Kaehlerian Manifolds of Positive Pure Symmetric Curvature Operator

Eriko Omachi

The Doctoral Research Course in Human Culture,
Ochanomizu University, 2-1-1, Bunkyo-ku, Tokyo, 112, Japan

§0. Introduction. Many works are known on Riemannian manifolds of positive curvature operator [1], [3], [4], [5], [6], [10], etc.. The analogies of that for Kaehlerian manifolds were studied first by S. Tachibana [11] introducing the σ-positiveness. The curvature operator of an $n(=2m)$ dimensional Kaehlerian manifold M^{2m} is said to be of σ-positive if it is not less than that of an m dimensional complex projective manifold CP^m with standard metric. He proved the following theorems A and B.

THEOREM A. *A* $2m$ *dimensional compact Kaehlerian manifold of* σ-*positive curvature operator is real cohomologically equivalent to* CP^m.

THEOREM B. *A compact Kaehler-Einstein manifold of* σ-*positive curvature operator is a space of constant holomorphic sectional curvature.*

Let $R_{a\bar{b}c}{}^d$ be components of the curvature tensor of a Kaehlerian manifold with respect to complex coordinates. K. Ogiue and S. Tachibana [8] have dealt with the following two curvature operators of Kaehlerian manifolds:

(i) $$\xi_{ab} \longrightarrow \sum R^c{}_{ab}{}^d \xi_{cd}$$

(ii) $$\xi_{a\bar{b}} \longrightarrow \sum R_{a\bar{b}c\bar{d}} \xi^{c\bar{d}}$$

It should be noticed that the positiveness of the operator (ii) is equivalent to the σ-positiveness. For a Kaehlerian manifold whose operator (i) is positive, they showed the following Theorem A′, which is an analogy of Theorem A;

THEOREM A′. *A* $2m$ *dimensional compact Kaehlerian manifold whose operator* (i) *is positive is real cohomologically equivalent to* CP^m.

The purpose of this paper is to study about such a manifold further, and show that, in the case when the manifold is harmonic, it is locally symmetric, under an additional condition.

Section 1 will be devoted to the notations and preliminary facts. The

operator (i) will be called a pure symmetric curvature (simply, PSC) one. We shall reconsider in Section 2 the result of Ogiue-Tachibana about positive PSC operator from the view point of real coordinate, and in Section 3 obtain some inequalities for later use. The main theorem will be given in Section 4 for harmonic Kaehlerian manifold of positive PSC operator.
The author expresses her hearty gratitude to Professor S. Tachibana for his suggestions and encouragement.

§ 1. Preliminaries. Let M^n $(n>1)$ be an n dimensional Riemannian manifold. We denote by g_{ji}, $R_{kji}{}^h$ and $R_{ji}=R_{kji}{}^k$ the components of the Riemannian metric g, the curvature tensor R and the Ricci tensor Ric with respect to the natural basis, and the summation convention are assumed. ∇ denoted the operator of the covariant differential, and $\Delta=d\delta+\delta d$ the Laplacian operator.

Let u be a p-form $u=(1/p!)u_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ with skew symmetric coefficients $u_{i_1\cdots i_p}$. The quadratic forms $F_p(u)$ of u are defined by

$$F_1(u)=R_{ji}u^ju^i\,,\qquad (p=1)\,,$$

$$F_p(u)=\frac{1}{(p-1)!}\left(R_{rs}u^{ri_2\cdots i_p}u^s{}_{i_2\cdots i_p}+\frac{p-1}{2}R_{rsjh}u^{rsi_3\cdots i_p}u^{jh}{}_{i_3\cdots i_p}\right),\quad (p\geqq 2)\,.$$

In the following, M^n means an $n=2m$ dimensional Kaehlerian manifold $(m>1)$. M^n admits a parallel tensor field $F=(F_i{}^h)$ such that

$$F_i{}^rF_r{}^j=-\delta_i{}^j\,,\qquad F_{ij}=F_i{}^rg_{rj}=-F_{ji}\,.$$

The curvature tensor and the Ricci tensor satisfy

$$F_l{}^rR_{rhji}=-F_h{}^rR_{lrji}\,,\qquad F_l{}^rR_{rh}=-F_h{}^rR_{lr}\,,\tag{1.1}$$
$$\frac{1}{2}F^{rr}R_{rsji}=F^{rs}R_{rjsi}=-S_{ji}=S_{ij}\,,$$

where

$$S_{ji}=F_j{}^rR_{ri}\,.$$

A tensor $u=(u_{ij})$ is said to be pure (in i, j) [15], if it satisfies

$$F_i{}^ku_{kj}=F_j{}^ku_{ik}\,,$$

and to be hybrid (in i, j) if

$$F_i{}^ku_{kj}=-F_j{}^ku_{ik}\,.$$

For example, g_{ji}, F_{ji}, R_{ji} and S_{ji} are hybrid, R_{kjih} is hybrid in k, j and in i, h respectively.

The curvature tensor $U=(U_{rhji})$ of M^n is defined by

$$U_{rhj}{}^i = R_{rhj}{}^i - \frac{S}{4m(m+1)}\Big(g_{hj}\delta_r{}^i - g_{rj}\delta_h{}^i + F_{hj}F_r{}^i - F_{rj}F_h{}^i - 2F_{rh}F_j{}^i\Big).$$

where $S(=R_i{}^i)$ is the scalar curvature.

The curvature tensor of a space of constant holomorphic sectional curvature k satisfies

$$R_{rhji} = \frac{1}{4}k(g_{hj}g_{ri} - g_{rj}g_{hi} + F_{hj}F_{ri} - F_{rj}F_{hi} - 2F_{rh}F_{ji}).$$

A Kaehlerian manifold is of constant holomorphic sectional curvature if and only if its U vanishes.

In a space of constant holomorphic sectional curvature k, it holds that

$$R_{rhji}u^{rj}u^{hi} = ku_{ji}u^{ji}$$

for a pure symmetric tensor $u=(u_{ji})$. A Kaehlerian manifold is said to be of positive pure symmetric curvature operator (or simply, positive PSC operator) if there exists a constant $k>0$ satisfying

$$R_{rhji}u^{rj}u^{hi} \geqq ku_{ji}u^{ji} \tag{1.3}$$

for any pure symmetric tensor $u=(u_{ji})$ at each point.

First, we have

PROPOSITION 1. *In a Kaehlerian manifold M^{2m} of positive PSC operator, the scalar curvature S satisfies*

$$S \geqq m(m+1)k, \tag{1.4}$$

where the equality holds if and only if the manifold is of constant holomorphic sectional curvature k.

PROOF. We take a point and consider every quantities at this point. Let $X=(X_i)$ be a unit vector and put $\tilde{X}=(\tilde{X}_j)=(F_j{}^rX_r)$. If we define $u=(u_{ji})$ by $u_{ji}=X_jX_i-\tilde{X}_j\tilde{X}_i$, u is pure and symmetric. Then (1.3) implies

$$\rho_H(X) \geqq k, \tag{1.5}$$

where $\rho_H(X)$ denotes the holomorphic sectional curvature for X, i.e. $\rho_H(X) = -R_{rhji}X^rX^j\tilde{X}^h\tilde{X}^i$. Next, let $X=(X_i)$ and $Y=(Y_i)$ be any unit vectors and consider a pure symmetric tensor $v=(v_{ji})$ given by

$$v_{ji} = X_jY_i + X_iY_j - \tilde{X}_j\tilde{Y}_i - \tilde{X}_i\tilde{Y}_j.$$

Then, from (1.3) we have

$$-R(X, \tilde{X}, Y, \tilde{Y}) \geqq \frac{k}{2}(1+\cos^2\theta_{XY}+\cos^2\theta_{X\tilde{Y}}),$$

where $\rho(X, Z)$ denotes the sectional curvature for X and Z, θ_{XZ} the argument between X and Z.

Especially, if Y is orthogonal to X and $\tilde{X}$, i.e., X and Y are anti-holomorphic to each other, it holds that

$$\rho(X, Y)+\rho(X, \tilde{Y})\geqq \frac{k}{2} .$$

Now, take a unit vector X_1 and put $X_{m+1}=\tilde{X}_1$. Next, take a unit vector X_2 orthogonal to X_1 and X_{m+1}, and put $X_{m+2}=\tilde{X}_2$. Continue this process and we get a basis $X_1, \cdots, X_m, X_{m+1}, \cdots, X_{2m}$. Let us calculate the Ricci tensor with respect to this basis, and we obtain

$$\begin{aligned} R_{11}&=\rho(X_1, X_{m+1})+\{\rho(X_1, X_2)+\rho(X_1, X_{m+2})\}+ \\ &\quad\cdots+\{\rho(X_1, X_m)+\rho(X_1, X_{2m})\} \\ &\geqq k+\frac{k}{2}(m-1)=\frac{k}{2}(m+1) . \end{aligned}$$

Similarly we have

$$R_{ii}\geqq \frac{k}{2}(m+1) , \qquad (i \text{ no sum}) ,$$

for $i=1, \cdots, 2m$ and

$$S=\sum_{i=1}^{2m} R_{ii}\geqq m(m+1)k . \qquad \square$$

§ 2. Betti numbers.

First we note the following

LEMMA. *In a Kaehlerian manifold M^{2m}, the condition* (1.3) *of positive PSC operator is equivalent to*

$$(R_{khji}-R_{krjs}F_h{}^rF_i{}^s)v^{kj}v^{hi}\geqq k(v_{rs}v^{rs}-F_r{}^nF_s{}^qv_{pq}v^{rs}) \tag{2.1}$$

where $v=(v_{ji})$ is any symmetric tensor.

PROOF. Let us put

$$u_{ji}=v_{ji}-F_j{}^rF_i{}^sv_{rs}$$

for a symmetric tensor $v=(v_{ji})$, then $u=(u_{ji})$ is pure symmetric. Assume M^{2m} to be of positive PSC operator, and substitution u into (1.3) implies (2.1). Conversely, (1.3) follows from (2.1) with pure v. $\square$

It follows from Lemma

PROPOSITION 2. *In a Kaehlerian manifold M^{2m} of positive PSC operator, the following formulae hold for any p-form u;*

$$F_1(u) \geqq \frac{m+1}{2} k u_r u^r , \tag{2.2}$$

$$\begin{aligned} F_p(u) \geqq \frac{1}{(p-1)!} \frac{k}{4} \{ &(2m-p+3) u_{i_1 \cdots i_p} u^{i_1 \cdots i_p} \\ &-(p-1) F^{tq} F^{rs} u_{tq i_3 \cdots i_p} u_{rs}{}^{i_3 \cdots i_p} \\ &-(p-1) F^{tr} F^{qs} u_{tq i_3 \cdots i_p} u_{rs}{}^{i_3 \cdots i_p} \} \qquad (p \geqq 2) . \end{aligned} \tag{2.3}$$

PROOF. Fix a point and take an orthonormal frame at the point. In the rest of this section, tensors are represented by the components with respect to this frame, with lower indices. We take the summation from 1 to n for every repeated indices.

The case of $p=1$. For a 1-form $u=(u_r)$, define a symmetric tensor $v^{(i)}=(v^{(i)}_{rs})$ for each $i=1, 2, \cdots, n$ as follows:

$$v^{(i)}_{rs} = u_r \delta_{si} + u_s \delta_{ri} .$$

We substitute $v^{(i)}$ to (2.1) for each i, and take summation with respect to i. The first term of the lefthand side becomes

$$\begin{aligned} R_{tqrs} v^{(i)}_{tr} v^{(i)}_{qs} &= R_{tqrs}(u_t \delta_{ri} + u_r \delta_{ti})(u_q \delta_{si} + u_s \delta_{qi}) \\ &= 2R_{tqrs}(u_t \delta_{ri} + u_r \delta_{ti}) u_q \delta_{si} \\ &= 2R_{qr} u_q u_r . \end{aligned}$$

For other terms, we have

$$\begin{aligned} -R_{tqrs} F_{hq} F_{ms} v^{(i)}_{tr} v^{(i)}_{hm} &= 6R_{rs} u_r u_s , \\ v^{(i)}_{rs} v^{(i)}_{rs} &= 2(n+1) u_r u_r , \\ -F_{rt} F_{sq} v^{(i)}_{tq} v^{(i)}_{rs} &= 2u_r u_r \end{aligned}$$

by virtue of (1.1), and (2.2) follows.

The case of $p \geqq 2$. Let $u=(u_{i_1 \cdots i_p})$ be any p-form, and define a symmetric tensor $v^{(i_1 \cdots i_p)} = (v^{(i_1 \cdots i_p)}_{rs})$ for each p-tuple $(i_1, \cdots, i_p)$ as follows:

$$v^{(i_1 \cdots i_p)}_{rs} = \sum_{k=1}^{p} (u_{i_1 \cdots i_{k-1} r i_{k+1} \cdots i_p} \delta_{s i_k} + u_{i_1 \cdots i_{k-1} s i_{k+1} \cdots i_p} \delta_{r i_k}) .$$

We put

$$\begin{aligned} a &= R_{tqrs} v^{(i_1 \cdots i_p)}_{tr} v^{(i_1 \cdots i_p)}_{qs} \\ b &= -R_{tqrs} F_{hq} F_{ms} v^{(i_1 \cdots i_p)}_{tr} v^{(i_1 \cdots i_p)}_{hm} \\ c &= v^{(i_1 \cdots i_p)}_{rs} v^{(i_1 \cdots i_p)}_{rs} \\ d &= -F_{rt} F_{sq} v^{(i_1 \cdots i_p)}_{tq} v^{(i_1 \cdots i_p)}_{rs} . \end{aligned}$$

Then it follows from (2.1) that

$$a+b\geqq k(c+d)\,, \tag{2.4}$$

and we shall calculate a, b, c and d.

At first, by the symmetric property in r, s of $v_{rs}^{(i_1\cdots i_p)}$ and the properties of curvature tensor, it holds that

$$\begin{aligned}
a&=2\sum_{k=1}^{p}\sum_{l=1}^{p}R_{tqrs}(u_{i_1\cdots i_{k-1}ti_{k+1}\cdots i_p}\delta_{ri_k}\\
&\quad+u_{i_1\cdots i_{k-1}ri_{k+1}\cdots i_p}\delta_{ti_k})u_{i_1\cdots i_{l-1}qi_{l+1}\cdots i_p}\delta_{si_l}\\
&=4\sum_{k<l}R_{tqi_ki_l}u_{i_1\cdots t\cdots i_p}u_{i_1\cdots q\cdots i_p}\qquad (t\rightarrow k,\ q\rightarrow l)\\
&\quad+2\sum_{k=1}^{p}R_{qr}u_{i_1\cdots r\cdots i_p}u_{i_1\cdots q\cdots i_p}\qquad (r\rightarrow k,\ q\leftarrow k)\\
&\quad+4\sum_{k<l}R_{i_kqri_l}u_{i_1\cdots r\cdots i_p}u_{i_1\cdots q\cdots i_p}\qquad (r\rightarrow k,\ q\rightarrow l)\\
&=2p(p-1)R_{tqrs}u_{tri_3\cdots i_p}u_{qsi_3\cdots i_p}+2pR_{tq}u_{ti_2\cdots i_p}u_{qi_2\cdots i_p}\\
&\quad+2p(p-1)R_{tqrs}u_{rsi_3\cdots i_p}u_{tqi_3\cdots i_p}\,,
\end{aligned}\tag{2.5}$$

where $(t\rightarrow k)$ means that t is at the k-th position i.e., $i_1\cdots i_{k-1}ti_{k+1}\cdots i_p$. As we have

$$R_{tqrs}u_{tri_3\cdots i_p}u_{qsi_3\cdots i_p}=\frac{1}{2}R_{tqrs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}\,,$$

(2.5) becomes

$$a=3p(p-1)R_{tqrs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}+2pR_{tq}u_{ti_2\cdots i_p}u_{qi_2\cdots i_p}\,. \tag{2.6}$$

Next, we calculate b in the same way. From the symmetric property of $v_{rs}^{(i_1\cdots i_p)}$ with respect to r and s, the skew-symmetric one of F and (1.1), we have

$$\begin{aligned}
b&=-2\sum_{k=1}^{p}\sum_{l=1}^{p}R_{tqrs}F_{hq}F_{ms}(u_{i_1\cdots t\cdots i_p}\delta_{ri_k}\\
&\quad+u_{i_1\cdots r\cdots i_p}\delta_{ti_k})u_{i_1\cdots h\cdots i_p}\delta_{mi_l}\qquad (t\rightarrow k,\ r\rightarrow k,\ h\rightarrow l)\\
&=-2\sum_{k=1}^{p}R_{i_kqi_ks}F_{hq}F_{rs}u_{i_1\cdots r\cdots i_p}u_{i_1\cdots h\cdots i_p}\qquad (r,h\rightarrow k)\\
&=6pR_{rs}u_{ri_2\cdots i_p}u_{si_2\cdots i_p}-2p(p-1)R_{tqrs}F_{hq}F_{ms}u_{tmi_3\cdots i_p}u_{rhi_3\cdots i_p}\,.
\end{aligned}\tag{2.7}$$

On the other hand, from the first Bianchi identity, the skew symmetric property of u and (1.1), it holds that

(2.8) $$R_{tqrs}F_{hq}F_{ms}u_{tmi_3\cdots i_p}u_{rhi_3\cdots i_p}=-\frac{1}{2}R_{tqrs}u_{tqi_3\cdots p}u_{rsi_3\cdots i_p}\,.$$

Making use of (2.8), (2.7) becomes

(2.9) $$b=6pR_{rs}u_{ri_2\cdots i_p}u_{si_2\cdots i_p}+p(p-1)R_{tqrs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}\,.$$

Hence from (2.6) and (2.9),

(2.10) $$a+b=(p-1)!\,8pF_p(u)\,.$$

The calculation of c and d are straightforward, and we obtain

(2.11) $$c=2p(2n-p+2)u_{i_1\cdots i_p}u_{i_1\cdots i_p}\,,$$

(2.12) $$d=2pu_{i_1\cdots i_p}u_{i_1\cdots i_p}-2p(p-1)F_{tq}F_{rs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}$$
$$-2p(p-1)F_{tr}F_{qs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}$$

and hence

(2.13) $$c+d=p(2m-p+3)u_{i_1\cdots i_p}u_{i_1\cdots i_p}$$
$$-2p(p-1)F_{tq}F_{rs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}\,.$$
$$-2p(p-1)F_{tr}F_{qs}u_{tqi_3\cdots i_p}u_{rsi_3\cdots i_p}\,.$$

Thus (2.3) is obtained from (2.4), (2.10) and (2.13), and Proposition 2 has been proved. □

The following corollary follows immediately from Proposition 2 as in [11], i.e., we have

COROLLARY (Ogiue-Tachibana [8]). *An $n(=2m)$ dimensional compact Kaehlerian manifold of positive PSC operator is real cohomologically equivalent to CP^m.*

§ **3. Some inequalities.** In this section, we prove two inequalities which are required in the proof of Theorem in § 4. For simplification, we put as follows:

$$\alpha=R_{kjih}R^{ihlm}R_{lm}{}^{kj}\,,\qquad \beta=R_{kjih}R^{klim}R^{j}{}_{l}{}^{h}{}_{m}\,.$$

$$\gamma=R_{kl}R^{kjih}R^{l}{}_{jih}\,,\qquad \rho=R_{kjih}R^{kjih}\,,\qquad \tau=R_{ki}R^{ki}\,.$$

Then, we have the following

PROPOSITION 3. *In a Kaehlerian manifold M^{2m} of positive PSC operator, the following inequality holds:*

(3.1) $$\alpha+2\gamma\geqq k(m\rho-2\tau)\,.$$

PROOF. Fix a point and take an orthonormal frame at the point. Tensors are represented with respect to this frame by the components with lower indices. We take the summation from 1 to n for every repeated indices. A 2-form $u^{(rs)}$ are defined by

$$u_{kj}^{(rs)}=R_{kjrs}$$

for each r and s. Apply (2.3) to $u^{(rs)}$ for each r and s, and take the summation for r and s. The lefthand side is $(1/2)\alpha+\gamma$. On the other hand, the righthand side is

$$\frac{k}{4}\{(2m+1)\rho-F_{tq}F_{rs}R_{tqih}R_{rsih}-F_{tr}F_{qs}R_{tqih}R_{rsih}\}$$

$$=\frac{k}{4}\{(2m+1)\rho-4\tau-\rho\}=\frac{k}{2}(m\rho-2\tau)$$

by (1.1). Therefore we have (3.1) from (2.3). □

PROPOSITION 4. *In a Kaehlerian manifold M^{2m} of positive PSC operator, the following inequality holds:*

$$-\alpha+4\beta\geqq 2k\rho\,. \tag{3.2}$$

PROOF. Fix a point and take an orthonormal frame at the point. Define a symmetric tensor $u^{(rs)}=(u_{kj}^{(rs)})$ by

$$u_{kj}^{(rs)}=R_{krjs}+R_{jrks}$$

for each r and s, apply (2.1) to $u^{(rs)}$ for each r and s, and take the summation for r and s from 1 to n. Let us denote

$$a=R_{khji}u_{kj}^{(rs)}u_{hi}^{(rs)}\,,\qquad b=-R_{kpjq}F_{hp}F_{iq}u_{kj}^{(rs)}u_{hi}^{(rs)}\,,$$

$$c=u_{kj}^{(rs)}u_{kj}^{(rs)}\,,\qquad d=-F_{kh}F_{ji}u_{kj}^{(rs)}u_{hi}^{(rs)}\,,$$

and from (2.1) we have

$$a+b\geqq k(c+d)\,. \tag{3.3}$$

We shall calculated a, b, c and d. At the first, by the symmetric property of $u^{(rs)}$, we have

$$a=2\beta+2R_{khji}R_{krjs}R_{irhs}\,.$$

From the Bianchi identity, it holds that

$$R_{khji}R_{kjsr}R_{hisr}=\frac{1}{2}\alpha \tag{3.4}$$

$$R_{khji}R_{krjs}R_{ihsr}=\frac{1}{2}R_{khji}R_{kjsr}R_{hisr}=\frac{1}{4}\alpha$$

$$R_{khji}R_{krjs}R_{irhs}=-R_{khji}R_{krjs}R_{ihsr}+\beta=-\frac{1}{4}\alpha+\beta\,,$$

and we get

$$a=-\frac{1}{2}\alpha+4\beta\,. \tag{3.5}$$

As for b, as the same way of a, we can denote

$$b=2b_1+2b_2\,,$$

where

$$b_1=-R_{ktjq}F_{ht}F_{iq}R_{krjs}R_{hris}$$

$$b_2=-R_{ktjq}F_{ht}F_{iq}R_{krjs}R_{irhs}\,.$$

Here we have from (1.1) and (3.4)

$$b_2=-\frac{1}{4}\alpha+\beta\,,$$

and from (1.1) and the Bianchi identity,

$$b_1=-R_{kjhi}R_{krjs}R_{hris}+b_2$$

$$R_{kjhi}R_{krjs}R_{hris}=\beta-b_2\,.$$

Hence

$$b=2\{-(\beta-b_2)+b_2\}+2b_2=-2\beta+6b_2=-\frac{3}{2}\alpha+4\beta\,. \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$a+b=-2\alpha+8\beta\,. \tag{3.7}$$

Next, we calculate c and d. It is easy to see that

$$c=2\rho+2R_{krjs}R_{jrks}=3\rho\,. \tag{3.8}$$

d can be calculated as

$$d=2d_1+2d_2$$

where

$$d_1=-F_{kh}F_{ji}R_{krjs}R_{hris}=0$$

$$d_2=-F_{kh}F_{ji}R_{krjs}R_{irhs}=R_{khjs}R_{kjhs}=\frac{1}{2}\rho\,,$$

which deduces

$$d=\rho\,. \tag{3.9}$$

From (3.8) and (3.9), we get

$$c+d=4\rho\,. \tag{3.10}$$

Therefore we conclude (3.2) from (3.3), (3.7) and (3.10). □

§ 4. **Harmonic Kaehlerian manifolds.** In this section, we shall consider harmonic Kaehlerian manifolds. An analytic Riemannian manifold M is said to be harmonic if for any point x_0 of M there is a normal neighborhood U in such a way that $\Delta\Omega$ is a function of Ω only, say $f(\Omega)$, where $\Omega=s^2/2$ and s is the geodesic distance measured from x_0. The local function $f(\Omega)$ is independent of the choice of x_0 and called the characteristic function of M.

S. Tachibana [9] showed an inequality which are satisfied by the first and second derivatives at $\Omega=0$ of $f(\Omega)$, and obtained a necessary and sufficient condition for the manifold to be of constant holomorphic sectional curvature. Then, Y. Watanabe [13] calculated the third derivative at $\Omega=0$ and an inequality which gives a necessary and sufficient condition for the manifold to be locally symmetric. In the process of obtaining these results, he used the Lichnerowicz's formula

$$\frac{1}{2}\Delta\rho=|\nabla R|^2-4R^{jikh}\nabla_j\nabla_h R_{ik}+K \tag{4.1}$$

where

$$K=2\gamma+\alpha+4\beta\,,$$

which is valid in Riemannian manifolds. He also showed that

$$-\alpha+2\beta=\frac{3S}{2m(m+1)}\left(\rho+\frac{1}{3m}S^2\right) \tag{4.2}$$

is valid in harmonic Kaehlerian manifolds. It was proved there that $|\nabla R|$, α and β are constant in harmonic Kaehlerian manifolds.

As a harmonic Kaehlerian manifold is Einsteinian, we have $R_{ji}=(S/2m)g_{ji}$ and

$$2\tau=S^2/m\,, \tag{4.3}$$

$$2\gamma=(S/m)\rho\,, \tag{4.4}$$

which and (4.2) together imply the following

PROPOSITION 5. *In an $n(=2m)$ dimensional harmonic Kaehlerian manifold $(m\geqq 2)$, the following formula holds:*

$$(m+1)(-\alpha+2\beta)-3\gamma=\frac{S}{m}\tau\,. \tag{4.5}$$

In the following, we assume that the harmonic Kaehlerian monifold M^{2m} under consideration is of positive PSC operator, and $m\geqq 5$. We calculate $(3.1)\times(m-5)+(3.2)\times(2m-1)+(4.5)\times(-2)$ by making use of Propositions 3, 4 and 5. The lefthand side becomes

$$(m-2)(\alpha+4\beta+2\gamma)=(m-2)K$$

and the righthand side becomes

$$\begin{aligned} &k(m-2)(m+1)\rho-2\left\{(m-5)k+\frac{S}{m}\right\}\tau \\ &=(m-2)(m+1)k\left(\rho-\frac{4}{m+1}\tau\right)-2(m+1)\left\{\frac{S}{m(m+1)}-k\right\}\tau \\ &=(m-2)(m+1)k|U|^2-2(m+1)\left\{\frac{S}{m(m+1)}-k\right\}\tau, \end{aligned}$$

where U is the tensor defined in Section 1 and $|U|^2=\rho-(2/m(m+1))S^2=\rho-(4/m+1)\tau$. Hence it holds that

$$(m-2)K\geqq(m+1)k\left\{(m-2)|U|^2-2\left(\frac{S}{m(m+1)k}-1\right)\tau\right\}.$$

We substitute (4.3) in the above inequality. Here, we notice that ρ is constant by virtue of (4.2). As $\nabla_j\nabla_h R_{ik}$ vanish, we have from (4.1)

$$|\nabla R|^2+K=0,$$

and

$$0\geqq-|\nabla R|^2=K\geqq(m+1)S^2k\left\{\frac{|U|^2}{S^2}-\frac{1}{m(m-2)}\left(\frac{S}{m(m+1)k}-1\right)\right\}. \tag{4.6}$$

Now, we assume

$$\frac{|U^2|}{S^2}\geqq\frac{1}{m(m-2)}\left(\frac{S}{m(m+1)k}-1\right).$$

Then the righthand side of (4.6) is non-negative, and it follows that $K=0$ i. e. M^{2m} is locally symmetric. Therefore we obtain the following

THEOREM. *A $2m(m\geqq5)$ dimensional harmonic Kaehlerian manifold with positive PSC operator is locally symmetric, if it satisfies the condition*

$$\frac{|U|^2}{S^2}\geqq\frac{1}{m(m-2)}\left(\frac{S}{m(m+1)k}-1\right),$$

where U is the curvature tensor given in Section 1, S is the scalar curvature.

References

[1] M. Berger, Sur les variétés à opérateur de courbure positif, C. R. Paris, **253** (1963), 2832–2834.

[2] E. Calabi and E. Vesentini, On compact, locally symmetric Kähler manifolds, Ann. of Math. **71** (1960), 472–507.

[3] S. Gallot, Sur les variétés à opérateur de courbure positif qui ont même première valeur propre du p-spectre que la sphère, C. R. Paris, **277** (1973), 457–459.

[4] S. Gallot et D. Meyer, Sur les variétés riemanniennes à opérateur de courbure positif ou nul, C. R. Paris, **274** (1972), 1557–1559.

[5] S. Gallot et D. Meyer, Sur la première valeur propre du p-spectre pour les variétés à opérateur de courbure positif, C. R. Paris, **276** (1973), 1619–1621.

[6] D. Meyer, Sur les variétés riemanniennes à opérateur de courbure positif, C. R. Paris, **272** (1971), 482–485.

[7] K. Ogiue and S. Tachibana, Les variétés riemanniennes dont l'opérateur de courbure restreint est positif sont des sphères d'homologie réelle, C. R. Paris, **289** (1979), 29–30.

[8] K. Ogiue and S. Tachibana, Kähler manifolds of positive curvature operator, Proc. Amer. Math. Soc. **78** (1980), 548–550.

[9] S. Tachibana, On the characteristic function of spaces of constant holomorphic curvature, Colloq. Math. **26** (1972), 149–155.

[10] S. Tachibana, A theorem on Riemannian manifolds of positive curvature operator, Proc. Japan Acad. **50** (1974), 301–302.

[11] S. Tachibana, On Kählerian manifolds of σ-positive curvature operator. Nat. Sc. Rep. Ochanomizu Univ. **25** (1974), 7–16.

[12] S. Tachibana and S. Yamaguchi, The first proper space of Δ for p-forms in compact Riemannian manifolds of positive curvature operator, J. Diff. Geom. **15** (1980), 51–60.

[13] Y. Watanabe, On the characteristic function of harmonic Kählerian spaces, Tohoku Math. J. **27** (1975), 13–24.

[14] S. Yamaguchi, On a theorem of Gallot-Meyer-Tachibana in Riemannian manifolds of positive curvature operator, TRU Math. **11** (1975), 17–22.

[15] K. Yano, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.