \chi_{\mathbf{R}^d\setminus\overline{D}}(x)$$

Proof. Let $X=(x,t)\in\mathbf{R}^d\times(0,T]$ and $t>\rho>0$. Applying (3.2) to $A_n\times(0,\rho)$, we have

$$\int_0^\rho ds \int_{\partial A_n} \langle \nabla_y W(X-Y), n_y\rangle d\sigma(y) - \int_{A_n\times\{0\}} W(X-Y)dy$$
$$+\int_{A_n\times\{\rho\}} W(X-Y)dy = 0.$$

Using the divergence theorem for A_n in $\mathbf{R}^d$, we have

$$\int_0^\rho ds \int_{A_n} \Delta_y W(X-Y)dy - \int_{A_n\times\{0\}} W(X-Y)dy$$
$$+\int_{A_n\times\{\rho\}} W(X-Y)dy = 0.$$

As $\rho\to t$ and $n\to\infty$, we obtain,

$$\lim_{n\to\infty} g_n(X) = \int_{D\times\{0\}} W(X-Y)dy - \chi_D(x).$$

On the other hand $g_n(X)=0$ for $t=0$. Hence $\lim_{n\to\infty} g_n(X)$ exists for each $X\in\mathbf{R}^d\times[0,T]$.

We next consider $\{h_n\}$. For $X=(x,t)\in\mathbf{R}^d\times(0,T]$ we take $r>\max\{2R,|x|\}$. Applying the Green formula to $D_0 = \mathrm{int}\, B_n\cap B(0,r)$ and (3.2) to int $B_n\cap B(0,r)$, we have, for positive number $\rho<t$ and a sufficiantly large number n,

$$\int_0^\rho \int_{B_n\cap B(O,r)} \Delta_y W(X-Y)dy$$
$$= \int_0^\rho ds \int_{|y|=r} \langle \nabla_y W(X-Y), n_y\rangle dy - \int_0^\rho ds \int_{\partial B_n} \langle \nabla_y W(X-Y), n_y\rangle dy$$
$$= \int_{(B_n\cap B(0,r))\times\{0\}} W(X-Y)dy - \int_{(B_n\cap B(0,r))\times\{\rho\}} W(X-Y)dy.$$

As $r\to\infty$, $\rho\to t$ and $n\to\infty$ we have

$$\lim_{n\to\infty} h_n(X) = \int_{(B_n\cap B(0,r))\times\{0\}} W(X-Y)dy - \chi_{\mathbf{R}^d\setminus\overline{D}}(x).$$

Thus we have the conclusion. □

4. Double layer heat potentials

We first mention two fundamental properties deduced from (1.4) for $F = \partial D$. For $f \in L^1(\mu)$ we define the maximal function $\mathcal{M}_\mu f$ of f with respect to μ as follows:

$$\mathcal{M}_\mu f(z) = \sup\{\frac{1}{\mu(B(z,r)\cap\partial D)}\int_{B(z,r)}|f(x)|d\mu(x) : r > 0\}$$

for each $z \in \partial D$. Noting that ∂D is a β-set, we obtain the following lemma by the same method as in the proof of Theorem 1 on p.1 in [S].

Lemma 4.1. *Let $p > 1$. Then*

$$\|\mathcal{M}_\mu f\|_p \le c\|f\|_p$$

for every $f \in L^p(\mu)$, where

$$\|f\|_p = \left(\int |f(z)|^p d\mu(z)\right)^{1/p}.$$

It is easy to see that the following lemma holds.

Lemma 4.2. (i) *Suppose $k > -\beta$ and set*

$$u_r(x) = \int_{\{|x-y|<r\}\cap\partial D} |x-y|^k d\mu(y) \quad \textit{for } x \in \partial D.$$

Then $|u_r| \le cr^{k+\beta}$ for every positive number r with $r \le 4R$.
(ii) *Suppose $k < -\beta$ and set*

$$v_r(x) = \int_{\{|x-y|>r\}\cap\partial D} |x-y|^k d\mu(y) \quad \textit{for } x \in \partial D.$$

Then $|v_r(x)| \le cr^{k+\beta}$ for every positive number r.

In [W1] we gave the following lemma.

Lemma B. *Let δ, k be non-negative numbers satisfying $d-\beta > \delta$ and $d-\delta-k > 0$. Then*

$$\int_{B(z,r)} \mathrm{dist}(y,\partial D)^{-\delta}|y-z|^{-k}dy \le cr^{d-\delta-k}$$

for every $z \in \partial D$ and every positive real number $r \le 4R$.

We next show that the double layer heat potential defined by (1.5) and (1.6) converges.

Lemma 4.3. *Let $0 \le \beta-(d-1) < \alpha < 1$, $p > 1$ and $f \in \Lambda^p_\alpha(\mu_T)$. Then Φf is caloric in $(\mathbf{R}^d \setminus \partial D) \times \mathbf{R}$.*

Proof. Set, for $X = (x,t) \in D \times \mathbf{R}$,

$$J_1(X) = \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \langle\nabla_y\mathcal{E}(f)(y,s), \nabla_y W(X-Y)\rangle dy \tag{4.1}$$

and let $X_0 = (x_0, t_0) \in D$. Choose $\rho > 0$ satisfying $\overline{B(x_0, 2\rho)} \subset D$. Noting that $d - \beta - p(1-\alpha) + p(1-\alpha)/q = d - \beta - 1 + \alpha > 0$ ($q = p/(p-1)$), we pick $\epsilon > 0$ such that

$$\frac{1-\alpha}{q} > \epsilon \text{ and } d - \beta - p(1-\alpha) + p\epsilon > 0. \tag{4.2}$$

If $X = (x,t) \in B(x_0, \rho) \times \mathbf{R}$, then we deduce from Lemmas 3.1, 1.1 and Lemma B

$$\begin{aligned}|J_1(X)| &\leq \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla_y \mathcal{E}(f)(y.s)| \delta(X,Y)^{-1-d} dy \\ &\leq c_1 \rho^{-1-d} \left(\int_0^T ds \int_{B(0,2R)} |\nabla_y \mathcal{E}(f)(y,s)|^p \mathrm{dist}(y, \partial D)^{p\epsilon} dy \right)^{1/p} \\ &\times \left(\int_0^T ds \int_{B(0,2R)} \mathrm{dist}(y, \partial D)^{-q\epsilon} dy \right)^{1/q} \leq c_2 \rho^{-1-d} \|f\|_{p,\alpha},\end{aligned}$$

whence J_1 converges locally uniformly in D. We denote by g_1 the integrand of the right-hand side of (4.1). Since

$$|\nabla_y \frac{\partial^2}{\partial x_i \partial x_j} W(X-Y)| \leq c_2 \delta(X,Y)^{-d-3} \text{ and } |\nabla_y \frac{\partial}{\partial t} W(X-Y)| \leq c_3 \delta(X,Y)^{-d-3},$$

we see that the integral of Lg_1 over $(\mathbf{R}^d \setminus \overline{D}) \times [0,T]$ also converges locally uniformly on D. Therefore J_1 satisfies the heat equation in $D \times \mathbf{R}$.

Next, set

$$J_2(X) = \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \mathcal{E}(f)(y,s) \Delta W(X-Y) dy.$$

Using Lemma 1.1, (iii), we can show by the above method that J_2 also converges locally uniformly in D and satisfies the heat equation. Thus we conclude that $\Phi f = J_1 + J_2$ has the same properties in $D \times \mathbf{R}$. We can show that Φf also has the same properties in $(\mathbf{R}^d \setminus \overline{D}) \times R$. □

5. Spaces of Hölder continuous functions

In this secion we consider double layer potentials of Hölder continuous boundary functions. To do so, let $0 < \alpha \leq 1$ and F be a closed set in $\mathbf{R}^d$. We denote by $\Lambda_\alpha(F \times [0,T])$ the normed space of all continuous functions f on $F \times [0,T]$ such that $f(\cdot, t)$ is α-Hölder continuous for every $t \in [0,T]$ with norm

$$\|f\|_{\infty,\alpha} = \sup_{X \in F \times [0,T]} |f(X)| + \sup_{x,y \in F, x \neq y, t \in [0,T]} \frac{|f(x,t) - f(y,t)|}{|x-y|^\alpha}.$$

Further let $0 < \alpha, \lambda \leq 1$. We also denote by $\Lambda_{\alpha,\lambda}(F \times [0,T])$ the normed space of all $f \in \Lambda_\alpha(F \times [0,T])$ such that f is λ-Hölder continuous with respect to the time variable with norm

$$\|f\|_{\infty,\alpha,\lambda} = \|f\|_{\infty,\alpha} + \sup_{x \in F, t,s \in [0,T], t \neq s} \frac{|f(x,t) - f(x,s)|}{|t-s|^\lambda}.$$

We first mention the properties of the extension $\mathcal{E}(f)$ of $f \in \Lambda_\alpha(S_D)$.

Lemma 5.1. *Let $1 > \alpha > \beta - (d-1) \geq 0$ and $f \in \Lambda_\alpha(S_D)$. Then $\mathcal{E}(f)$ is α-Hölder continuous with respect to the space variables and*

$$|\frac{\partial \mathcal{E}(f)}{\partial y_i}(y,s)| \leq c\|f\|_{\infty,\alpha} \mathrm{dist}(y, \partial D)^{\alpha-1}, \quad |\frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_k}(y,s)| \leq c\|f\|_{\infty,\alpha} \mathrm{dist}(y, \partial D)^{\alpha-2} \tag{5.1}$$

for every $(y,s) \in (\mathbf{R}^d \setminus \partial D) \times [0,T]$. *Especially if* $f \in \Lambda_{\alpha,\lambda}(S_D)$, *then*

$$|\mathcal{E}(f)(x,t) - \mathcal{E}(f)(y,s)| \leq c\|f\|_{\infty,\alpha,\lambda}(|x-y|^\alpha + |t-s|^\lambda) \tag{5.2}$$

for every X, $Y \in \mathbf{R}^d \times [0,T]$.

Proof. To show (5.1), let $Q \in \mathcal{V}(\mathbf{R}^d \setminus \partial D)$. Further, let a be the fixed point on ∂D satisfying $\mathrm{dist}(a,Q) = \mathrm{dist}(\partial D, Q)$ and l be the side length of Q. Suppose $Q \cap Q_j^* \neq \emptyset$ and take $y' \in Q \cap Q_j^*$. Using the same notations as in the proof of Lemma 1.1, we have, for $z \in B(a^{(j)}, \eta l_j)$,

$$|z-a| \leq |z-a^{(j)}| + |a^{(j)} - y'| + |y'-a| \leq c_1 l$$

and hence, for $y \in Q$,

$$\begin{aligned}
|\nabla_y \mathcal{E}_0(f)(y,s)| &= |\nabla_y(\mathcal{E}_0(f)(y,s) - f(a,s))| \\
&\leq c_2 \sum \frac{\chi_{Q_j^*}(y)}{l_j^{\beta+1}} \int_{B(a^{(j)},\eta l_j)} |f(z,s) - f(a,s)| d\mu(z) \\
&\leq c_3 \|f\|_{\infty,\alpha} \sum \frac{\chi_{Q_j^*}(y)}{l_j^{\beta+1}} \int_{B(a^{(j)},\eta l_j)} |z-a|^\alpha d\mu(z) \\
&\leq c_4 \|f\|_{\infty,\alpha} l^{\alpha-1} \leq c_5 \|f\|_{\infty,\alpha} \mathrm{dist}(y,\partial D)^{\alpha-1}.
\end{aligned}$$

Similarly we obtain

$$|\mathcal{E}_0(f)(y,s)| \leq c_6 \left(\|f\|_{\infty,\alpha} \mathrm{dist}(y,\partial D)^\alpha + \|f\|_\infty\right)$$

and

$$\left|\frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_k}(y,s)\right| \leq c_7 \|f\|_{\infty,\alpha} \mathrm{dist}(y,\partial D)^{\alpha-2}.$$

From these we deduce (5.1).

Next let $y \in \mathbf{R}^d \setminus \partial D$ and $w \in \partial D$. Choose $Q \in \mathcal{V}(\mathbf{R}^d \setminus \partial D)$ satisfying $y \in Q$. If $|y-w| < 5\sqrt{d}l$, then

$$\begin{aligned}
&|\mathcal{E}_0(f)(y,s) - f(w,s)| \\
&\leq c_7 \sum \frac{\chi_{Q_j^*}(y)}{l_j^\beta} \int_{B(a^{(j)},\eta l_j)} |f(z,s) - f(w,s)| d\mu(z) \\
&\leq c_8 \|f\|_{\infty,\alpha} l^\alpha \leq c_9 \|f\|_{\infty,\alpha} \mathrm{dist}(y,\partial D)^\alpha \leq c_{10} \|f\|_{\infty,\alpha} |y-w|^\alpha.
\end{aligned}$$

If $|y-w| \geq 5\sqrt{d}l$, then

$$\begin{aligned}
|\mathcal{E}_0(f)(y,s) - f(w,s)| &\leq |\mathcal{E}_0(f)(y,s) - f(a,s)| + |f(a,s) - f(w,s)| \\
&\leq c_{11} \|f\|_{\infty,\alpha}(|y-a|^\alpha + |a-w|^\alpha) \leq c_{12}|y-w|^\alpha,
\end{aligned}$$

whence

$$|\mathcal{E}(f)(y,s) - f(w,s)| \leq c_{13}|y-w|^\alpha \tag{5.3}$$

for $y \in \mathbf{R}^d \setminus \partial D$ and $w \in \partial D$.

On account of (5.1) and (5.3) we can show that $\mathcal{E}(f)$ is α-Hölder continuous with respect to the space variable by the same method as in the proof of Theorem 3 in Chapter 6 in **[S]**.

Finally assume that $f \in \Lambda_{\alpha,\lambda}(S_D)$. Then

$$|\mathcal{E}_0(f)(y,t) - \mathcal{E}_0(f)(y,s)|$$
$$\leq c_{12} \sum \frac{\chi_{Q_j}(y)}{l_j^{\beta}} \int_{B(a^{(j)},\eta l_j)} |f(z,t) - f(z,s)| d\mu(z) \leq c_{13} \|f\|_{\infty,\alpha,\lambda} |t-s|^{\lambda}.$$

From this and the α-Hölder continuity of $\mathcal{E}(f)(\cdot, s)$ we deduce (5.2). □

We prepare several lemmas to study the boundary behavior of $\mathcal{E}(f)$ for $f \in \Lambda_\alpha(S_D)$.

Lemma 5.2. *Let $1 > \alpha > \beta - (d-1) \geq 0$, $r > 0$, $k \geq 2$ and $d + \alpha + 1 - k > 0$. The function defined by*

$$g(X) = \int_0^T ds \int_{B(0,r)} \operatorname{dist}(y, \partial D)^{\alpha-1} \delta(X,Y)^{-k} dy$$

is bounded on $\mathbf{R}^d \times [0,T]$.

Proof. Let $X = (x,t) \in \mathbf{R}^d \times [0,T]$, $Y = (y,s) \in B(0,r) \times [0,T]$ and choose $z \in \partial D$ satisfying $\operatorname{dist}(x, \partial D) = |x - z|$. Set

$$A = \{y \in B(0,r); |y-z| \leq 3|x-y|\} \text{ and } B = \{y \in B(0,r); |y-z| > 3|x-y|\}.$$

put

$$g(X) = \int_0^T \int_A + \int_0^T \int_B \equiv I_1(X) + I_2(X).$$

Further, choose $\epsilon > 0$ satisfying $d + \alpha + 1 - k - 2\epsilon > 0$. Then

$$I_1(X) \leq \int_0^T |t-s|^{-1+\epsilon} ds \int_A \operatorname{dist}(y, \partial D)^{\alpha-1} |y-x|^{-k+2-2\epsilon} dy$$
$$\leq c_1 \int_{B(z,r+R)} \operatorname{dist}(y, \partial D)^{\alpha-1} |y-z|^{-k+2-2\epsilon} dy.$$

Lemma B yields the boundedness of I_1.

Next, if $y \in B$, then $|z - x| \geq 2|x-y|$. For $w \in \partial D$, we have

$$|y - w| \geq |w - x| - |x - y| \geq |z - y| - |x - y| > |x - y|,$$

which shows $\operatorname{dist}(y, \partial D) \geq |x - y|$. Since

$$3|x-y| \leq |y-z| \leq |y| + |z| \leq r + R,$$

we have

$$I_2(X) \leq \int_0^T |t-s|^{-1+\epsilon} ds \int_{|x-y| \leq (r+R)/3} |x-y|^{\alpha-1-k+2-2\epsilon} dy.$$

Thus we see that I_2 is also bounded. □

Lemma 5.3. *Let $0 \leq \beta - (d-1) < \alpha < 1$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$, and define*

$$J_3(X) = \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)) \Delta_y W(X-Y) dy.$$

Then the function J_1 defined by (4.1) and J_3 are continuous on $\mathbf{R}^d\times[0,T]$. Furthermore the functions replaced the integral over $\mathbf{R}^d \setminus \overline{D}$ with that over D in the definitions of J_1 and J_3, respectively are also continuous.

Proof. To prove that J_1 is continuous at $Z \in \mathbf{R}^d \times [0,T]$, let $X = (x,t) \in \mathbf{R}^d \times [0,T]$ and choose $\epsilon > 0$ satisfying $\alpha > \epsilon$. Lemmas 3.1 and 5.1 imply

$$\begin{aligned}
&|J_1(X) - J_1(Z)| \\
&\le \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y \mathcal{E}(f)(y,s)||\nabla_y(W(X-Y) - W(Z-Y))|dy \\
&\le c_1\delta(X,Z)^\epsilon J(X,Z),
\end{aligned}$$

where

$$\begin{aligned}
J(X,Z) = &\int_0^T ds \int_{B(0,2R)} \operatorname{dist}(y,\partial D)^{\alpha-1}\delta(X,Y)^{-1-d-\epsilon}dy \\
&+ \int_0^T ds \int_{B(0,2R)} \operatorname{dist}(y,\partial D)^{\alpha-1}\delta(Z,Y)^{-1-d-\epsilon}dy.
\end{aligned}$$

On account of the previous lemma we see that $J(X,Z)$ is bounded on $(\mathbf{R}^d \times [0,T]) \times (\mathbf{R}^d \times [0,T])$. Therefore J_1 is continuous at Z.

We next consider J_3. To show that J_3 is continuous at $Z = (z,t_0) \in \mathbf{R}^d \times [0,T]$, take a point $X \in B(z,1) \times [0,T]$ and set $r = 2\max\{2R, |z|+1\}$. Then

$$\begin{aligned}
&|J_3(X) - J_3(Z)| \\
&\le \left|\int\int_A (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X))(\Delta_y W(X-Y) - \Delta_y W(Z-Y))dyds\right| \\
&+ \left|\int\int_B (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X))(\Delta_y W(X-Y) - \Delta_y W(Z-Y))dyds\right| \\
&+ \left|\int\int_C (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X))(\Delta_y W(X-Y) - \Delta_y W(Z-Y))dyds\right| \\
&+ \left|(\mathcal{E}(f)(Z) - \mathcal{E}(f)(X)) \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \Delta_y W(Z-Y)dy\right| \\
&\equiv I_1(X,Z) + I_2(X,Z) + I_3(X,Z) + I_4(X,Z),
\end{aligned}$$

where

$$\begin{aligned}
A &= \{Y = (y,s) \in (\mathbf{R}^d \setminus \overline{D}) \times [0,T]; \delta(X,Y) \le 3\delta(X,Z), |y| \le r\}, \\
B &= \{Y = (y,s) \in (\mathbf{R}^d \setminus \overline{D}) \times [0,T]; \delta(X,Y) > 3\delta(X,Z), |y| < r\}.
\end{aligned}$$

and

$$C = \{Y = (y,s) \in (\mathbf{R}^d \setminus \overline{D}) \times [0,T]; |y| > r\}.$$

Since $Z \mapsto \int_0^T \int_{\mathbf{R}^d\setminus\overline{D}} \Delta_y W(Z-Y)dy$ is bounded on $\mathbf{R}^d\times[0,T]$ by Lemma 3.2, we see that $I_4(X,Z) \to 0$ as $X \to Z$.

To estimate I_1 and I_2, choose $\epsilon > 0$ satisfying $\alpha - 2\epsilon > 0$. On account of Lemmas 3.1 and 5.1, we obtain

$$
\begin{aligned}
& I_1(X,Z) \\
& \le c_2 \|f\|_{\infty,\alpha,\alpha/2} \iint_A \delta(X,Y)^\alpha \delta(X,Z)^\epsilon \left(\delta(X,Y)^{-d-2-\epsilon} + \delta(Y,Z)^{-d-2-\epsilon}\right) dyds \\
& \le c_3 \|f\|_{\infty,\alpha,\alpha/2} \delta(X,Z)^\epsilon \{ \int_0^T ds \int_{|x-y|\le 3\delta(X,Z)} \delta(X,Y)^{\alpha-d-2-\epsilon} dy \\
& + \int_0^T ds \int_{|z-y|\le 4\delta(X,Z)} \delta(Z,Y)^{\alpha-d-2-\epsilon} dy \}
\end{aligned}
$$

Thus we see that $I_1(X,Z) \to 0$ as $X \to Z$.

To estimate I_2, pick ρ satisfying $0<\rho<1$ and $\alpha+\rho>1$. If $Y \in B$, then

$$\frac{3}{4}\delta(Y,Z) \le \delta(X,Y) \le \frac{3}{2}\delta(Y,Z).$$

Hence

$$
\begin{aligned}
& |I_2(X,Z)| \\
& \le c_4 \|f\|_{\infty,\alpha,\alpha/2} \iint_B \delta(X,Y)^\alpha \delta(X,Z) \left(\delta(X,Y)^{-d-3} + \delta(Y,Z)^{-d-3}\right) dyds \\
& \le c_5 \|f\|_{\infty,\alpha,\alpha/2} \delta(X,Z) \iint_B \delta(Y,Z)^{\alpha-d-3} dy.
\end{aligned}
$$

Noting that $\delta(Y,Z) \ge 2\delta(X,Z)$ for $Y \in B$ and

$$
\begin{aligned}
& \iint_B \delta(Y,Z)^{\alpha-d-3} dyds \\
& \le \int_0^T |r-s|^{-1+\epsilon} ds \int_{|y-z|>\sqrt{2}\delta(X,Z)} |y-z|^{-d-1+\alpha-2\epsilon} dy \\
& + \int_{|r-s|>2\delta(X,Z)^2} |r-s|^{-1-\rho/2} ds \int_{B(0,r)} |y-z|^{-d-1+\alpha+\rho} dy \\
& \le c_6 \left(\delta(X,Z)^{-1+\alpha-2\epsilon} + \delta(X,Z)^{-\rho}\right),
\end{aligned}
$$

we obtain

$$|I_2(X,Z)| \le c_7 \|f\|_{\infty,\alpha,\alpha/2} \left(\delta(X,Z)^{\alpha-2\epsilon} + \delta(X,Z)^{1-\rho}\right),$$

whence $I_2(X,Z) \to 0$ as $X \to Z$.

Finally we estimate I_3. If $Y \in C$, then

$$|y-z| \ge |y|-|z| > r/2+1 \quad \text{and } |x-y| \ge |y-z|-|x-z| > r/2,$$

which imply

$$
\begin{aligned}
& I_3(X,Z) \\
& \le c_8 \|f\|_{\infty,\alpha,\alpha/2} \delta(X,Z) \int_0^T \int_C \delta(X,Y)^\alpha \left(\delta(X,Y)^{-d-3} + \delta(Y,Z)^{-d-3}\right) dyds \\
& \le c_9 \|f\|_{\infty,\alpha,\alpha/2} \delta(X,Z) \{ \int_{|x-y|>r/2} |x-y|^{-d-3+\alpha} dy + \int_{|y-z|>r/2-1} |z-y|^{-d-3} dy \} \\
& \le c_{10} \|f\|_{\infty,\alpha,\alpha/2} \delta(X,Z),
\end{aligned}
$$

which shows that I_3 is continuous at Z. Similarly we can show the second conclusion. Therefore we have the conclusion. $\square$

Lemma 5.4. *Let $0 \le \beta-(d-1) < \alpha < 1$ and $g \in \Lambda_{\alpha,\alpha/2}(\mathbf{R}^d \times [0,T])$ such that $g(\cdot,t) \in C^1(\mathbf{R}^d)$, $\operatorname{supp} g(\cdot,t) \subset B(0,r_0)$ for every $t \in [0,T]$ and $\frac{\partial g}{\partial x_j}(x,\cdot)$ is bounded on $[0,T]$ for every $x \in \mathbf{R}^d$. Let $X=(x,t) \in \mathbf{R}^d \times (0,T]$ and set, for $0<\rho\le T$,*

$$\begin{aligned}A_\rho g(X) = &\int_0^\rho ds \int_{\mathbf{R}^d\setminus\overline{D}} \langle \nabla_y g(Y), \nabla_y W(X-Y)\rangle dy\\ &+\int_0^\rho ds \int_{\mathbf{R}^d\setminus\overline{D}} (g(Y)-g(X))\Delta_y W(X-Y)dy\\ &+g(X)\int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}} W(X-Y)dy\end{aligned}$$

and

$$\begin{aligned}B_\rho g(X) = -&\int_0^\rho ds \int_D \langle \nabla_y g(Y), \nabla_y W(X-Y)\rangle dy\\ &-\int_0^\rho ds \int_D (g(Y)-g(X))\Delta_y W(X-Y)dy\\ &-g(X)\int_{D\times\{0\}} W(X-Y)dy.\end{aligned}$$

Then

$$A_T g(X) = B_T g(X) + g(X) \text{ for } X \in \mathbf{R}^d \times (0,T] \tag{5.4}$$

Proof. To simplify the notation, we use $A_\rho(X)$ and $B_\rho(X)$ instead of $A_\rho g(X)$ and $B_\rho g(X)$, respectively. We first show (5.4) for a bounded piecewise smooth domain D_0. Let $X=(x,t)$ and set, for $0<\rho<t$,

$$I_\rho(X) = -\int_0^\rho ds \int_{\partial D_0} g(Y)\langle \nabla_y W(X-Y), n_y\rangle d\sigma(y).$$

The Green formula for D_0 yields

$$\begin{aligned}I_\rho(X) = -&\int_0^\rho ds \int_{D_0} \langle \nabla g(Y), \nabla_y W(X-Y)\rangle dy\\ &-\int_0^\rho ds \int_{D_0} (g(Y)-g(X))\Delta_y W(X-Y)dy\\ &-g(X)\int_0^\rho ds \int_{D_0} \Delta_y W(X-Y)dy\end{aligned} \tag{5.5}$$

From (3.2) we deduce

$$\begin{aligned}&\int_0^\rho ds \int_{D_0} \Delta_y W(X-Y)dy\\ &=\int_{D_0\times\{0\}} W(X-Y)dy - \int_{D_0\times\{\rho\}} W(X-Y)dy,\end{aligned}$$

whence

$$\int_0^t ds \int_{D_0} \Delta_y W(X-Y)dy = \int_{D_0\times\{0\}} W(X-Y)dy - \chi_{D_0}(x).$$

This and (5.5) imply

$$I_t(X) = B_t(X) + g(X)\chi_{D_0}(x) \text{ for } X \in (\mathbf{R}^d \setminus \partial D_0) \times (0,T]. \tag{5.6}$$

Similarly, using the Green formula for $B(0,r)\setminus \overline{D}_0$ and $r \to \infty$, we obtain

$$I_t(X) = A_t(X) - g(X)\chi_{\mathbf{R}^d\setminus\overline{D}_0}(x)$$

for $X \in (\mathbf{R}^d \setminus \partial D_0) \times (0,T]$. This and (5.6) lead to

$$A_t(X) = B_t(X) + g(X) \text{ for } X \in (\mathbf{R}^d \setminus \partial D_0) \times (0,T].$$

Noting that $A_t(X) = A_T(X)$ and $B_t(X) = B_T(X)$, we obtain (5.4) for $X \in (\mathbf{R}^d\setminus\partial D) \times (0,T]$. Since A_T, B_T and the functions $X \mapsto \int_{D_0\times\{0\}} W(X-y)dy$, $X \mapsto \int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}} W(X-Y)dy$ are continuous on $\mathbf{R}^d \times (0,T]$, (5.4) holds for a bounded piecewise smooth domain $D = D_0$.

We next show (5.4) for a bounded domain such that ∂D is a β-set. Let $X \in D$. We use (5.4) for $D_0 = A_n$. Since

$$\int_0^T \int_{\mathbf{R}^d} |\nabla g(Y)||\nabla_y W(X-Y)|dyds < \infty,$$
$$\int_0^T ds \int_{\mathbf{R}^d} |g(Y) - g(X)||\Delta_y W(X-Y)|dy < \infty$$

and

$$\int_{\mathbf{R}^d\times\{0\}} W(X-Y)dy < \infty,$$

we obtain (5.4) for the domain D as $n \to \infty$. □

Lemma 5.5. *Let $0 \le \beta - (d-1) < \alpha$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$. Then (5.4) holds for $g = \mathcal{E}(f)$.*

Proof. Let $f \in \Lambda_{\alpha,\alpha/2}(S_D)$ and $\{v_m\}$ be a mollifier on $\mathbf{R}^d$ such that supp $v_m \subset B(0,1/m)$. We define, for $Y = (y,s) \in \mathbf{R}^d \times [0,T]$,

$$g_m(Y) = \big(\mathcal{E}(f)(\cdot,s) * v_m\big)(y).$$

Lemma 5.4 yields

$$A_T g_m(X) = B_T g_m(X) + g_m(X) \text{ for } X \in \mathbf{R}^d \times (0,T].$$

Since $g_m(X) \to \mathcal{E}(f)(X)$ as $m \to \infty$, it suffices to show that

$$A_T g_m(X) \to A_T\mathcal{E}(f)(X) \tag{5.7}$$

and

$$B_T g_m(X) \to B_T\mathcal{E}(f)(X) \tag{5.8}$$

for $X \in \mathbf{R}^d \times (0,T]$ as $m \to \infty$.

To show (5.7), we write

$$\begin{aligned}
&|A_T g_m(X) - A_T\mathcal{E}(f)(X)| \\
&\le \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y g_m(Y) - \nabla_y \mathcal{E}(f)(Y)||\nabla_y W(X-Y)|dy \\
&+ \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |g_m(Y) - g_m(X) - \mathcal{E}(f)(Y) + \mathcal{E}(f)(X)||\Delta_y W(X-Y)|dy \\
&+ |g_m(X) - \mathcal{E}(f)(X)| \int_{(\mathbf{R}^d\setminus\overline{D})\times\{0\}} W(X-Y)dy \\
&\equiv I_{m1}(X) + I_{m2}(X) + I_{m3}(X).
\end{aligned}$$

To estimate $I_{m1}(X)$, we write

$$\begin{aligned}
I_{m1}(X) &= \int_0^T \int_{\mathrm{dist}(y,\partial D)\le 2/m} + \int_0^T \int_{\mathrm{dist}(y,\partial D)>2/m} \\
&\equiv I_{m11}(X) + I_{m12}(X)
\end{aligned}$$

and

$$\begin{aligned}
I_{m11}(X) &\le \int_0^T ds \int_{\mathrm{dist}(y,\partial D)\le 2/m} |\nabla_y g_m(Y)||\nabla_y W(X-Y)|dy \\
&+ \int_0^T ds \int_{\mathrm{dist}(y,\partial D)\le 2/m} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y W(X-Y)|dy \\
&\equiv I_{m111}(X) + I_{m112}(X).
\end{aligned}$$

Noting that

$$\begin{aligned}
|\nabla_y g_m(Y)| &\le c_1\|f\|_{\infty,\alpha} \int \mathrm{dist}(y-z,\partial D)^{\alpha-1} v_m(z)dz \\
&\le c_1\|f\|_{\infty,\alpha} \int_{B(w,3/m)} \mathrm{dist}(u,\partial D)^{\alpha-1} v_m(y-u)du,
\end{aligned}$$

where w is a point in ∂D such that $\mathrm{dist}(y,\partial D) = |y-w|$. Hence, together with Lemma B and Lemma 5.2,

$$\begin{aligned}
I_{m111}(X) &\le c_2\|f\|_{\infty,\alpha}\left(\frac{3}{m}\right)^{\alpha-1} \int_0^T ds \int_{\mathrm{dist}(y,\partial D)\le 2/m} \delta(X,Y)^{-d-1}dy \\
&\le c_3\|f\|_{\infty,\alpha} m^{-\epsilon} \int_0^T ds \int_{B(0,R+1)} \delta(X,Y)^{-d-1}\mathrm{dist}(y,\partial D)^{\alpha-1-\epsilon}dy \\
&\le c_4\|f\|_{\infty,\alpha} m^{-\epsilon}.
\end{aligned}$$

Since

$$I_{m112}(X) \le c_5\|f\|_{\infty,\alpha} m^{-\epsilon} \int_0^T ds \int_{B(0,R+1)} \mathrm{dist}(y,\partial D)^{\alpha-1-\epsilon}\delta(X,Y)^{-d-1}dy,$$

we see that $I_{m11}(X) \to 0$ as $m \to \infty$.

We next estimate I_{m12}. Suppose $\mathrm{dist}(y,\partial D) > 2/m$. Noting that

$$\left|\frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_j}(y)\right| \le c_6\|f\|_{\infty,\alpha}\mathrm{dist}(y,\partial D)^{\alpha-2},$$

we have

$$\begin{aligned}
&|\frac{\partial g_m}{\partial y_j}(Y) - \frac{\partial \mathcal{E}(f)}{\partial y_j}(y,s)| \\
&\le c_6\|f\|_{\infty,\alpha} \int |\frac{\partial \mathcal{E}(f)}{\partial y_j}(y-z,s) - \frac{\partial \mathcal{E}(f)}{\partial y_j}(y,s)|v_m(z)dz \\
&\le c_7\|f\|_{\infty,\alpha}\frac{1}{m}\mathrm{dist}(y,\partial D)^{\alpha-2}.
\end{aligned}$$

Choose $\epsilon > 0$ satisfying $\alpha - 3\epsilon > 0$ and $\alpha - \epsilon > \beta - (d-1)$. Then

$$\begin{aligned}
&I_{m12}(X) \\
&\le c_8\|f\|_{\infty,\alpha}\frac{1}{m}\int_0^T ds \int_{\mathrm{dist}(y,\partial D)>2/m} \mathrm{dist}(y,\partial D)^{\alpha-2}\delta(X,Y)^{-d-1}dy \\
&\le c_9\|f\|_{\infty,\alpha} m^{-\epsilon}\int_0^T ds \int_{B(O,2R)} \mathrm{dist}(y,\partial D)^{\alpha-1-\epsilon}\delta(X,Y)^{-d-1}dy.
\end{aligned}$$

Using Lemma 5.2, we have

$$I_{m12}(X) \le c_{10}\|f\|_{\infty,\alpha} m^{-\epsilon},$$

whence $I_{m12}(X) \to 0$ as $m \to \infty$. Thus we conclude that $I_{m1}(X) \to 0$ as $m \to \infty$.

We next consider I_{m2}. To do so, we put

$$\begin{aligned}
I_{m2}(X) &\le \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\Delta_y W(X-Y)|dy \\
&\times \int |\mathcal{E}(f)(y-z,s) - \mathcal{E}(f)(y,s) - \mathcal{E}(f)(x-z,s) + \mathcal{E}(f)(x,s)|v_m(z)dz \\
&= \int_0^T \int_A \int + \int_0^T \int_B \int \equiv I_{m21}(X) + I_{m22}(X),
\end{aligned}$$

where

$$A = \{y \in \mathbf{R}^d \setminus \overline{D}; |x-y| \le 2/m\} \quad B = \{y \in \mathbf{R}^d \setminus \overline{D}; |x-y| > 2/m\}.$$

Then

$$\begin{aligned}
I_{m21}(X) &\le \int_0^T ds \int_A |\Delta_y W(X-Y)|dy \\
&\times \int (|\mathcal{E}(f)(y-z,s) - \mathcal{E}(f)(x-z,s)| + |\mathcal{E}(f)(y,s) - \mathcal{E}(f)(x,s))|v_m(z)dz \\
&\le c_{11}\|f\|_{\infty,\alpha}\int_0^T ds \int_A \delta(X,Y)^{-d-2}|x-y|^\alpha dy \\
&\le c_{12}\|f\|_{\infty,\alpha} m^{-\epsilon}\int_0^T |t-s|^{-1+\epsilon}ds \int_{|x-y|\le 2/m} |x-y|^{-d+\alpha-3\epsilon}dy
\end{aligned}$$

and

$$\begin{aligned}
&I_{m22}(X) \le c_{13}\int_0^T ds \int_B \delta(X,Y)^{-d-2}dy \\
&\int (|\mathcal{E}(f)(y-z,s) - \mathcal{E}(f)(y,s)| + |\mathcal{E}(f)(x-z,s) - \mathcal{E}(f)(x,s))|v_m(z)dz \\
&\le c_{14}\|f\|_{\infty,\alpha} m^{-\alpha}\int_0^T |t-s|^{-1+\epsilon}ds \int_{|x-y|>2/m} |x-y|^{-d-2\epsilon}dy \\
&\le c_{15}\|f\|_{\infty,\alpha} m^{-\alpha+2\epsilon}.
\end{aligned}$$

Therefore we see that $I_{m2}(X) \to 0$ as $m \to \infty$.

Finally we have

$$I_{m3}(X) \le |g_m(X) - \mathcal{E}(f)(X)| \quad \to 0 \ (m \to \infty).$$

Thus we obtain (5.7). Similarly we can show (5.8). □

The nontangential limits of Φf are as follows.

Lemma 5.6. *Let $p > 1$, $0 \le \beta - (d-1) < \alpha < 1$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$. Then, for each $Z \in \partial D \times [0,T]$,*

$$\lim_{X \to Z, X \in D \times (0,T)} \Phi f(X) = Kf(Z) + \frac{1}{2} f(Z) \tag{5.9}$$

and

$$\lim_{X \to Z, X \in (\mathbf{R}^d \setminus \overline{D}) \times (0,T)} \Phi f(X) = Kf(Z) - \frac{1}{2} f(Z). \tag{5.10}$$

Proof. Let $X \in D \times (0,T]$. Using Lemma 3.2, we have $\Phi f(X) = A_T f(X)$. Since $A_T f$ is continuous on $\mathbf{R}^d \times (0,T]$ by Lemma 5.3, we have

$$\lim_{X \to Z, X \in D \times (0,T)} \Phi f(X) = A_T f(Z). \tag{5.11}$$

On the other hand Lemma 5.5 yields

$$Kf(Z) = \frac{1}{2}(A_T f(Z) + B_T f(Z)) = A_T f(Z) - \frac{1}{2} f(Z),$$

which and (5.11) lead to (5.9). Similarly we can show (5.10). □

6. Boundedness of the operator K

In this section we investigate the boundedness of the operator K defined by (1.9). The proof of the following lemma is easy.

Lemma 6.1. *Let $y \in \mathbf{R}^d \setminus \partial D$ and z_y be a boundary point satisfying* $\mathrm{dist}(y, \partial D) = |y - z_y|$. *Then*

$$|y - w| \ge \frac{1}{2}|z_y - w| \quad \textit{for all } w \in \partial D.$$

Lemma 6.2. *Let $p > 1$, $0 \le \beta - (d-1) < \alpha < 1$ and $f \in \Lambda^p_\alpha(\mu_T)$. Further define*

$$Uf(Z) = \int_0^T ds \int |\nabla_y \mathcal{E}(f)(y,s)| \delta(Z,Y)^{-d-1} dy$$

for $Z \in S_D$. Then

$$\|Uf\|_p \le \|f\|_{p,\alpha}.$$

Proof. We choose $\epsilon > 0$ satisfying (4.2) and choose $b > 0$ satisfying $1 - \alpha - \epsilon q - 3bq > 0$. Let $Z = (z,r) \in S_D$. Since $B(0,2R) \subset B(z,3R)$, we have, by Lemma B,

$$\int_{B(0,2R)} \mathrm{dist}(y, \partial D)^{-\epsilon q} |y - z|^{1-d-3bq} dy \le c_1 (3R)^{1-\epsilon q - 3bq}.$$

This and Hölder's inequality imply

$$|Uf(Z)| \leq c_3 \int_0^T \int |r-s|^{-1+b} |\nabla_y \mathcal{E}(f)(y,s)| |z-y|^{-d+1-2b} dy ds$$
$$\leq c_2 \left(\int_0^T |r-s|^{-1+b} ds \int |\nabla_y \mathcal{E}(f)(y,s)|^p \operatorname{dist}(y,\partial D)^{p\epsilon} |y-z|^{1-d+bp} dy \right)^{1/p}.$$

Therefore we have, by Lemma 6.1,

$$\int_0^T \int |Uf(Z)|^p d\mu(z) dr$$
$$\leq c_3 \int_0^T ds \int |\nabla_y \mathcal{E}(f)(y,s)|^p \operatorname{dist}(y,\partial D)^{p\epsilon} dy \int_0^T |r-s|^{-1+\epsilon} dr \int |w_y - z|^{1-d+bp} d\mu(z),$$

where w_y is a point in ∂D satisfying $\operatorname{dist}(y,\partial D) = |y-w_y|$. Noting that $1-d+bp > -\beta$, we obtain

$$\int_0^T \int |Uf(Z)|^p d\mu(z) dr \leq c_4 \int_0^T ds \int |\nabla_y \mathcal{E}(f)(y,s)|^p \operatorname{dist}(y,\partial D)^{p\epsilon} dy.$$

Since $d-\beta+p(1-\alpha)+p\epsilon > 0$, Lemma 1.1, (iv) leads to the conclusion. □

We note that $\mathcal{E}(f)$ has the following properties.

Lemma 6.3. *Let $p>1$ and $0 \leq \beta-(d-1) < \alpha < 1$. Then*

$$\int_0^T dr \int \left(\int_0^T ds \int_{\mathbf{R}^d \setminus \partial D} \frac{|\mathcal{E}(f)(y,s) - f(z,s)|}{\delta(Y,Z)^{d+2}} dy \right)^p d\mu(z) \leq c \|f\|^p_{p,\alpha}$$

for every $f \in \Lambda^p_\alpha(\mu_T)$.

Proof. We write

$$|\mathcal{E}(f)(y,s) - f(z,s)| \leq |\phi_0(y)(\mathcal{E}_0(f)(y,s) - f(z,s))|$$
$$+ |(1-\phi_0(y)) f(z,s)| \equiv I_1(Y,Z) + I_2(Y,Z).$$

Let us consider $I_1(Y,Z)$. To do so, let $Q \in \mathcal{V}_k(\mathbf{R}^d \setminus \partial D)$ and $y \in Q$. Denote by a (resp. l) the center (resp. the side length) of Q. For $Y=(y,s) \in Q$ we note that

$$|\mathcal{E}_0(f)(y,s) - f(z,s)| \leq c_1 \sum_j \phi_j^*(y) l_j^{-\beta} \int_{B(a^{(j)},\eta l_j)} |f(w,s) - f(z,s)| d\mu(w)$$
$$\leq c_2 l^{-\beta} \int_{B(a,bl)} |f(w,s) - f(z,s)| d\mu(w),$$

where b is a constant satisfying $b \geq 6\sqrt{d}$, independent of Q and f.

First assume that $|z-a| \leq 3bl$. Then

$$I_1(Y,Z) \leq c_3 l^\alpha \left(\int_{B(a,bl)} \frac{|f(w,s) - f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w) \right)^{1/p}$$

and

$$\int_Q I_1(Y,Z)^p \delta(Y,Z)^{-d-2-p\alpha+3p\epsilon} dy \tag{6.1}$$
$$\le c_3|s-t|^{-1+p\epsilon} l^{p\alpha} \int_Q |y-z|^{-d-p\alpha+p\epsilon} dy \int_{B(a,bl)} \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w)$$
$$\le c_4|s-r|^{-1+p\epsilon} l^{p\epsilon} \int_{B(a,bl)} \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w),$$

where ϵ is a positive real number satisfying $\alpha - 3\epsilon > 0$.

We next assume that $|z-a| > 3bl$. Since $|z-w| \le b_2|y-z|$, we have

$$I_1(Y,Z)\delta(Y,Z)^{-(d+2)/p-\alpha+3\epsilon}$$
$$\le c_5\delta(Y,Z)^{-(d+2-\beta)/p+3\epsilon} l^{-\beta/p} \left(\int_{B(a,bl)} \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w) \right)^{1/p}$$

and hence

$$\int_Q I_1(Y,Z)^p \delta(Y,Z)^{-(d+2)-p(\alpha-3\epsilon)} dy \tag{6.2}$$
$$\le c_6 \int_Q \delta(Y,Z)^{-(d+2-\beta)+3\epsilon p} l^{-\beta} dy \int_{B(a,bl)} \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w)$$
$$\le c_7|r-s|^{-1+p\epsilon} l^{\epsilon p} \int_{B(a,bl)} \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w).$$

From (6.1), (6.2) and Lemma 2.1 we deduce

$$\sum_{Q\in\mathcal{V}_K(\mathbf{R}^d\setminus\partial D)} \int_Q I_1(Y,Z)^p \delta(Y,Z)^{-d-2-p(\alpha-3\epsilon)} dy$$
$$\le c_8(2^{-k})^{\epsilon p}|r-s|^{-1+p\epsilon} \sum_{Q\in\mathcal{V}_K(\mathbf{R}^d\setminus\partial D)} \int_{B(a,bl)} \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w)$$
$$\le c_9(2^{-k})^{\epsilon p}|r-s|^{-1+p\epsilon} \int \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w),$$

whence

$$\int_0^T ds \int_{B(0,2R)\setminus\partial D} I_1(Y,Z)^p \delta(Y,Z)^{-d-2-p(\alpha-3\epsilon)} dy$$
$$\le c_{10} \int_0^T |r-s|^{-1+p\epsilon} ds \int \frac{|f(w,s)-f(z,s)|^p}{|w-z|^{\beta+\alpha p}} d\mu(w).$$

On the other hand, since

$$\left(\int_0^T ds \int_{B(0,2R)\setminus\partial D} I_1(Y,Z)\delta(Y,Z)^{-d-2} dy \right)^p$$
$$\le \int_0^T \int_{B(0,2R)\setminus\partial D} I_1(Y,Z)^p \delta(Y,Z)^{-d-2-p(\alpha-3\epsilon)} ds dy \left(\int_0^T \int_{B(0,2R)} \delta(Y,Z)^{-d-2+q(\alpha-3\epsilon)} ds dy \right)^{p/q},$$

we have

$$\int_0^T dr \int d\mu(z) \left(\int_0^T ds \int_{(B(0,2R)\setminus\partial D} I_1(Y,Z)\delta(Y,Z)^{-d-2} dy \right)^p \le c_{11} \|f\|_{p,\alpha}^p.$$

We next estimate $I_2(Y,Z)$. Noting that $1 - \phi_0(y) = 0$ on $B(0,R)$ and

$$\int_0^T ds \int I_2(Y,Z)\delta(Y,Z)^{-d-2} dy$$

$$\le c_{12} \int_0^T |f(z,s)| ds \int_{|y-z| \ge R/2} |y-z|^{-d-2} dy \le c_{13} R^{-2} \left(\int_0^T |f(z,s)|^p ds \right)^{1/p},$$

we obtain

$$\int_0^T dr \int d\mu(x) \left(\int_0^T ds \int_{B(0,2R)\setminus\partial D} I_2(Y,Z)\delta(Y,Z)^{-d-2} dy \right)^p \le c_{14} \|f\|_p^p.$$

Thus we have the conclusion. □

Lemma 6.4. *Let $p > 1$ and $\lambda > 0$. Set, for a function f in $L^p(\mu_T)$,*

$$J_f = \int d\mu(z) \int_0^T \int_0^T \frac{|f(z,s) - f(z,r)|^p}{|s-r|^{1+p\lambda}} ds dr.$$

If $J_f < \infty$, then

$$\int_0^T dr \int d\mu(z) \left(\int_0^T ds \int_{\mathbf{R}^d\setminus\partial D} \frac{|f(z,s) - f(z,r)|}{\delta(Y,Z)^{d+2}} dy \right)^p \le c(J_f + \|f\|_p^p).$$

Proof. Set

$$I(z,r) = \int_0^T ds \int_{\mathbf{R}^d\setminus\partial D} |f(z,s) - f(z,r)| \delta(Y,Z)^{-d-2} dy.$$

It suffices to show that

$$I(z,r) \le c_1 \Big(\int_0^T \frac{|f(z,s) - f(z,r)|^p}{|s-r|^{p\lambda}} ds \Big)^{1/p} + c_1 \Big(\int_0^T |f(z,r)|^p ds \Big)^{1/p} + c_1 |f(z,r)|. \tag{6.3}$$

To show this we write

$$I(z,r) = \int_0^T ds \int_{B(0,2R)\setminus\partial D} + \int_0^T ds \int_{\mathbf{R}^d\setminus B(0,2R)} \equiv I_1(z,r) + I_2(z,r).$$

Choose $\epsilon > 0$ satisfying $\lambda - \epsilon > 0$. Then

$$I_1(z,r) \le \int_0^T |s-r|^{1/p+\lambda-1-\epsilon} \frac{|f(z,s) - f(z,r)|}{|s-r|^{1/p+\lambda}} ds \int_{B(z,3R)} |z-y|^{-d+2\epsilon} dy$$

$$\le c_2 \left(\int_0^T \frac{|f(z,s) - f(z,r)|^p}{|s-r|^{1+p\lambda}} ds \right)^{1/p} \left(\int_0^T |s-r|^{-1+(\lambda-\epsilon)q} ds \right)^{1/q}$$

$$\le c_3 \left(\int_0^T \frac{|f(z,s) - f(z,r)^p|}{|s-r|^{1+p\lambda}} ds \right)^{1/p}.$$

On the other hand we obtain

$$I_2(z,r) \le c_4 \int_0^T |f(z,s)-f(z,r)|ds \int_{|y-z|>R} |z-y|^{-d-2}dy$$
$$\le c_5 \int_0^T |f(z,s)-f(z,r)|ds \le c_5\Big(\int_0^T |f(z,s)|^p ds\Big)^{1/p} + c_5|f(z,r)|.$$

Thus we see that (6.3) holds. □

Lemma 6.5. *Let* $0 \le \beta-(d-1) < \alpha < 1$, $1 \ge \lambda > 0$ *and* $p>1$. *Then the operator* K *defined by (1.9) is a bounded operator from* $\Lambda^p_{\alpha,\lambda}(\mu_T)$ *to* $L^p(\mu_T)$.

Proof. Noting that the function

$$x \mapsto \int_{\mathbf{R}^d\times\{0\}} W(X-Y)dy$$

is bounded on ∂D and using Lemmas 6.2, 6.3 and 6.4, we have the conclusion. □

We next estimate the nontangential maximal function of Φf.

Lemma 6.6. *Suppose* $1>\alpha>\beta-(d-1)\ge 0$, $p>1$, $1>\lambda>0$ *and* $f\in\Lambda^p_{\alpha,\lambda}(\mu_T)$, *and define, for* $Z\in S_D$,

$$(\Phi f)^*(Z) := \sup\{|\Phi f(X)| : x\in\Gamma_\tau(Z)\cap B(Z,\epsilon_0)\times(0,T)\}$$

and

$$(\Phi f)^{**}(Z) := \sup\{|\Phi f(X)| : x\in\Gamma^e_\tau(Z)\cap B(Z,\epsilon_0)\times(0.T)\}.$$

Then

$$\|(\Phi f)^*\|_p \le c\|f\|_{p,\alpha,\lambda} \quad \textit{and} \quad \|(\Phi f)^{**}\|_p \le c\|f\|_{p,\alpha,\lambda}.$$

Proof. We claim that

$$\delta(X,Y)\ge c_1\delta(Y,Z) \quad \text{for every } X\in\Gamma_\tau(Z).$$

To see this, set

$$A=\{Y=(y,s)\in(\mathbf{R}^d\setminus\overline{D})\times(0,T);\delta(Y,Z)\le 2\delta(X,Z)\}$$

and

$$B=\{Y=(y,s)\in(\mathbf{R}^d\setminus\overline{D})\times(0,T);\delta(Y,Z)> 2\delta(X,Z)\}$$

If $Y\in A$, then

$$\delta(X,Y)\ge\delta(X;S_D)\ge\frac{\delta(X,Z)}{1+\tau}\ge\frac{\delta(Y,Z)}{2(1+\tau)}.$$

If $Y\in B$, then

$$\delta(X,Y)\ge\delta(Y,Z)-\delta(Z,X)\ge\frac{\delta(Y,Z)}{2}.$$

Thus we see that the claim is true.

Using the claim, we have, for J_1 defined by (4.1),

$$\sup_{X\in\Gamma_\tau(Z)} |J_1(X)| \le c_1\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y\mathcal{E}(f)(Y)|\delta(Y,Z)^{-d-1}dy. \tag{6.2}$$

Lemma 6.2 yields

$$\left(\int_0^T \int J_1^*(Z)^p d\mu(z) dr\right)^{1/p} \le \|f\|_{p,\alpha}.$$

We next set

$$J_2(X) = \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(X-Y) dy.$$

We write

$$\begin{aligned} J_2(X) &= \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(y,s) - f(z,s)) \Delta_y W(X-Y) dy \\ &+ (f(z,s) - f(z,r)) \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \Delta_y W(X-Y) dy \\ &+ \lim_{n\to\infty} f(z,r) \int_0^T ds \int_{B_n} \Delta_y W(X-Y) dy \\ &\equiv J_{21}(X) + J_{22}(X) + J_{23}(X). \end{aligned}$$

Using the claim, we have

$$\begin{aligned} J_{21}^*(Z) &\le c_3 \int_0^T ds \int |\mathcal{E}(f)(y,s) - f(z,s)| \delta(Y,Z)^{-d-2} dy, \\ J_{22}^*(Z) &\le c_4 \int_0^T |f(z,s) - f(z,r)| ds \int \delta(Y,Z)^{-d-2} dy \end{aligned}$$

and

$$J_{23}^*(Z) \le c_5 |f(z,r)|.$$

Lemmas 6.2, 6.3 and 6.4 yield

$$\int_0^T \int (\Phi f)^*(Z) d\mu(z) ds \le c_6 \|f\|_{p,\alpha,\lambda}.$$

Similarly we can also show the estimate of $(\Phi f)^{**}$. □

7. Approximation of Hölder continuous functions

In this section we approximate $f \in \Lambda^p_{\alpha,\alpha/2}(\mu_T)$ by functions in $\Lambda_{1,1}(S_D)$. To do so we choose C^∞-functions ψ, ψ_0 on $\mathbf{R}^d$, $\mathbf{R}$, respectively such that

$$\psi = 1 \text{ on } B(0,1), \quad 0 \le \psi \le 1, \quad \operatorname{supp} \psi \subset B(0,2)$$

and

$$\psi_0 = 1 \text{ on } [-1,1], \quad 0 \le \psi_0 \le 1, \quad \operatorname{supp} \psi \subset [-2,2].$$

For each natural number n and each $X = (x,t) \in \partial D \times [0,T]$ define

$$a_n(X) = \int \psi_0(n^2(t-s)) ds \int \psi(n(x-y)) d\mu(y)$$

and

$$h_n(X,Y) = h_n(x,t;y,s) = \frac{1}{a_n(X)} \psi_0(n^2(t-s)) \psi(n(x-y)).$$

Further, define, for $f \in \Lambda^p_{\alpha,\alpha/2}(\mu_T)$,

$$H_n f(X) = \int_0^T \int h_n(X,Y) f(Y) ds d\mu(y).$$

We note that $H_n 1 = 1$.

Lemma 7.1. *Let $0 \le \beta - (d-1) < \alpha < 1$ and $f \in \Lambda^p_{\alpha,\alpha/2}(\mu_T)$. Then $H_n f \in \Lambda_{1,1}(S_D)$ and*

$$\|H_n f - f\|_{p,\alpha,\alpha/2} \to 0 \quad \text{as } n \to \infty.$$

Proof. We note that $a_n(X) \ge c_1 n^{-\beta-2}$ for every $X \in S_D$. We first show that $H_n f \in \Lambda_{1,1}(S_D)$. Indeed,

$$\begin{aligned}
|h_n(x,t;y,s) - h_n(z,t;y,s)| &\le \left|\frac{\psi_0(n^2(t-s))}{a_n(x,t)} (\psi(n(x-y)) - \psi(n(z-y)))\right| \\
&+ \left|\psi_0(n^2(t-s))\psi(n(z-y)) \frac{a_n(z,t) - a_n(x,t)}{a_n(x,t)a_n(z,t)}\right| \\
&\le c_2 |x-z| n^{\beta+3} \left(\psi_0(n^2(t-s)) + \psi_0(n^2(t-s))\psi(n(z-y))\right)
\end{aligned}$$

and

$$\begin{aligned}
|h_n(x,t;y,s) - h_n(x,r;y,s)| &\le \left|\frac{\psi_0(n^2(t-s)) - \psi_0(n^2(r-s))}{a_n(x,t)} \psi(n(x-y))\right| \\
&+ \left|\psi_0(n^2(r-s))\psi(n(x-y)) \frac{a_n(x,r) - a_n(x,t)}{a_n(x,t)a_n(x,r)}\right| \\
&\le c_3 |t-r| n^{\beta+4} \left(\psi(n(x-y)) + \psi_0(n^2(r-s))\psi(n(x-y))\right).
\end{aligned}$$

Therefore we see that $H_n f \in \Lambda_{1,1}(S_D)$.

We next show that

$$\|H_n f - f\|_p \to 0 \quad \text{as } n \to \infty. \tag{7.1}$$

To do so, set

$$J_n(X) \equiv H_n f(X) - f(X) = \int_0^T \int h_n(X,Y)(f(Y) - f(X)) ds d\mu(y).$$

Then

$$\begin{aligned}
|J_n(X)| &\le c_4 \frac{n^{-\beta/p-\alpha}}{a_n(X)} \int_{|t-s|\le 2/n^2} ds \int_{|x-y|\le 2/n} \frac{|f(y,s) - f(x,s)|}{|y-x|^{\beta/p+\alpha}} d\mu(y) \\
&+ c_4 \frac{n^{-2/p-\alpha}}{a_n(X)} \int_{|t-s|\le 2/n^2} \frac{|f(x,s) - f(x,t)|}{|s-t|^{1/p+\alpha/2}} ds \int_{|x-y|\le 2/n} d\mu(y) \\
&\le c_5 n^{-\alpha} \mathcal{M} u_x(t) + c_5 n^{-\alpha} \left(\int_{|s-t|\le 2/n^2} \frac{|f(x,s) - f(x,t)|^p}{|s-t|^{1+p\alpha/2}} ds \right)^{1/p},
\end{aligned}$$

where

$$u_x(s) = \begin{cases} \left(\int_{|x-y|\le 2/n} \frac{|f(y,s) - f(x,s)|^p}{|y-x|^{\beta+p\alpha}} d\mu(y) \right)^{1/p} & \text{if } s \in [0,T] \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}&\int_0^T dt \int |J_n(X)|^p d\mu(x)\\&\le c_6 n^{-p\alpha} \int d\mu(x) \int_0^T \mathcal{M}u_x(t)^p dt + c_6 n^{-p\alpha} \int d\mu(x) \int_0^T \int_0^T \frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}} ds dt\\&\le c_7 n^{-p\alpha} \|f\|^p_{p,\alpha,\alpha/2},\end{aligned}$$

which shows (7.1).

We next show that

$$\int d\mu(x) \int_0^T \int_0^T \frac{|H_n f(x,t) - f(x,t) - H_n f(x,r) + f(x,r)|^p}{|t-r|^{1+p\alpha/2}} dt dr \to 0. \tag{7.2}$$

To do so, set

$$A = \{(t,r) \in [0,T] \times [0,T]; |t-r| \le 4/n^2\}$$

and

$$B = \{(t,r) \in [0,T] \times [0,T]; |t-r| > 4/n^2\}.$$

Let $(t,r) \in A$. Since $H_n 1 = 1$, we write

$$\begin{aligned}&|H_n f(x,t) - f(x,t) - H_n f(x,r) + f(x,r)|\\&\le \left|\int_0^T ds \int (h_n(x,t;y,s) - h_n(x,r;y,s))(f(y,s) - f(x,s)) d\mu(y)\right|\\&+ \left|\int_0^T ds \int (h_n(x,t;y,s) - h_n(x,r;y,s))(f(x,s) - f(x,t)) d\mu(y)\right|\\&+ |f(x,t) - f(x,r)| \equiv I_1 + I_2 + I_3.\end{aligned}$$

Noting that

$$\begin{aligned}&|h_n(x,t;y,s) - h_n(x,r;y,s)|\\&\le c_8 n^{\beta+4} |t-r| \chi_{B(x,2/n)}(y) \chi_{B(t,8/n^2)}(s),\end{aligned} \tag{7.3}$$

we have

$$\begin{aligned}I_1 &\le c_9 n^{\beta+4} n^{-\beta/p-\alpha} |t-r| \int_{|t-s|\le 8/n^2} ds \int_{|x-y|\le 2/n} \frac{|f(y,s)-f(x,s)|}{|y-x|^{\beta/p+\alpha}} d\mu(y)\\&\le c_{10} n^{2-\alpha} |t-r| \mathcal{M}u_x(t).\end{aligned}$$

Hence

$$\begin{aligned}&\int d\mu(x) \iint_A I_1^p |t-r|^{-1-p\alpha/2} dt dr\\&\le c_{11} n^{p(2-\alpha)} \int d\mu(x) \int \mathcal{M}u_x(t)^p dt \int_{|t-r|\le 4/n^2} |t-r|^{p-1-p\alpha/2} dr\\&\le c_{12} \int d\mu(x) \int_0^T dt \int_{|x-y|\le 2/n} \frac{|f(y,s)-f(x,s)|^p}{|y-x|^{\beta+p\alpha}} d\mu(y).\end{aligned} \tag{7.4}$$

Similarly we have

$$I_2 \le c_{13} n^{\beta+4} |t-r| \int_{|s-t|\le 8/n^2} |f(x,s)-f(x,t)| ds \int_{|x-y|\le 2/n} d\mu(y)$$
$$\le c_{14} n^{2-\alpha} |t-r| \left(\int_{|s-t|\le 8/n^2} \frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}} ds \right)^{1/p},$$

whence

$$\int d\mu(x) \int_0^T \int_0^T \frac{(I_2 \chi_A(t,r))^p}{|t-r|^{1+p\alpha/2}} dt dr \tag{7.5}$$
$$\le c_{14} n^{2p-p\alpha} \int d\mu(x) \int_0^T dt \int_{|s-t|\le 8/n^2} \frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}} ds$$
$$\int_{|t-r|\le 4/n^2} |t-r|^{p-1-p\alpha/2} dr$$
$$\le c_{15} \int d\mu(x) \int_0^T \int_{|s-t|\le 8/n^2} \frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}} ds dt.$$

We also have

$$\int d\mu(x) \int_0^T \int_0^T \frac{(I_3 \chi_A(t,r))^p}{|t-r|^{1+p\alpha/2}} dt dr$$
$$\le \int d\mu(x) \int_0^T \int_{|t-r|\le 4/n^2} \frac{|f(x,t)-f(x,r)|^p}{|t-r|^{1+p\alpha/2}} ds dt,$$

whence, together with (7.4) and (7.5),

$$\int d\mu(x) \int_0^T \int_A \frac{|H_n f(x,t) - f(x,t) - H_n f(x,t) + f(x,r)|^p}{|t-r|^{1+p\alpha/2}} dt dr \to 0 \tag{7.6}$$

as $n \to \infty$.

Let $(t,r) \in B$. We write

$$|H_n f(x,t) - f(x,t) - H_n f(x,r) + f(x,t)|$$
$$\le |\int ds \int h_n(x,t;y,s)(f(y,s)-f(x,t)) d\mu(y)|$$
$$+ |\int ds \int h_n(x,r;y,s)(f(y,s)-f(x,r)) d\mu(y)| \equiv I_4 + I_5$$

and

$$I_4 \le \int ds \int h_n(x,t;y,s) |f(y,s)-f(x,s)| d\mu(y)$$
$$+ \int ds \int h_n(x,t;y,s) |f(x,s)-f(x,t)| d\mu(y) \equiv I_{41} + I_{42}.$$

Then

$$I_{41} \le c_{16} n^{2-\alpha} \int_{|t-s|\le 2/n^2} ds \left(\int_{|x-y|\le 2/n} \frac{|f(y,s)-f(x,s)|^p}{|x-y|^{\beta+p\alpha}} d\mu(y) \right)^{1/p}$$
$$\le c_{17} n^{-\alpha} \mathcal{M} u_x(t).$$

Hence

$$\int d\mu(x)\int_0^T\int_0^T \frac{I_{41}^p\chi_B(t,r)}{|t-r|^{1+p\alpha/2}}dtdr \tag{7.7}$$
$$\leq c_{18}n^{-\alpha p}\int d\mu(x)\int \mathcal{M}u_x(t)^p dt\int_{|t-r|\geq 4/n^2}|t-r|^{-1-p\alpha/2}dr$$
$$\leq c_{19}\int d\mu(x)\int_0^T dt\int_{|x-y|\leq 2/n}\frac{|f(y,t)-f(x,t)|^p}{|x-y|^{\beta+p\alpha}}d\mu(y).$$

On the other hand we have

$$I_{42}\leq c_{20}n^{\beta+2}n^{-2/p-\alpha}\int_{|x-y|\leq 2/n}d\mu(y)\int_{|t-s|\leq 2/n^2}\frac{|f(x,s)-f(x,t)|}{|s-t|^{1/p+\alpha/2}}ds$$
$$\leq c_{21}n^{-\alpha}\left(\int_{|t-s|\leq 2/n^2}\frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}}ds\right)^{1/p},$$

whence

(7.8)
$$\int d\mu(x)\int_0^T\int_0^T \frac{I_{42}^p\chi_B(t,r)}{|t-r|^{1+p\alpha/2}}dtdr$$
$$\leq c_{22}n^{-p\alpha}\int d\mu(x)\int_0^T dt\int_{|t-s|\leq 2/n^2}\frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}}ds\int_{|r-t|>4/n^2}|r-t|^{-1-p\alpha/2}dr$$
$$\leq c_{23}\int d\mu(x)\int_0^T dt\int_{|t-s|\leq 2/n^2}\frac{|f(x,s)-f(x,t)|^p}{|s-t|^{1+p\alpha/2}}ds.$$

Similarly we can estimate I_5. From (7.7) and (7.8) we deduce

$$\int d\mu(x)\iint_B\frac{|H_nf(x,t)-f(x,t)-H_nf(x,r)+f(x,r)|^p}{|t-r|^{1+p\alpha/2}}dtdr\to 0$$

as $n\to\infty$. Thus we see that (7.2) holds.

We next show that

$$\int_0^T dt\iint\frac{|H_nf(x,t)-f(x,t)-H_nf(y,t)+f(y,t)|^p}{|x-y|^{\beta+p\alpha}}dxdy\to 0 \tag{7.9}$$

as $n\to\infty$. To do so, set

$$C=\{(x,z)\in\partial D\times\partial D;|x-z|\leq 4/n\}\quad E=\{(x,z)\in\partial D\times\partial D;|x-z|>4/n\}.$$

Let $(x,z)\in C$. We write

$$|H_nf(x,t)-f(x,t)-H_nf(z,t)+f(z,t)|$$
$$\leq|\int_0^T ds\int(h_n(x,t;y,s)-h_n(z,t;y,s))(f(y,s)-f(x,s))d\mu(y)|$$
$$+|\int_0^T ds\int(h_n(x,t;y,s)-h_n(z,t;y,s))(f(x,s)-f(x,t))d\mu(y)|$$
$$+|f(x,t)-f(z,t)|\equiv I_6+I_7+I_8.$$

Since

$$|h_n(x,t:y,s)-h_n(z,t;y,s)|\le c_{24}n^{\beta+3}|x-z|\chi_{B(t,2/n^2)}(s)\chi_{B(x,6/n)}(y), \tag{7.10}$$

we have

$$\begin{aligned} I_6 &\le c_{25}n^{3-\alpha}|x-z|\int_{|t-s|\le 2/n} ds\left(\int_{|x-y|\le 6/n}\frac{|f(y,s)-f(x,s)|^p}{|x-y|^{\beta+p\alpha}}d\mu(y)\right)^{1/p}\\ &\le c_{26}n^{1-\alpha}|x-z|\mathcal{M}v_x(t), \end{aligned}$$

where

$$v_x(s)=\begin{cases}\left(\int_{|x-y|\le 6/n}\frac{|f(y,s)-f(x,s)|^p}{|y-x|^{\beta+p\alpha}}d\mu(y)\right)^{1/p} & \text{if } s\in[0,T]\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\begin{aligned} &\int_0^T dt\iint_C\frac{I_6^p}{|x-z|^{\beta+p\alpha}}d\mu(x)d\mu(z)\\ &\le c_{27}n^{p(1-\alpha)}\int d\mu(x)\int\mathcal{M}v_x(t)^p dt\int_{|x-z|\le 4/n}|x-z|^{-\beta-p\alpha+p}d\mu(z)\\ &\le c_{28}\int d\mu(x)\int u_x(t)^p dt\\ &\le c_{28}\int_0^T ds\iint_{|x-y|\le 6/n}\frac{|f(y,s)-f(x,s)|^p}{|x-y|^{\beta+p\alpha}}d\mu(y)d\mu(x). \end{aligned}$$

Using (7.10), we also have

$$I_7\le c_{29}n^{1-\alpha}|x-z|\left(\int_{|t-s|\le 2/n^2}\frac{|f(x,s)-f(x,t)|^p}{|t-s|^{1+p\alpha/2}}ds\right)^{1/p},$$

whence

$$\begin{aligned} &\int_0^T dt\iint_C\frac{I_7^p}{|x-z|^{\beta+p\alpha}}d\mu(x)d\mu(z)\\ &\le c_{29}n^{(1-\alpha)p}\int d\mu(x)\int_0^T dt\int_{|t-s|\le 2/n^2}\frac{|f(x,s)-f(x,t)|^p}{|t-s|^{1+p\alpha/2}}ds\int_{|x-z|\le 4/n}|x-z|^{-\beta-p\alpha+p}d\mu(x)\\ &\le c_{30}\int d\mu(x)\int_0^T\int_{|t-s|\le 2/n^2}\frac{|f(x,s)-f(x,t)|^p}{|t-s|^{1+p\alpha/2}}dsdt. \end{aligned}$$

Therefore we obtain

$$\int_0^T dt\iint_C\frac{|H_nf(x,t)-f(x,t)-H_nf(z,t)+f(z,t)|^p}{|x-z|^{\beta+p\alpha}}d\mu(x)d\mu(y)\to 0$$

as $n\to\infty$.

We next assume that $(x,z)\in E$. We write

$$\begin{aligned} &|H_nf(x,t)-f(x,t)-H_nf(z,t)+f(z,t)|\\ &\le|\int_0^T ds\int h_n(x,t;y,s)(f(y,s)-f(x,t))d\mu(y)|\\ &+|\int_0^T ds\int h_n(z,t;y,s)(f(y,s)-f(z,t))d\mu(y)|\\ &\equiv I_9+I_{10} \end{aligned}$$

and

$$\begin{aligned} I_9 &\leq \left| \int_0^T ds \int h_n(x,t;y,s)(f(y,s)-f(x,s))d\mu(y) \right| \\ &+ \left| \int_0^T ds \int h_n(x,t;y,s)(f(x,s)-f(x,t))d\mu(y) \right| \\ &\equiv I_{91} + I_{92}. \end{aligned}$$

Then

$$\begin{aligned} I_{91} &\leq c_{31} n^{2-\alpha} \int_{|t-s|\leq 2/n^2} ds \left(\int_{|y-x|\leq 2/n} \frac{|f(y,s)-f(x,s)|^p}{|y-x|^{\beta+p\alpha}} d\mu(y) \right)^{1/p} \\ &\leq c_{32} n^{-\alpha} \mathcal{M} u_x(t), \end{aligned}$$

whence

$$\begin{aligned} &\int_0^T dt \iint_E \frac{I_{91}^p}{|x-z|^{\beta+p\alpha}} d\mu(x) d\mu(z) \\ &\leq c_{32} n^{-p\alpha} \int d\mu(x) \int \mathcal{M} u_x(t)^p dt \int_{|x-z|>4/n} |x-z|^{-\beta-p\alpha} d\mu(z) \\ &\leq c_{33} \int d\mu(x) \int u_x(t)^p dt \\ &= c_{33} \int_0^T dt \int d\mu(x) \int_{|y-x|\leq 2/n} \frac{|f(y,s)-f(x,s)|^p}{|y-x|^{\beta+p\alpha}} d\mu(y). \end{aligned}$$

Similarly we can estimate I_{10}. Thus we obtain (7.9). From (7.1), (7.2) and (7.9) we deduce

$$\|H_n f - f\|_{p,\alpha,\alpha/2} \to 0 \text{ as } n \to \infty.$$

□

Let us now prove our theorem.

Proof of Theorem. Let $f \in \Lambda^p_{\alpha,\alpha/2}(\mu_T)$. On account of Lemma 7.1 we can choose a sequence $\{f_n\} \subset \Lambda_{1,1}(S_D)$ such that $\|f_n - f\|_{p,\alpha,\alpha/2} \to 0$. Lemma 5.6 yields

$$\lim_{X\to Z, X\in D\times(0,T)} \Phi f_n(X) = K f_n(Z) + \frac{1}{2} f_n(Z). \tag{7.11}$$

Since $\|f_n - f\|_{p,\alpha,\alpha/2} \to 0$ and $\|Kf_n - Kf\|_p \to 0$ by Lemma 6.5, we can choose a subsequence $\{g_k\}$ of $\{f_n\}$ such that

$$\lim_{k\to\infty} g_k(Z) = f(Z) \quad \text{and} \quad \lim_{k\to\infty} K g_k(Z) = K f(Z) \tag{7.12}$$

$\mu_T - a.e. Z \in S_D$. Set

$$F_1 = \{Z \in S_D : |f(Z)| = +\infty \text{ or } |Kf(Z)| = +\infty\}$$

and denote by F_2 the set of all points $Z \in S_D$ at which both of equalities in (7.12) do not hold. Put $F = F_1 \cup F_2$ and

$$E_b = \{Z \in S_D : \limsup_{X\to Z, X\in\Gamma_r(Z)} |\Phi f(X) - (Kf(Z) + \frac{f(Z)}{2})| > b\}$$

for $b > 0$. Note that, for $Z \in S_D \setminus F$,

$$|\Phi f(X) - (Kf(Z) + \frac{f(Z)}{2})| \leq |\Phi(f - g_k)(X)| + |\Phi g_k(X) - Kg_k(Z) - \frac{g_k(Z)}{2}| + |K(g_k - f)(Z) + \frac{g_k(Z) - f(Z)}{2}|.$$

If $Z \in E_b \setminus F$, then, by (7.11),

$$b \leq \limsup_{X \to Z, X \in \Gamma_r(Z)} |\Phi(f - g_k)(X)| + |K(f - g_k)(Z)| + \frac{1}{2}|f(Z) - g_k(Z)|,$$

whence, together with Lemmas 6.5 and 6.6,

$$\begin{aligned} &\mu_T(E_b \setminus F) \\ &\leq \frac{c_1}{b^p}\left(\int_{E_b} (\Phi(f - g_k))^{*p} d\mu_T + \int_{E_b} |K(f - g_k)|^p d\mu_T + \frac{1}{2^p}\int_{E_b} |f - g_k|^p d\mu_T\right) \\ &\leq \frac{c_2}{b^p}\|f - g_k\|^p_{p,\alpha,\alpha/2}. \end{aligned}$$

As $k \to \infty$, we see that $\mu_T(E_b \setminus F) = 0$. Since $\mu_T(F) = 0$, we conclude that $\mu_T(E_b) = 0$ for every $b > 0$. Thus we see that the first equality of the conclusion holds. Similarly we can show the second equality of the conclusion. □

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