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On the Curvature Operator of the Second Kind

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Introduction

In a Riemannian manifold, the Riemannian curvature tensor defines two kind curvature operators: the operators $\hat{R}$ of the first kind acting on 2-forms and $\mathring{R}$ of the second kind acting on symmetric 2-tensors. A Riemannian manifold of positive curvature operator $\hat{R}$ has been studied from many derections. In [6], we know that a compact connected Riemannian manifold of positive curvature operator of the 1st kind is of constant sectional curvature if $\delta R=0$. The purpose of this paper is to study a similar problem in a Riemannian manifold of positive restoicted $\mathring{R}$, where we say *restricted $\mathring{R}$ to be positive* if $\mathring{R}$ is positive on a space of traceless symmetric 2-tensors ([5]).

If $\hat{R}$ or restricted $\mathring{R}$ is positive, then the sectional curvature is also positive (§3). But, we cannot yet know any relations between properties of positive $\hat{R}$ and positive restricted $\mathring{R}$ in general. In the last section, we shall investigate relations between them under some conditions.

§1. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold, and R be the Riemannian curvature tensor. With respect to a local coordinate system, we express by $R_{ijk}{}^h$ and $R_{jk}=R_{rjk}{}^r$ the component of the Riemannian curvature tensor and the Ricci tensor, respectively.

By virtue of Bianchi's identity, the following equalities hold:

LEMMA 1.1. *Put*

$$\sum R_{rjkl}R_{sikl}R_{rjsi}=\alpha, \qquad \sum R_{rjkl}R_{riks}R_{ijsl}=\beta$$

where $\sum$ means the summation for all indices. Then we can represent

$$\text{(i)} \qquad 2\sum R_{rjkl}R_{rkis}R_{ijsl}=\sum R_{rjkl}R_{sikl}R_{rsji}=\frac{1}{2}\alpha$$

(ii) $$\sum R_{rjkl}R_{rski}R_{slij}=\beta-\frac{1}{4}\alpha\,.$$

The curvaure operator of the 2nd kind $\mathring{R}$ is defined ([5]) as an operator acting on a symmetric 2-tensor:

$$\zeta=(\zeta_{ij}) \longmapsto R(\zeta)=(R_{risj}\zeta^{rs})\,.$$

Denoting by $x\odot y$ a symmetric tensor: $x\odot y=(1/2)(x\otimes y+y\otimes x)$ for $x,\ y \in T_p(M)$, we can represent $\mathring{R}$ as

$$\langle\mathring{R}(x\odot y), u\odot v\rangle=\frac{1}{2}\{R(x, u, y, v)+R(x, v, y, u)\}\,,$$

where $R(x, u, y, v)=g(R(x, u)y, v)=R_{ijkl}x^iu^jy^kv^l$.

Let $\mathcal{S}^2$ be the space of symmetric 2-tensors with the canonical inner product $\langle\ \rangle$ and $\mathcal{S}_0^2$ the space of traceless symmetric 2-tensors. For an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of $T_p(M)$, we can take bases $\{e_i\odot e_j\}_{i\leq j\leq n}$ of $\mathcal{S}^2$ and $\{e_i\odot e_j, e_a\overline{\odot}e_a\}_{\substack{i\lneq j\leq n\\ a<n}}$ of $\mathcal{S}_0^2$, respectively, where $\overline{\odot}$ is defined by

$$e_i\,\overline{\odot}\,e_i=e_i\odot e_i-e_n\odot e_n\,.$$

Hereafter, components of tensors are taken with respect to an orthonormal basis.

§ 2. Theorem

We are going to prove the following:

THEOREM 2.1. *Let (M^n, g) be a compact connected Riemannian manifold of positive restricted curvature operator of the 2nd kind. If it satisfies*

$$\delta R=0\,,$$

then (M^n, g) is a space of constant curvature.

PROOF. For each numbers i, j, k, l, we define symmetric 2-tensors $\zeta^{(ijkl)}$ as follows:

$$\zeta^{(ijkl)}=\eta^{(ijkl)}-\eta^{(klij)}$$

where we put

$$\eta^{(ijkl)}=\sum_{r=1}^{n}(R_{rjkl}e_r\odot e_i-R_{rikl}e_r\odot e_j)\,.$$

Since it is also represented as

$$\zeta^{(ijkl)}$$

$$=\sum_{r\neq i}^{n} R_{rjkl}e_r\odot e_i-\sum_{r(\neq j)}^{n} R_{rikl}e_r\odot e_j-\sum_{r(\neq k)}^{n} R_{rlij}e_r\odot e_k+\sum_{r(\neq l)}^{n} R_{rkij}e_r\odot e_l$$

$$+R_{ijkl}(e_i\overline{\odot}\, e_i+e_j\overline{\odot}\, e_j-e_k\overline{\odot}\, e_k-e_l\overline{\odot}\, e_l)\,,$$

$\zeta^{(ijkl)}$ is traceless.

We now calculate $\sum\langle R(\zeta^{(ijkl)}),\zeta^{(ijkl)}\rangle$. On account of

$$\langle\mathring{R}(e_i\odot e_j),\, e_k\odot e_l\rangle=\frac{1}{2}(R_{ikjl}+R_{iljk})\,,$$

and of Lemma 1.1, we have

$$R_{rjkl}R_{sikl}\langle\mathring{R}(e_r\odot e_i),\, e_s\odot e_j\rangle=-\frac{3}{4}\alpha\,,$$

$$R_{rjkl}R_{slij}\langle\mathring{R}(e_r\odot e_i),\, e_s\odot e_k\rangle=-\frac{1}{2}\beta+\frac{1}{4}\alpha\,.$$

Hence,

$$\sum\langle\mathring{R}(\zeta^{(ijkl)}),\,\zeta^{(ijkl)}\rangle$$

$$=2\sum\{\langle\mathring{R}(\eta^{(ijkl)}),\,\eta^{(ijkl)}\rangle-\langle\mathring{R}(\eta^{(ijkl)}),\,\eta^{(klij)}\rangle\}$$

$$=4\sum\{R_{rjkl}R_{sjkl}\langle\mathring{R}(e_r\odot e_i),\, e_s\odot e_i\rangle-R_{rjkl}R_{sikl}\langle\mathring{R}(e_r\odot e_i),\, e_s\odot e_j\rangle$$

$$-2R_{rjkl}R_{slij}\langle\mathring{R}(e_r\odot e_i),\, e_s\odot e_k\rangle\}$$

$$=2\sum\left\{R_{rjkl}R_{sjkl}R_{rs}+\frac{1}{2}\alpha+2\beta\right\},$$

namely,

$$\sum\langle\mathring{R}(\zeta^{(ijkl)}),\,\zeta^{(ijkl)}\rangle=2K$$

where we set K as

$$K=\sum\left\{R_{rjkl}R_{sjkl}R_{rs}+\frac{1}{2}R_{rjkl}R_{sikl}R_{rjsi}+2R_{rjkl}R_{sjil}R_{rski}\right\}.$$

The function K is accordingly positive for non-zero $\zeta^{(ijkl)}$ from our assumption for $\mathring{R}$. On the other hand, in a compact Riemannian manifold it holds that ([1], [2])

$$\int_M\{K-|\delta R|^2\}d\sigma=-\frac{1}{2}\int_M|\nabla R|^2 d\sigma\,.$$

Hence, $\delta R=0$ implies $\nabla R=K=0$, and then $\zeta^{(ijkl)}$ are zero for each i, j, k, l. In particular, from $\zeta^{(ijil)}=0$, we have $R_{ijil}(e_j\overline{\odot}\, e_j-e_i\overline{\odot}\, e_i)=0$. Consequently, if $n\geqq 3$, $g(R(x,y)x,z)$ vanishes for any orthogonal vectors $\{x, y, z\}$, and this fact implies, as well known, that the space is of constant curvature. In the case $n=2$, $\nabla R=0$ means that the scalar curvature is constant, namely the curvature is constant. □

§ 3. Remarks on difference between the curvature operators

The curvature operator (of the first kind) is an operator $\hat{R}$ acting on the space of 2-forms with the canonical inner product $\langle\ ,\ \rangle$ defined by:

$$\omega=(\omega_{ij}) \longmapsto \hat{R}(\omega)=(-R_{ijkh}\omega^{ij}) .$$

In this section we discuss relations among the following properties:

(i) the sectional curvature is positive,

(ii) the curvature operator of the 1st kind $\hat{R}$ is positive,

(iii) the restricted curvature operator of the 2nd kind $\mathring{R}$ is positive.

LEMMA 3.1. *In an n-dimensional Riemannian manifold* (i) *follows from* (ii) *or* (iii).

This fact is seen from the following expressions:

$$\langle\hat{R}(x\wedge y), x\wedge y\rangle=4g(R(x, y)y, x) ,$$

$$\langle\mathring{R}(x\odot y), x\odot y\rangle=2g(R(x, y)y, x) ,$$

for any $x, y\in T_pM$. □

Now, if $n=2$, the properties (i), (ii), (iii) are obviously equivalent to each other.

If $n=3$, (i) is equivalent to (ii).

In fact, (i) implies (ii) as following: With respect to an orthonormal basis consists of eigenvectors of the Ricci operator, any 2-form ζ satisfies

$$\langle\hat{R}(\zeta), \zeta\rangle=-R_{ijkl}\zeta^{ij}\zeta^{kl}=2R_{jiij}(\zeta^{ij})^2$$

because of $R_{ijki}=R_{jk}=0$ for different i, j, k.

REMARK. In a conformally flat Riemannian manifold, (i) and (ii) are equivalent.

By Weinstein ([7]), it has been shown that on a submanifold (M^n, g) in an $(n+2)$-dimensional Euclidean space E^{n+2} $(n\geqq 3)$, (i) is equivalent to (ii). In case when M^n is, in particular, a hypersurface of E^{n+1}, it holds that

$$g(R(e_i, e_j)e_j, e_i)=\lambda_i\lambda_j \qquad (i\neq j)$$

and

$$\hat{R}(e_i\wedge e_j)=2\lambda_i\lambda_j e_i\wedge e_j ,$$

with respect to an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ consisting of eigenvectors of the shape operator corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Hence (i) implies (ii), that is, (i) and (ii) are equivalent on account of Lemma 3.1.

However, (ii) does not implies (iii). In fact, we can give an example of a space of (ii) which does not satisfy (iii) as following:

Let M^n ($n \geqq 3$) be an elliptic hypersurface in E^{n+1} with the cartesian coordinate system $\{x_0, x_1, \cdots, x_n\}$ given by

$$\sum_{\alpha=0}^{n-1} x_\alpha^{\ 2} + c^2 x_n^{\ 2} = 1 \qquad (0 < |c| = \text{const} < 1).$$

We consider at $p=(1, 0, \cdots, 0)$. As the eigenvalues of the shape operator are $\{1, 1, \cdots, c^4\}$, $\langle \mathring{R}(\zeta), \zeta \rangle_p$ is negative for the tensor

$$\zeta = \sum_{a=1}^{n-1} \left(\frac{\partial}{\partial x_a} \otimes \frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_n} \otimes \frac{\partial}{\partial x_n} \right)_p .$$

On the other hand, since M^n is of positive sectional curvature, (ii) is satisfied as we remarked above. □

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