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ON $\alpha\beta$-EINSTEIN ALMOST GENERALIZED HOPF MANIFOLDS

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Abstract

We consider a locally conformal Kähler manifold satisfying a property for Ricci tensor. From a view point of the transversal geometry, we give a sufficient condition to be integrable for a compact almost generalized Hopf manifold with such a property.

Introduction.

A locally conformal almost Kähler manifold (l.c.a.K.-manifold) has been defined by Vaisman [9] as an almost Hermitian manifold (M, J, g) whose metric is, in local, conformal to an almost Kähler metric: namely, for each point $p \in M$, there exist a neighbourhood U and a function σ on U such that $e^{\sigma}g_{|U}$ is an almost Kähler metric of (U, J). If, in particular, its almost complex structure J is integrable, M is called a locally conformal Kähler manifold (l.c.K.-manifold). Such a Hermitian manifold is characterized by the existence of a closed global 1-form called the Lee form which satisfies the relation (1) in §1. The Hopf manifold is a typical example of an l.c.K.-manifold, and in this case the Lee form is parallel.

Locally conformal (almost) Kähler manifolds with parallel Lee form are henceforce called (almost) generalized Hopf manifolds. As is known, if an almost generalized Hopf manifold satisfies Einstein property, it is necessarily Kählerian.

The purpose of this paper is to study an almost generalized Hopf manifold with a certain property for the Ricci tensor, called $\alpha\beta$-Einstein property. A foliation canonically defined in an almost generalized Hopf manifold will be discussed and from such point of view, we give a sufficient condition for the almost complex structure of an almost generalized Hopf manifold with $\alpha\beta$-Einstein property to be integrable (Theorem 4.5).

1 An l.c.a.K.-manifold

Let (M^{2n}, J, g) be a $2n$-dimensional almost Hermitian manifold and Ω the fundamental 2-form defined by $\Omega(X, Y) = g(JX, Y)$, that is, $\Omega = \frac{1}{2}J_{ij}dx^i \wedge dx^j$ $(J_{ij} = J_i{}^r g_{rj})$ in terms of a local coordinate system $\{x^1, x^2, \cdots, x^{2n}\}$. A *locally conformal almost Kähler manifold* (abbreviated as an *l.c.a.K.-manifold*) ([9]) is an almost Hermitian manifold which admits a closed 1-form α satisfying

$$d\Omega = 2\alpha \wedge \Omega \tag{1}$$

i.e.,

$$\nabla_i J_{jk} + \nabla_j J_{ki} + \nabla_k J_{ij} = 2(\alpha_i J_{jk} + \alpha_j J_{ki} + \alpha_k J_{ij}).$$

This 1-form α is called the *Lee form* and represented in the form

$$\alpha = \frac{-1}{2(n-1)}\delta\Omega \circ J \ \left(= \frac{1}{2(n-1)}\nabla^r J_{rk} J_i{}^k \, dx^i\right). \tag{2}$$

REMARK. If $n \geq 3$, a 1-form α satisfying (1) is closed necessarily.

In an l.c.a.K.-manifold (M^{2n}, J, α, g), let β be a 1-form defined by $\beta = i(\alpha^{\#})\Omega = \frac{-1}{2(n-1)}\delta\Omega$, where $i(X)$ is the interior product operator with respect to X. We set $A = \alpha^{\#}, B = \beta^{\#}$, hereafter. Since $\alpha = -i(B)\Omega$ is closed and $i(B)d\Omega = 0$, the Lie derivative $\mathcal{L}_B\,\Omega = (i(B)d + di(B))\Omega$ vanishes, namely,

$$(\nabla_B\Omega)(X,Y) + \Omega(\nabla_X B, Y) + \Omega(X, \nabla_Y B) = 0. \tag{3}$$

If J is integrable, (M^{2n}, J, α, g) is called a *locally conformal Kähler manifold* (an *l.c.K.-manifold*).
As a characterization of an l.c.K.-manifold, we have

PROPOSITION 1.1 *Let (M^{2n}, J, g) be an almost Hermitian manifold and H the tensor field defined by*

$$H(X,Y) = (\nabla_X J)Y - \alpha(JY)X - g(X,Y)J\alpha^{\#} + \alpha(Y)JX - \Omega(X,Y)\alpha^{\#}$$

where $\alpha = \frac{-1}{2(n-1)}\delta\Omega \circ J$.
Then, (M^{2n}, J, g) is an l.c.K.-manifold iff $H = 0$ and α is closed.

PROOF. In [2], we know that H vanishes in an l.c.k.-manifold.
Conversely, in an almost Hermitian manifold with the property $H = 0$, it may be easily checked that the Nijenhuis tensor vanishes and (1) is valid. The condition $d\alpha = 0$ is necessary only when $n = 2$. □

2 An almost generalized Hopf manifold

An l.c.a.K.(resp. an l.c.K.)-manifold is called an *almost generalized Hopf* (resp. a *generalized Hopf)-manifold* if its Lee form α is parallel and β is Killing (Cf. [10]). Notice that in an *l.c.K.*-manifold, β is necessarily a Killing form if α is parallel.
We are going to prepare the following formulas:

LEMMA 2.1 *In an almost generalized Hopf manifold, the following identities are valid:*

$$\nabla_A J = 0, \qquad \nabla_B J = 0, \tag{4}$$
$$\nabla\beta = -c\Omega + \alpha \wedge \beta \qquad (c = |\alpha|^2), \tag{5}$$
$$\frac{1}{2}\delta d\beta(X) = \rho(B,X) = 2(n-1)c\beta(X), \tag{6}$$
$$R(X,B)B = c(cX - \alpha(X)A - \beta(X)B) \tag{7}$$

for $X \in TM$, where R and ρ denote the curvature tensor and the Ricci tensor, respectively.

PROOF. Since $\nabla A = 0$ and $B = JA$ is a Killing vector, the following relations are valid:

$$\nabla_A B = \nabla_B B = 0, \tag{8}$$

$$R(A,X)Y = 0, \qquad (\nabla^2 B)(X,Y) = -R(B,X)Y \tag{9}$$

for $X, Y \in TM$, where ∇^2 means $(\nabla^2 B)(X,Y) = \nabla_X(\nabla_Y B) - \nabla_{\nabla_X Y} B$.

Differentiating $\beta = i(A)\Omega$, we have

$$\nabla_A \Omega - 2\nabla\beta = 2(c\,\Omega - \alpha \wedge \beta) \tag{10}$$

on account of $d\beta = 2\nabla\beta$ and $i(A)(\alpha \wedge \Omega) + \alpha \wedge i(A)\Omega = c\Omega$. Furthermore, differentiating (10) covariantly by A and making use of (8),(9), we have

$$(\nabla^2\Omega)(A,A) = 2c(\nabla_A\Omega).$$

Thereby, because $< \nabla_A\Omega, \Omega >= 0$ and $< (\nabla^2\Omega)(A,A),\ \Omega >= -|\nabla_A\Omega|^2$, the former of (4) is now clear, and then (5) follows from (10). The latter of (4) comes from (3) and (5). Differentiating (5) and making use of (9), we obtain

$$\begin{aligned} R(B,X)Y &= c((\nabla_X J)Y + \alpha(Y)JX - \Omega(X,Y)A) \\ &\quad -\alpha(X)(\alpha(Y)B - \beta(Y)A), \end{aligned} \tag{11}$$

which implies (6) and (7).

3 $\alpha\beta$-Einstein l.c.a.K.-manifolds

DEFINITION. An l.c.a.K.-manifold is said to be $\alpha\beta$ *-Einstein* if the Ricci tensor is represented as

$$\rho = \lambda g + \mu\alpha \otimes \alpha + \nu\beta \otimes \beta \tag{12}$$

where λ, μ, ν are functions.

Notice that $\alpha\beta$-Einstein property is invariant by homothetic change of the metric.

In an almost generalized Hopf manifold which is not almost Kählerian (i.e., $\alpha \neq 0$), λ, μ, ν in (12) satisfy the relations

$$\lambda + c\nu = 2(n-1)c, \qquad \lambda + c\mu = 0 \tag{13}$$

$$\tau = 2(n-1)(\lambda + c)$$

because of (6) and (8), where τ is the scalar curvature.

Differentiating (12) and making use of $d\tau = -2\delta\rho$, we know that

LEMMA 3.1 *If a $2n$-dimensional almost generalized Hopf manifold ($\alpha \neq 0$) is $\alpha\beta$-Einstein and $n \geq 3$, then λ, μ, ν in (12) and the scalar curvature are necessarily constant.*

EXAMPLE. Let N^{2n-1} be a contact Riemannian manifold with a contact metric structure (φ, η, h), i.e., a tensor field φ, a unit 1-form (contact form) η and a Riemannian metric h satisfying the relations:

$$|\eta| = 1,$$

$$\varphi^2 = -I + \xi \otimes \eta \ \text{ where } \xi = \eta^{\#},$$

$$2h(\varphi X, Y) = (d\eta)(X,Y) \ \text{ for } X, Y \in TN.$$

A K-contact manifold $N^{2n-1}(\varphi, \eta, h)$ is a contact Riemannian manifold whose contact structure η is a Killing 1-form. If moreover, η is a special Killing form, namely, the Killing form η satisfies

$$(\nabla^2\eta)(X,Y;Z) = \eta(Y)h(X,Z) - h(X,Y)\eta(Z), \tag{14}$$

$N^{2n-1}(\varphi,\eta,h)$ is called a Sasakian manifold.

Now, let $N^{2n-1}(\varphi,\eta,h)$ be a contact Riemannian manifold and consider the Riemannian product manifold $M^{2n}=N^{2n-1}\times S^1$. Let $\mathbf{u}$ be the unit tangent vector of S^1 and ω its dual form. Denote by J a linear map on the tangent space $T_{(P,t)}M$ at $(P,t)\in N\times S^1$ defined by

$$JX=\varphi X+\eta(X)\mathbf{u}\quad \text{for } X\in T_PN \tag{15}$$

$$J\mathbf{u}=-\xi$$

identifying T_PN naturally with a subspace of $T_{(P,t)}M$. Then J and the product metric give an almost Hermitian structure on M^{2n}. Let Ω be the fundamental 2-form of J. Then

$$(d\Omega)(X,Y,Z)=0$$

$$(d\Omega)(X,Y,\mathbf{u})=(d\eta)(X,Y)$$

hold for $X,Y,Z\in TN$, that is,

$$d\Omega=2\omega\wedge\Omega.$$

$M^{2n}(J,g)$ is hence an l.c.a.K-manifold with parallel Lee form ω.

In the case when N^{2n-1} is a K-contact manifold, $-\eta=i(\mathbf{u})\Omega$ satisfies evidently the Killing property as a form on M^{2n}, and hence M^{2n} is an almost generalized Hopf manifold.

As is known, for a K-contact structure, the Sasakian property (14) means that the almost complex structure J induced above is integrable. So, $M^{2n}(J,\omega,g)$ is a generalized Hopf manifold if N^{2n-1} is a Sasakian manifold. For example, the number space R^{2n-1} admits a Sasakian structure (η,h) defined by

$$2\eta=dz-\sum_{a=1}^{n-1}y_a dx_a,$$

$$h=\frac{1}{4}\sum_{a=1}^{n-1}(dx_a^2+dy_a^2)+\eta^2$$

$\{x_a,y_a,z\}_{a=1,2,..,n-1}$ being cartesian coordinate system, and then the product manifold $M^{2n}=R^{2n-1}\times S^1$ is a generalized Hopf manifold. Taking a quotient of R^{2n-1}, we have also an example of compact one (Cf.[1],[6]).

A K-contact manifold $M^{2n-1}(\varphi,\eta,h)$ whose Ricci tensor ρ_N is expressed as

$$\rho_N=ag+b\eta\otimes\eta\qquad (a+b=2(n-1))$$

is said to be an η-Einstein manifold ([6]).

The following is clear by calculation:

PROPOSITION 3.2 *If $N^{2n-1}(\varphi,\eta,g)$ is an η-Einstein K-contact manifold, then $M^{2n}=N^{2n-1}(\varphi,\eta,h)\times S^1$ with the structure (J,g) defined by* (15) *is an almost generalized Hopf manifold with $\alpha\beta$-Einstein property.*

The example of the product manifold $M^{2n}=R^{2n-1}\times S^1$ mentioned above is an $\alpha\beta$-Einstein generalized Hopf manifold whose Ricci tensor of the product metric g is written as

$$\rho=-2g+2n\eta\otimes\eta+2\omega\otimes\omega,$$

4 An almost generalized Hopf manifold from a view point of the transversal geometry

The next purpose is to discuss an almost generalized Hopf manifold from a point of view of the transversal geometry. We begin by considering a foliation $\mathcal{F}$ in a Riemannian manifold (M, g) (Cf.[8]).

Let L be the tangent bundle of $\mathcal{F}$, $Q = TM/L$ the normal bundle and π the projection: $TM \to Q$. TM splits orthogonally as $TM = L \oplus L^{\perp}$. Hence identifying $L^{\perp}$ with Q by $\sigma : Q \to L^{\perp}$ and putting the induced metric by $g_Q = \sigma^* g_{|L^{\perp}}$, we can identify (Q, g_Q) with $(L^{\perp}, g_{|L^{\perp}})$. Denote by $\theta(X)(X \in \Gamma L)$ the transversal Lie derivative acting on ΓQ defined as

$$\theta(X)s = \pi[X, Y_s] \qquad s \in \Gamma Q$$

where $Y_s \in \Gamma TM, \pi(Y_s) = s$. A section $s \in \Gamma Q$ which satisfies $\theta(X)s = 0$ for any $X \in \Gamma L$ is said to be holonomy invariant.

If g is bundle-like, i.e., g_Q is holonomy invariant, $\mathcal{F}$ is called a Riemannian foliation. In this case, a connection ∇' in Q defined by

$$\nabla'_X s = \begin{cases} \pi[X, Z_s] & \text{for } X \in \Gamma L \\ \pi \nabla_X Z_s & \text{for } X \in \Gamma L^{\perp} \end{cases} \tag{16}$$

where $Z_s = \sigma(s) \in \Gamma L^{\perp}$, is metrical and torsionfree.

A Riemannian foliation $\mathcal{F}$ is called an almost Kähler foliation ([4]) if it satisfies the following conditions:

(i) there exists a holonomy invariant almost complex structure J_Q on the normal bundle Q, namely, a homomorphism J_Q such that

$$J_Q^2 = -\mathrm{id}, \quad \theta(X)J_Q = 0 \text{ for } X \in \Gamma L,$$

(ii) $g_Q(J_Q s, J_Q t) = g_Q(s, t)$ for $s, t \in \Gamma Q$,

(iii) the 2-form Φ of M defined by

$$\Phi(X, Y) = g_Q(J_Q \pi X, \pi Y) \ (X, Y \in \Gamma TM)$$

is closed.

A Riemannian foliation $\mathcal{F}$ which satisfies (i), (ii) and

(iii') $\nabla' J_Q = 0$

is called a *Kähler foliation* ([5]).

Now let us consider a foliation on an almost Hermitian manifold (M, J, g). Let $\mathcal{F}$ be a Riemannian foliation with respect to g and assume $JL^{\perp} = L^{\perp}$. If we define a homomorphism J_Q of Q by $J_Q Z = JZ$ for $Z \in \Gamma L^{\perp}$ where we identify $(Q, g_Q) = (L^{\perp}, g_{|L^{\perp}})$, it satisfies $J_Q^2 = -\mathrm{id}$ and g_Q is hermitian with respect to J_Q. Since the transversal Lie derivative of J_Q is

$$\begin{aligned} (\theta(X)J_Q)Z &= \theta(X)(J_Q Z) - J_Q \theta(X) Z \\ &= \pi[X, JZ] - J_Q \pi[X, Z] = \pi([X, JZ] - J[X, Z]), \end{aligned}$$

for $X \in \Gamma L, Z \in \Gamma L^{\perp}$, it follows that

PROPOSITION 4.1 *The homomorphism J_Q is holonomy invariant if and only if*

$$\pi([X, JZ] - J[X, Z]) = 0 \tag{17}$$

for $X \in \Gamma L, Z \in \Gamma L^{\perp}$.

Returning to the case of an almost generalized Hopf manifold (M, J, g), let L be the distribution generated by $\{A, B\}$. Since L is involutive, it defines a foliation $\mathcal{F}$. In this case, $JL^{\perp} = L^{\perp}$ holds evidently.

PROPOSITION 4.2 *In an almost generalized Hopf manifold, the foliation defined by $\{A, B\}$ is an almost Kähler foliation.*

PROOF. (i) Since A, B are Killing vector fields, $(\theta(fX)g_Q)(Z, W) = 0$ holds for $X = A, B$, where $Z, W \in \Gamma L^{\perp}$ and f is a function.
(ii) By virtue of (4), the equation (17) holds for $Z \in \Gamma L^{\perp}$ and $X = A, B$.
(iii) Owing to (5), we have

$$\Phi(X,Y) = g_Q(J_Q \pi X, \pi Y) = \frac{-1}{2c}(d\beta)(X,Y)$$

for $X, Y \in \Gamma TM$. This completes the proof. □

Now, we know that the tensor H of an almost generalized Hopf manifold may be written as

$$H(X,Y) = \begin{cases} \pi((\nabla_X J)Y) = (\nabla'_X J_Q)Y & \text{if } X, Y \in \Gamma L^{\perp} \\ 0 & \text{if } X \text{ or } Y \in \Gamma L \end{cases}$$

because of

$$\beta(\nabla_X Y) = -(\nabla_X \beta)(Y) = cg(JX, Y) \quad \text{for } X, Y \in \Gamma L^{\perp}. \tag{18}$$

Hence, by virtue of Proposition 1.1, we obtain the following.

THEOREM 4.3 *An almost generalized Hopf manifold is a generalized Hopf manifold if and only if the foliation defined by $\{A, B\}$ is transversally Kähler.*

Let R' be the curvature tensor of ∇': for $X, Y \in \Gamma TM, s \in \Gamma Q$,

$$R'(X,Y)s = [\nabla'_X, \nabla'_Y]s - \nabla'_{[X,Y]}s.$$

Let ρ' and τ' the Ricci tensor, the scalar curvature of ∇' respectively defined as usual.

LEMMA 4.4 *In an almost generalized Hopf manifold, ρ' is expressed as*

$$\rho'(X,Y) = \rho(X,Y) + 2cg(X,Y) \quad \textit{for } X, Y \in \Gamma L^{\perp}.$$

PROOF. Let $\{E_1, ..., E_{2(n-1)}\}$ be a local orthonormal basis of $\Gamma L^{\perp}$. For $X, Y, Z \in \Gamma L^{\perp}$, it holds that

$$\begin{aligned} \nabla'_X Y &= \nabla_X Y - \frac{1}{c}(\alpha(\nabla_X Y)A + \beta(\nabla_X Y)B) \\ &= \nabla_X Y - g(JX, Y)B. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla'_Z \nabla'_X Y &= \pi \nabla_Z (\nabla_X Y - g(JX, Y)B), \\ R'(E_i, X)Y &= \pi R(E_i, X)Y + cg(JX, Y)JE_i - cg(JE_i, Y)JX + 2cg(JX, E_i)JY \end{aligned}$$

where we used $\nabla_Z B = -cJZ$ and $\pi[E_i, X] = [E_i, X] - \frac{1}{c}\beta([E_i, X])B = [E_i, X] + 2g(JX, E_i)B$. Then making use of (7),(9), we obtain

$$\begin{aligned} \rho'(X,Y) &= \sum_{i=1}^{2n-2} g_Q(R'(E_i, X)Y, E_i) \\ &= \rho(X,Y) - \frac{1}{c} g(R(B, X)Y, B) + 3cg(X,Y) \\ &= \rho(X,Y) + 2cg(X,Y) \qquad \square \end{aligned}$$

If an almost generalized Hopf manifold is $\alpha\beta$-Einstein, it follows from Lemma 4.4 that

$$\rho' = \frac{\tau'}{2(n-1)} g_Q \quad \text{and} \quad \tau' = \tau + 2(n-1)c. \tag{19}$$

In general, a foliation $\mathcal{F}$ in a Riemannian manifold (M, g) is said to be *transversally Einstein* if $\rho' = kg_Q$ holds for some constant $k(= \frac{|\tau'|}{q})$, where q is the codimension of $\mathcal{F}$. $\mathcal{F}$ is also said to be *harmonic* if all leaves are minimal submanifolds of (M, g). In an almost generalized Hopf manifold, the foliation defined above is evidently harmonic, because the leaves are totally geodesic.

On the other hand, by Morimoto [4] the following fact has been showed.

THEOREM. *In a compact orientable Riemannian manifold, if a harmonic almost Kähler foliation $\mathcal{F}$ is transversally Einstein and $\tau' \geq 0$, then $\mathcal{F}$ is a Kähler foliation.*

Thus, as an immediate consequence of Theorem 4.3 and (19), we obtain the following

THEOREM 4.5 *If a compact almost generalized Hopf manifold (M^{2n}, J, g) $(n \geq 3)$ is $\alpha\beta$-Einstein and the scalar curvature τ satisfies $\tau \geq -2(n-1)c$, then (M^{2n}, J, g) is a generalized Hopf manifold.*

REMARK. The above fact in the case of $c = 0$ was obtained by Sekigawa [7] as a sufficient condition for almost Kähler manifold to be Kählerian.

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