Boundary Behavior of Harmonic Functions in Lipschitz and C^1-domains

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§ 1. Intsoduction

Let U be the unit disk in $\boldsymbol{R}^d$. Fatou's theorem asserts that for every $p \geqq 1$ and $f \in L^p(\partial U)$ the Poisson integral of f converges to f non-tangentially on ∂U except for a set of surface measure zero. Similar results with respect to the harmonic measure at a fixed point have been obtained for the upper half space, C^1-domains and Lipschitz domains (cf. [6], [3], [4], [5]).

On the other hand it is well-known that in $\boldsymbol{R}_1^{d+1}$ the Poisson integral of the Bessel potential $G_\alpha * f$ of each $f \in L^p(\boldsymbol{R}^d)$ converges not only non-tangentially but also tangentially except for a set of a suitably dimensional Hausdorff measure zero (cf. [1]).

We are interested in the following problem:

Let D be a bounded domain in $\boldsymbol{R}^d$ and $d-1>\beta>0$. Find a class of functions on ∂D such that the Poisson integral Hf of each function f in the class converges to f non-tangentially on ∂D except for a set of β-dimensional Hausdorff measure zero.

We have treated this problem in [8] in case D is the upper half space, by using a countably sublinear functional.

In this paper we will consider the same problem in case D is a C^1-domain or a Lipschitz domain.

More precisely, let D be a bounded open set in $\boldsymbol{R}^d$. The open ball with center X and radius r is denoted by $B(X, r)$. For $P \in \partial D$ the set $B(P, r) \cap \partial D$ is denoted by $A(P, r)$ and called a surface ball.

Let β be a positive real number such that $0<\beta \leqq d-1$. Recall that the β-dimensional Hausdorff outer measure was introduced by L. Carleson and defined by

$$M_\beta(E)=\inf\left\{\sum_{j=1}^{\infty} r_j^\beta : E \subset \bigcup_{j=1}^{\infty} B(X_j, r_j)\right\}.$$

We regard M_β as an outer measure on ∂D and define an 'upper integral'

γ_β with respect to M_β as follows:

$$\gamma_\beta(f)=\inf\left\{\sum_{j=1}^{\infty} b_j M_\beta(E_j): b_j\in \boldsymbol{R}^+, E_j\subset\partial D, \sum_{j=1}^{\infty} b_j\chi_{E_j}\geqq f\right\}$$

for each extended real-valued nonnegative function f on ∂D.

Denote by $J(\partial D)$ the family of all extended real-valued functions on ∂D and define

$$\gamma_\beta(f):=\gamma_\beta(|f|)$$

for each $f\in J(\partial D)$. Further, set

$$\mathscr{B}(\gamma_\beta)=\{f\in J(\partial D): \gamma_\beta(f)<+\infty\}.$$

It is easy to see that $\mathscr{B}(\gamma_\beta)$ contains the family $C(\partial D)$ of all continuous real-valued functions on ∂D.

Let us denote by $\mathscr{L}(\gamma_\beta)$ the family of all Borel measurable functions f in $\mathscr{B}(\gamma_\beta)$ such that $\gamma_\beta(f-f_n)\to 0$ for some sequence $\{f_n\}\subset C(\partial D)$.

The family $\mathscr{L}(\gamma_\beta)$ and the countably sublinear functional γ_β play the corresponding roles to $L^p(\partial D)$ and the L^p-norm, respectively.

The Poisson integral Hf of a Borel measurable function f on ∂D is defined by

$$Hf(X)=\int f d\omega_X,$$

if it is well-defined, where ω_X is the harmonic measure of D evalueted at $X\in D$.

In § 4 we will prove that, if D is a bounded Lipschitz domain in $\boldsymbol{R}^d$ and $0<2\beta<d-1$, then the Poisson integral Hf of each $f\in\mathscr{L}(\gamma_\beta)$ converges to f non-tangentially on ∂D except for a set of β-dimensional Hausdorff measure zero. We will also prove that, especially, if D is a bounded C^1-domain, the same conclusion is obtained for every real number β satisfying $0<\beta<d-1$.

§ 2. A Fatou type theorem

Let D be an bounded open set in $\boldsymbol{R}^d$ and let $0<\beta<d-1$. From the definition of γ_β in § 1 it follows that

$$\gamma_\beta(f)=\inf\left\{\sum_{j=1}^{\infty} b_j r_j^\beta: b_j\geqq 0, \sum_{j=1}^{\infty} b_j\chi_{A(P_j,r_j)}\geqq |f|\right\}.$$

The functional γ_β has the following properties:

(c_1) $\gamma_\beta(f)=\gamma_\beta(|f|)$,

(c_2) $\gamma_\beta(bf)=|b|\gamma_\beta(f)$ for each $b\in\boldsymbol{R}$,

(c_3) $f,\ f_n\geqq 0\ f\leqq\sum_{n=1}^{\infty} f\Rightarrow\gamma_\beta(f)\leqq\sum_{n=1}^{\infty}\gamma_\beta(f_n)$.

A subset E satisfying $\gamma_\beta(\chi_E)=0$ is said to be γ_β-polar. It is easy to see that if $\gamma_\beta(f)<+\infty$, then the set $E:=\{Q\in\partial D: |f(Q)|=+\infty\}$ is γ_β-polar.

Furthermore we introduce a set function μ_β on ∂D. Define, for an open set G relative to ∂D,

$$\mu_\beta(G)=\sup\{M_\beta(K): K \text{ is compact, } K\subset G\}$$

and for a subset E of ∂D

$$\mu_\beta(E)=\inf\{\mu_\beta(G): G=G_1\cap\partial D, G_1 \text{ is open, } E\subset G\}.$$

We can easily show that

$$E\subset E_1\cup E_2 \Longrightarrow \mu_\beta(E)\leqq\mu_\beta(E_1)+\mu_\beta(E_2).$$

The relations of three set functions M_β, μ_β and the function $E\mapsto\gamma_\beta(\chi_E)$ are as follows:

LEMMA 2.1. *Let E be a subset of ∂D. Then*

$$c\mu_\beta(E)\leqq\gamma_\beta(\chi_E)\leqq M_\beta(E),$$

where c is a constant independent of E.

PROOF. The inequality of the right-hand side is an easy consequence of the definitions. To show the inequality of the left-hand side, suppose that $\chi_E\leqq\sum_j b_j\chi_{A(P_j,r_j)}$ and $0<\varepsilon<1$. Further, set

$$G_1=\{X\in\boldsymbol{R}^d: \sum_j b_j\chi_{B(P_j,r_j)}>1-\varepsilon\} \quad\text{and}\quad G=G_1\cap\partial D.$$

Then G_1 is open and $E\subset G$. Choose $A(P_o, r_o)$ such that $\partial D\subset A(P_o, r_o)$ and let K be any compact subset of G. Then by [2, Theorem 1, § II] there exists a positive measure ν such that

$$\operatorname{supp}\nu\subset K, \qquad M_\beta(K)\leqq c_1\nu(K)$$

and

$$\nu(B(X,r))\leqq r^\beta$$

for every $X\in\boldsymbol{R}^d$ and $r>0$, where c_1 is a constant depending only on d and β. We have

$$\begin{aligned}M_\beta(K)\leqq c_1\nu(K)&\leqq c_1\int(\sum_j b_j\chi_{A(P_j,r_j)}+\varepsilon\chi_{A(P_o,r_o)})d\nu\\&\leqq c_1(\sum_j b_j r_j^\beta+\varepsilon r_o^\beta),\end{aligned}$$

which leads to

$$\mu_\beta(E)\leqq\mu_\beta(G)\leqq c_1(\sum_j b_j r_j^\beta+\varepsilon r_o^\beta).$$

As $\varepsilon\to0$, we have $\mu_\beta(E)\leqq c_1\sum_j b_j r_j^\beta$ and hence

$$\mu_\beta(E) \leqq c_1 \gamma_\beta(\chi_E)$$

for each $E \subset \partial D$. Q. E. D.

Recall that for a Borel set E $M_\beta(E) > 0$ if and only if $M_\beta(K) > 0$ for some compact subset of E (cf. [2, Theorem 3, § II]). From Lemma 2.1 we deduce

LEMMA 2.2. *For a Borel set $E \subset \partial D$ the following assertions are equivalent:*

(*i*) *E is of β-dimensional Hausdorff measure zero,*

(*ii*) $M_\beta(E) = 0$,

(*iii*) $\gamma_\beta(E) = 0$,

(*iv*) $\mu_\beta(E) = 0$.

The functional γ_β has the following property similar to the L^p-norm.

LEMMA 2.3. *Assume that $\gamma_\beta(f_n) \to 0$ for a sequence $\{f_n\} \subset \mathscr{B}(\gamma_\beta)$ (resp. $\{f_n\} \subset C(\partial D)$). Then there exists an subsequence $\{g_j\}$ of $\{f_n\}$ with the following property:*

For each $\varepsilon > 0$ there is a subset (resp. a closed subset) F of ∂D such that $\mu_\beta(\partial D \setminus F) < \varepsilon$ and $\{g_j\}$ converges to 0 uniformly on F.

PROOF. Noting that $\{f_n\}$ is a Cauchy sequence in $\mathscr{B}(\gamma_\beta)$ we can choose a subsequence $\{g_j\}$ of $\{f_n\}$ such that

$$\sum_{j=1}^{\infty} 2^j \gamma_\beta(g_j - g_{j+1}) < +\infty .$$

Set

$$F_n = \{Q \in \partial D : \ |g_j(Q) - g_{j+1}(Q)| \leqq 2^{-j}, j = n, n+1, \cdots\}$$

Then $\{g_j\}$ has the desired property. In fact, since

$$\partial D \setminus F_n \subset \bigcup_{j=n}^{\infty} \{Q : |g_j(Q) - g_{j+1}(Q)| > 2^{-j}\},$$

we have

$$\mu_\beta(\partial D \setminus F_n) \leqq \frac{1}{c} \gamma_\beta(\partial D \setminus F_n) \leqq \frac{1}{c} \sum_{j=n}^{\infty} 2^j \gamma_\beta(g_j - g_{j+1})$$

by Lemma 2.1. Noting that for $j \geqq n$

$$g_j = \sum_{k=j}^{\infty} (g_k - g_{k+1}) \quad \text{on } F_n ,$$

we see that $\{g_j\}$ converges to 0 uniformly on F_n. Especially if $\{g_j\} \subset C(\partial D)$, then F_n is closed. Q. E. D.

If a property holds on ∂D except for set vanishing μ_β, we say it holds μ_β-q. e. on ∂D.

We next consider a Fatou type lemma with respect to μ_β and γ_β. Suppose that to each $X\in D$ and $f\in\mathcal{L}(\gamma_\beta)$ thre corresponds a real number $\Phi_f(X)$ satisfying the following properties:

(a_1) $b\in \boldsymbol{R}^+\Rightarrow\Phi_{bf}(X)=b\Phi_f(X)$,

(a_2) $f+g=h$ μ_β-q. e. on $\partial D\Rightarrow\Phi_f(X)+\Phi_g(X)=\Phi_h(X)$.

Moreover, assume that to each $P\in\partial D$ there corresponds a filter $\mathcal{F}_P$ of subsets of D, converging to P. Under these assumptions we have

LEMMA 2.4. *Set*

$$F_{f,b}=\{P\in\partial D:\ \mathcal{F}_P-\limsup_{X\to P}|\Phi_f(X)|>b\}$$

and suppose that there is a constant c such that

$$\mu_\beta(F_{f,b})\leqq\frac{c}{b}\gamma_\beta(f) \tag{2.1}$$

for every real number $b>0$ and $f\in\mathcal{L}(\gamma_\beta)$. Moreover suppose that

$$\mathcal{F}_P-\lim_{X\to P}\Phi_f(X)=f(P)\quad \mu_\beta\text{-q. e.}\quad \text{on } \partial D \tag{2.2}$$

for every f in a dense subset $\mathcal{H}$ of $C(\partial D)$ with respect to γ_β. Then (2.2) *holds for every $f\in\mathcal{L}(\gamma_\beta)$.*

PROOF. Let $f\in\mathcal{L}(\gamma_\beta)$ and choose a sequence $\{f_n\}\subset\mathcal{H}$ such that $\gamma_\beta(f-f_n)\to 0$. On account of Lemma 2.3 we can assume that for each $\varepsilon>0$ there is a set F_ε such that $\mu_\beta(\partial D\setminus F_\varepsilon)<\varepsilon$ and $\{f_n\}$ converges uniformly to f on F_ε. Set

$$E_0=\bigcup_n\{Q:\ |f_n(Q)|=+\infty\}\cup\{Q:\ |f(Q)|=+\infty\}$$

and denote by E_n the set of all points $P\in\partial D$ at which (2.2) does not hold for $f=f_n$. Then $\gamma_\beta(\chi_E)=0$ for the set E defined by $\cup_{n=0}^{\infty}E_n$ and hence $\mu_\beta(E)=0$ by Lemma 2.1. For $P\in F_\varepsilon\setminus E$ we have

$$\mathcal{F}_P-\limsup_{X\to P}|\Phi_f(X)-f(P)|\leqq\mathcal{F}_P-\limsup_{X\to P}|\Phi_{f-f_n}(X)|+|f_n(P)-f(P)|\,.$$

Put

$$G_b=\{P:\ \mathcal{F}_P-\limsup_{X\to P}|\Phi_f(X)-f(P)|>b\}$$

and set $\delta=2^{-1}b$. There is a natural number n_o such that $|f_n-f|<\delta$ on F_ε for each $n\geqq n_o$. If $P\in(F_\varepsilon\cap G_b)\setminus E$, then

$$\mathcal{F}_P-\limsup_{X\to P}|\Phi_{f-f_n}(X)|>\delta\,,$$

which and (2.1) lead to

$$\mu_\beta(G_b) \leqq \mu_\beta(F_{f-f_n,\delta}) + \mu_\beta(\partial D \backslash F_\varepsilon) + \mu_\beta(E)$$
$$\leqq \frac{c}{\delta}\gamma_\beta(f-f_n) + \varepsilon .$$

As $n \to \infty$ we hεve $\mu_\beta(G_b) \leqq \varepsilon$ and hence $\mu_\beta(G_b)=0$. Thus we see that (2.2) holds. Q. E. D.

§ 3. Estimates for maximal functions

Hereafter we suppose that $d \geqq 3$ and D is a bounded Lipschitz domain in $\boldsymbol{R}^d$. Recall that a bounded domain D is called a Lipschitz domain if for each $P \in \partial D$ there exists a ball B with center P, a coordinate system of $\boldsymbol{R}^d$ and a Lipschitz function ϕ with the Lipschitz constant τ such that in these coordinates,

$$B \cap D = B \cap \{(x, t) : \ x \in \boldsymbol{R}^{d-1}, t \in \boldsymbol{R}, t > \phi(x)\}$$

and

$$B \cap \partial D = B \cap \{(x, \phi(x)) : \ x \in \boldsymbol{R}^{d-1}\} .$$

Note that in the above definition it is assumed that the radius of the ball B and the Lipschitz constant τ are chosen independently of $P \in \partial D$. In addition, if the function ϕ can be taken to be a C^1-function, D is called a C^1-domain.

Let us denote by σ the surface measure of a bounded Lipschitz domain D in $\boldsymbol{R}^d$ and assume that $D \subset B(0, 2^{-1}R)$. The following estimate of a surface ball with respect to σ is well-known.

LEMMA 3.1. *There is a constant $c>0$ such that*

$$\sigma(A(P, r)) \leqq c r^{d-1}$$

for all $P \in \partial D$ and all positive real numbers $r \leqq R$.

The maximal function Mf of $f \in L^1(\sigma)$ is defined as follows:

$$Mf(P) = \sup\left\{\frac{1}{r^{d-1}}\int_{A(P,r)} |f| d\sigma : \ r>0\right\} .$$

We estimate the maximal function of $\chi_{A(P,r)}$ by γ_β.

LEMMA 3.2. *Let p, β be real numbers such that $\beta < d-1$, $1 \leqq p < (d-1)/\beta$ and A_o be the surface ball with center P_o and radius r. Then*

$$\gamma_\beta((M\chi_{A_o})^{1/p}) \leqq c r^\beta , \tag{3.1}$$

where c is a constant, independent of P_o and r.

PROOF. Putting $A_1 = A(P_o, 2r)$, we claim that

$$(M\chi_{A_o})(P) \leqq \frac{2^{d-1}\sigma(A_o)}{|P-P_o|^{d-1}} \tag{3.2}$$

for all $P \in \partial D \backslash A_1$. Indeed, since

$$d(P, A_o) := \inf\{|P-Q\}: Q \in A_o\} \geqq (|P-P_o|-r)$$
$$\geqq 2^{-1}|P-P_o|,$$

we obtain

$$(M\chi_{A_o})(P) \leqq \frac{\sigma(A_o)}{d(P, A_o)^{d-1}} \leqq \frac{2^{d-1}\sigma(A_o)}{|P-P_o|^{d-1}},$$

which shows (3.2).

Next, set

$$E_j = \left\{P \in \partial D \backslash A_1 : (M\chi_{A_o})(P) > \frac{c}{2^j}\right\}$$

for $j \geqq 0$ and the positive real number c in Lemma 3.1. If $P \in E_j$, then, by (3.2) and Lemma 3.1, we have

$$|P-P_o| \leqq 2^{1+j/(d-1)}r \tag{3.3}$$

for all $P \in E_j$. Denoting by r_j the right-hand side of (3.3), we obtain

$$(M\chi_{A_o})^{1/p} \leqq \sum_{j=1}^{\infty}\left(\frac{c}{2^{j-1}}\right)^{1/p}\chi_{E_j \backslash E_{j-1}} + c^{1/p}\chi_{A_1}$$

$$= \sum_{j=1}^{\infty}\left(\frac{c}{2^j}\right)^{1/p}(2^{1/p}-1)\chi_{E_j} + c^{1/p}\chi_{A_1}$$

$$\leqq \sum_{j=1}^{\infty} c^{1/p}(2^{1/p}-1)2^{-j/p}\chi_{A(P_o, r_j)} + c^{1/p}\chi_{A_1}.$$

Therefore, setting $c_1 = c^{1/p}(2^{1/p}-1)$, we have

$$\gamma_\beta((M\chi_{A_o})^{1/p}) \leqq c_1 \sum_{j=1}^{\infty} 2^{-j/p} r_j^\beta + c^{1/p}(2r)^\beta$$

$$\leqq 2^\beta r^\beta\left(\sum_{j=1}^{\infty} c_1 2^{(\beta/(d-1)-1/p)j} + c^{1/p}\right).$$

Noting that $\beta/(d-1)-1/p<0$, we obtain (3.1). Q. E. D.

Furthermore we have an estimate of weak type for maximal functions with respect to μ_β and γ_β.

LEMMA 3.3. *Let p, β be positive real numbers satisfying $\beta < d-1$, $1 \leqq p < (d-1)/\beta$ and $f \in \mathcal{L}(\gamma_\beta)$. Set*

$$E_b = \{Q \in \partial D : (M|f|^p\}(Q)^{1/p} > b\}.$$

Then

$$\mu_\beta(E_b) \leqq \frac{c}{b} \gamma_\beta(f),$$

where c is a constant independent of f and b.

PROOF. We first claim

$$\gamma_\beta((M|f|^p)^{1/p}) \leqq c\gamma_\beta(f) \tag{3.5}$$

for all $f \in \mathcal{L}(\gamma_\beta)$. In fact, suppose that $|f| \leqq \sum_j b_j \chi_{A_j}$ where A_j are surface balls with radius r_j. From

$$(M|f|^p)^{1/p} \leqq \sum_j b_j (M\chi_{A_j})^{1/p}$$

and Lemma 3.2 we have

$$\gamma_\beta((M|f|^p)^{1/p}) \leqq c_1 \sum_j b_j r_j^\beta,$$

which implies (3.5).

We next show (3.4). Noting that $\gamma^{-1}(M|f|^p)^{1/p} > 1$ on E_b, we have

$$c_2\mu_\beta(E_b) \leqq r_\beta(\chi_{E_b}) \leqq \frac{1}{b}\gamma_\beta((M|f|^p)^{1/p}) \leqq \frac{c_3}{b}\gamma_\beta(f)$$

by (3.5) and Lemma 2.1. Thus we obtain (3.4). Q. E. D.

§ 4. Non-tangential limits of harmonic functions

Let D be a bounded Lipschitz domain and suppose that to each $P \in \partial D$ there is associated a cone $\Gamma(P)$ with vertex a P. According to B. E. J. Dahlberg we say that $\{\Gamma(P)\}_{P\in\partial D}$ is a regular family of cones if we can partition ∂D into finite many subsets $\{F_i\}$ with the following property:

To each E_i there are cones Γ_{ij} $(j=1,2,3)$ with vertex at 0 such that for each $P \in E_i$

$$\Gamma_{i1}+P \subset \Gamma(P)\bar{\Gamma}_{i2}\setminus\{0\}+P \subset \Gamma_{i3}+P \subset D.$$

Using Lemmas 3.4 and 3.5 in [4] and a patching argument, we can show the following theorem (cf. Proof of Lemma 3.1 in [4]).

THEOREM A. *Suppose $\{\Gamma(P)\}_{P\in\partial D}$ is a regular family of cones. If $2 \leqq p < +\infty$ and $f \in L^p(\sigma)$, then there are positive real numbers δ, c such that*

$$\begin{aligned} f^*(P) &:= \sup\{|Hf(X)| : X \in \Gamma(P), |X-P| < \delta\} \\ &\leqq c(M|f|^p)(P)^{1/p} \end{aligned} \tag{4.1}$$

for every $P \in \partial D$. Especially, if D is a C^1-domain, then (4.1) *holds for every positive real number p and $f \in L^p(\sigma)$.*

Our theorem is as follows:

THEOREM. *Let D be a bounded Lipschitz domain in $\boldsymbol{R}^d$ and $\{\Gamma(P)\}_{P\in\partial D}$ be a regular family of cones. Further, let $0<2\beta<d-1$. Then for each $f\in\mathcal{L}(\gamma_\beta)$ there exists a set E of β-dimensional Hausdorff measure zero such that*

$$\lim_{X\to P, X\in\Gamma(P)} Hf(X)=f(P)$$

for every $P\in\partial D\backslash E$. Especially, if D is a C^1-domain, then we have the same conclusion for every β satisfying $0<\beta<d-1$.

PROOF. Set $E_b=\{Q\in\partial D: f^*(Q)>b\}$, where f^* is the function defined in (4.1). By (4.1) and Lemma 3.3 we see that there is a constant c such that

$$\mu_\beta(E_b)\leqq\frac{c}{b}\gamma_\beta(f)$$

for every $f\in\mathcal{L}(\gamma_\beta)$.

On the other hand, it is well-known that every boundary point of D is a regular point with respect to the Dirichlet problem. Therefore we have

$$\lim_{X\to P, X\in\Gamma(P)} Hf(X)=f(P) \tag{4.2}$$

for every $f\in C(\partial D)$. Since $C(\partial D)$ is dense in $\mathcal{L}(\gamma_\beta)$, Lemma 2.4 yields that for every $f\in\mathcal{L}(\gamma_\beta)$ there is a subset E of ∂D such that (4.2) holds at every $P\in\partial D\backslash E$ and $\mu_\beta(E)=0$. Thus we have the conclusion by Lemma 2.2.

Q. E. D.

§ 5. Properties of $\mathcal{L}(\gamma_\beta)$

In this section we will study what functions belong to $\mathcal{L}(\gamma_\beta)$. A function f on ∂D is called M_β-quasicontinuous (resp. μ_β-quasicontinuous) if for each $\varepsilon>0$ there is a closed set $F\subset\partial D$ such that the restriction of f to F is continuous and $M_\beta(\partial D\backslash F)<\varepsilon$ (resp. $\mu_\beta(\partial D\backslash F)<\varepsilon$). We have

PROPOSITION 5.1. *If $f\in\mathcal{L}(\gamma_\beta)$, then $\gamma_\beta(f)<+\infty$ and f is μ_β-quasicontinuous. Conversely, if a Borel measurable function f is M_β-quasicontinuous and $\gamma_\beta(f)<+\infty$, then $f\in\mathcal{L}(\gamma_\beta)$.*

PROOF. Let $f\in\mathcal{L}(\gamma_\beta)$ and choose a sequence $\{f_n\}\subset C(\partial D)$ such that $\gamma_\beta(f-f_n)\to 0$. By Lemma 2.3 we can assume that there is, for esch $\varepsilon>0$, a closed set $F\subset\partial D$ such that $\mu_\beta(\partial D\backslash F)<\varepsilon$ and $\{f_n\}$ converges to f uniformly on F. Therefore the restriction of f to F is continuous and hence f is μ_β-quasicontinuous.

Conversely, suppose that a Borel measurable function f is M_β-quasicontinuous and $\gamma_\beta(f)<+\infty$. We can assume $f\geqq 0$ and $f\neq 0$. Choose $u=\sum_j b_j\chi_{A_j}$ such that

$$f\leqq u \quad \text{and} \quad \sum_j b_j r_j^\beta<+\infty,$$

where A_j are surface balls with radius r_j. If we set

$$f_n=\min\left(f, \sum_{j=1}^n b_j\right),$$

then f_n is also M_β-quasicontinuous. Therefore there is a nonnegative continuous function g_n such that

$$\|g_n\|_\infty\leqq\|f_n\|_\infty \quad \text{and} \quad M_\beta(G_n)\leqq\frac{1}{n\|F_n\|_\infty},$$

where $\|\ \|_\infty$ is the sup-norm on ∂D and $G_n=\{Q\in\partial D: f_n(Q)\neq g_n(Q)\}$. Noting that

$$|f-f_n|\leqq\sum_{j=n+1}^\infty b_j\chi_{A_j} \quad \text{and} \quad |f_n-g_n|\leqq\|f_n\|_\infty\chi_{G_n},$$

we have, together with Lemma 2.1,

$$\gamma_\beta(f-g_n)\leqq\gamma_\beta(f-f_n)+\gamma_\beta(f_n-g_n)\leqq\sum_{j=n+1}^\infty b_j r_j^\beta+\frac{1}{n},$$

which shows that $f\in\mathcal{L}(\gamma_\beta)$. Q. E. D.

Let us next show that the restriction of $g*f$ to ∂D for a suitable $g\in L^1(\boldsymbol{R}^d)$ and $f\in L^p(\boldsymbol{R}^d)$ belongs to $\mathcal{L}(\gamma_\beta)$. We prepare the following lemma, which can be proved by using a covering lemma (cf. [7, Lemma, p. 9]).

LEMMA 5.1. *If $f\in L^1(\mathbf{R}^d)$, then there is a function $\sum_{j=1}^\infty b_j\chi_{B_j}$ such that*

$$|f|\leqq\sum_j b_j\chi_{B_j} \quad \text{and} \quad \sum_j b_j m(B_j)<+\infty, \tag{5.1}$$

where $b_j\in\boldsymbol{R}^+$, B_j is an open ball in $\boldsymbol{R}^d$ and m is the Lebesque measure on $\boldsymbol{R}^d$.

LEMMA 5.2. *Let β, p positive real numbers such that $\beta\leqq d-1$ and $p\geqq 1$. Suppose that g is a function in $L^1((\boldsymbol{R}^d)$ such that*

$$|g|\leqq\sum_i a_i\chi_{E_i} \quad \text{and} \quad \sum_i a_i r_j^\beta<+\infty$$

for $a_i\in\boldsymbol{R}^+$ and open balls E_i with radius r_i. Then the restriction of $g*f$ to ∂D belongs to $\mathcal{L}(\gamma_\beta)$ for every $f\in L^p(\boldsymbol{R}^d)$.

PROOF. We may assume that both f and g are nonnegative and $f\in$

$L^1(\boldsymbol{R}^d)$. Indeed every function $f \in L^p(\boldsymbol{R}^d)$ can be written as $f = f_1 + f_2$, where $f_1 \in L^1(\boldsymbol{R}^d)$, $f_2 \in L^\infty(\boldsymbol{R}^d)$, and hence $g*f_2$ is continuous.

Set

$$g_n = \min\left(g,\ \sum_{i=1}^{n} a_i \chi_{E_i}\right).$$

Then $g_n * f$ is continuous and $0 \leqq g - g_n \leqq \sum_{i=n+1}^{\infty} a_i \chi_{E_i}$.

We shall show that

$$\gamma_\beta(h_n) \longrightarrow 0 \quad \text{as } n \to \infty, \tag{5.2}$$

where h_n is the restriction of $(g - g_n)*f$ to ∂D. By Lemma 5.1 there is a function $\sum_j b_j \chi_{B_j}$ satisfying (5.1), where B_j are open balls with radius ρ_j. Setting $s = d/\beta$ and $t = s/(s-1)$, we have

$$\begin{aligned} 0 \leqq (g - g_n)*f &\leqq \left(\sum_{i=n+1}^{\infty} a_i \chi_{E_i}\right) * \left(\sum_{j=1}^{\infty} b_j \chi_{B_j}\right) \\ &\leqq \sum_{i=n+1}^{\infty} a_i \sum_{j=1}^{\infty} b_j \chi_{E_i + B_j} \min(m(B_j), m(E_i)^{1/s} m(B_j)^{1/t}), \end{aligned}$$

where $E_i + B_j = \{X + X\colon X \in E_i,\ Y \in B_j\}$. If $(E_i + B_j) \cap \partial D \neq \emptyset$, then we take the surface ball F_{ij} with center $P_{ij} \in (E_i + B_j) \cap \partial D$ and radius $2(g_i + g_j)$. From

$$(g - g_n)*f \leqq \sum_{i=n+1}^{\infty} a_i \sum_{j=1}^{\infty} b_j \chi_{F_{ij}} \min(m(B_j), m(E_i)^{1/s} m(B_j)^{1/t})$$

it follows that

$$\begin{aligned} \gamma_\beta((g - g_n)*f) &\leqq 2^\beta \sum_{i=n+1}^{\infty} a_i \sum_{j=1}^{\infty} b_j (r_i + \rho_j)^\beta \min(m(B_j), m(E_i)^{1/s} m(B_j)^{1/t}) \\ &\leqq 4^\beta \sum_{i=n+1}^{\infty} a_i \sum_{j=1}^{\infty} b_j (r_j^\beta m(B_j) + \rho_j^\beta m(E_i)^{1/s} m(B_j)^{1/t}), \\ &\leqq c \left(\sum_{i=n+1}^{\infty} a_i r_i^\beta\right) \left(\sum_{j=1}^{\infty} b_j m(B_j)\right), \end{aligned}$$

where c is a constant depending only on d and β. Thus we have shown that (5.2) *holds and the restriction of* $g*f$ *to* ∂D *belongs to* $\mathcal{L}(\gamma_\beta)$.

Q. E. D.

Especially we have

PROPOSITION 5.2. *Let* α, β, p *be positive real numbers such that* $\alpha < d$, $\beta \leqq d-1$, $d < \alpha + \beta$ *and* $p \geqq 1$. *Further, let* g *be a radial nonnegative, decreasing, lower semicontinuous function in* $L^1(\boldsymbol{R}^d)$ *such that* $|g| = O(|X|^{\alpha - d})$ *as* $|X| \to 0$. *Then the restriction of* $g*f$ *to* ∂D *belongs to* $\mathcal{L}(\gamma_\beta)$ *for every* $f \in L^p(\boldsymbol{R}^d)$.

PROOF. Set

$$E_n=\{X: |X|^{\alpha-d}>2^n\}$$

for $n\geqq 1$. Then E_n is an open ball with radius $r_n=2^{-n(d-\alpha)}$ and g can be written as $g=g_1+g_2$, where g_2 is bounded and g_1 satisfies

$$0\leqq g_1\leqq c\sum_{n=1}^{\infty}2^n\chi_{E_n}$$

for some constant c. Therefore we obtain

$$\gamma_\beta(g_1)\leqq c\sum_{n=1}^{\infty}2^n r_n^\beta\leqq c\sum_{n=1}^{\infty}2^{(1-\beta/(d-\alpha))n}<+\infty,$$

because $1-\beta/(d-\alpha)<0$. On account of Lemma 5.2 we see that the restric. tion of g_1*f to ∂D belongs to $\mathcal{L}(\gamma_\beta)$. Since g_2*f is continuous, we have the conclusion. Q. E. D.

From the above proposition we deduce

COROLLARY. *Let α, β, p be positive real numbers satisfying the same conditions as in Proposition* 5.2. *If G_α in the Bessel kernel with order α, i.e., the function which belongs to $L^1(\boldsymbol{R}^d)$ and whose Fourier transformation is $(1+|\xi|^2)^{-\alpha/2}$, then the restriction of $G_\alpha*f$ to ∂D belongs to $\mathcal{L}(\gamma_\beta)$ for every $f\in L^1(\boldsymbol{R}^d)$. In addition, if $\alpha p<d$, then the same conclusion is obtained for the Riesz kernel R_α with order α, i.e., $R_\alpha(X)=|X|^{\alpha-d}$.*

REMARK. Let β, λ be real numbers such that $0<\beta\leqq d-1$, $\lambda>-\beta$. Then we can show, by similar methods to those in the proofs of Lemma 5.2 and Proposition 5.2, that the function

$$P\longmapsto\int|P-Q|^\lambda f(Q)d\sigma(Q)$$

belongs to $\mathcal{L}(\gamma_\beta)$ for every $f\in L^1(\sigma)$.

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