On the Incremental Integration

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(Received September 3, 1985)

Introduction. This is a continuation of our recent papers on integration theory. We shall be concerned with a new integration which generalizes that of Denjoy and which the author proposes to name incremental integration. By means of a concrete example it will be shown that this integration is strictly wider than that of Denjoy. It appears to us that on the whole the incremental integral is easier to handle than the ones hitherto considered by us, though there remains a number of undecided problems on the new integral.

§ 1. Functions IC, GIC, IR, GIR.

The sets considered in this paper will exclusively be linear. A *function*, by itself, will signify any mapping of the real line $\boldsymbol{R}$ into itself, unless another meaning is obvious from the context. We shall denote by $\boldsymbol{N}$ the set of the positive integers and by $\boldsymbol{M}$ that of the nonnegative integers. The letter δ will always represent a positive number, even when not specified so. A set $(\mathfrak{F}_\sigma)$ will synonymously be called *sigma-closed set*.

Given a function $\varphi(x)$, a compact set Q, and a number $\delta>0$, let H stand for a generic open interval (if existent) contiguous to Q. We shall denote by $\varphi(Q;\delta)$ the sum of the increments $\varphi(H)$ for all the H with $|H|>\delta$, where a possible void sum means zero. Thus defined, $\varphi(Q;\delta)$ is a finite real number, since there exists only a finite number of intervals H with $|H|>\delta$.

Given a function $\varphi(x)$ and a closed set S, let W denote a generic closed interval (if existent) whose end points belong to S. The function will be said to *vary incrementally* on S, if the following conditions are fulfilled, where $\delta>0$ as above:

(i) If the interval W is kept fixed, then

$$\varphi(S\cap W;\delta)\longrightarrow\varphi(W)\qquad\text{as}\quad\delta\longrightarrow 0.$$

(ii) There exists a positive number M independent of both W and δ, such that $|\varphi(S\cap W;\delta)|\leqq M$ for every W and every δ.

We find at once that when this is the case, the function $\varphi(x)$ must be a constant over each closed interval (if existent) contained in S.

Obviously, *if two functions vary incrementally on a closed set, so does also any linear combination, with constant coefficients, of these functions.*

EXAMPLE 1. We shall construct a function which varies incrementally on the compact set $Q=\{1, q_0, q_1, \cdots\cdot\}$, where $q_m=m/(m+1)$ for each $m\in\boldsymbol{M}$. Let $\varphi(x)$ be any function such that

$$\varphi(1)=\log 2 \quad \text{and} \quad \varphi(q_m)=\sigma_m \quad \text{for } m\in\boldsymbol{M},$$

σ_m denoting the mth partial sum of the infinite series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdot=\log 2,$$

so that $\sigma_0=0$ in particular. This function is clearly continuous on Q.

We remark firstly that the sequence $\langle q_n-q_{n-1}; n\in\boldsymbol{N}\rangle$ converges to 0 in a decreasing manner. In fact, we have

$$q_n-q_{n-1}=\frac{1}{n(n+1)} \qquad \text{for} \quad n\in\boldsymbol{N}.$$

Let us show that the function $\varphi(x)$ varies incrementally on the set Q, i.e. fulfils the above conditions (i) and (ii), where the set S is specialized to Q. Consider any closed interval $W=[u, v]$ with $u\in Q$ and $v\in Q$.

re (i): We distinguish two cases, according as $v<1$ or $v=1$. In the former case, $Q\cap W$ is a finite set and hence condition (i) is trivial. In the latter case, the result is immediate from the remark made just now.

re (ii): Keeping W and δ fixed, we shall show that $|\varphi(Q\cap W; \delta)|\leqq 1$. We may plainly suppose that $\varphi(Q\cap W; \delta)\neq 0$. On account of the above remark, there then exist two integers i and j such that $0\leqq i<j$ and that $\varphi(Q\cap W; \delta)=\sigma_j-\sigma_i$. But we have $0\leqq\sigma_m\leqq 1$ for every $m\in\boldsymbol{M}$, as is readily verified. Hence the result.

EXAMPLE 2. Consider the infinite sequence $\langle a_1, a_2, \cdots\cdot\rangle$, where

$$a_{2n-1}=\frac{4}{n(n+1)} \quad \text{and} \quad a_{2n}=\frac{1}{n(n+1)}$$

for $n\in\boldsymbol{N}$. This sequence has the property that

$$a_{4n-3}>a_{4n-1}>a_{2n}>a_{4n+1} \qquad \text{for} \quad n\in\boldsymbol{N}.$$

Further, if r_m denotes for each $m\in\boldsymbol{M}$ the mth partial sum of the infinite series $a_1+a_2+\cdots\cdot$, so that $r_0=0$ in particular, we have

$$\lim_{m\to\infty} r_m=\sum_{n=1}^{\infty} a_n=5\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=5.$$

Hence the set $R=\{5, r_0, r_1, \cdots\}$ is compact.

The function $\varphi(x)$ of Example 1 is continuous on the set Q. We shall now construct a function which varies incrementally on the set R without being continuous on R.

Let $\psi(x)$ be any function such that

$$\psi(5)=\log 2\sqrt{2} \quad \text{and} \quad \psi(r_m)=\sigma_m \quad \text{for } m\in \boldsymbol{M},$$

where σ_m has the same meaning as in Example 1. This function is evidently discontinuous at $x=5$. We proceed to show that the function $\psi(x)$ fulfils the conditions (i) and (ii), where the set S is specialized to R.

re (i): Consider any closed interval $W=[u, v]$ with u and v belonging to R. We shall show that $\psi(R\cap W;\delta)\longrightarrow\psi(W)$ as $\delta\to 0$. It suffices to treat the case in which $v=5$. Without loss of generality we may assume further that $u=0$, so that $W=[0,5]\supset R$. On account of the relation $a_{4n-3}>a_{4n-1}>a_{2n}>a_{4n+1}$ mentioned above, we find easily that

$$\begin{aligned}\lim_{\delta\to 0}\psi(R;\delta)&=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)\\ &=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-2}+\frac{1}{4n-1}-\frac{1}{4n}\right)+\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)\\ &=\frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\frac{3}{2}\log 2=\psi(5)=\psi(W).\end{aligned}$$

re (ii): Denoting the nth partial sum of the infinite series

$$1+\frac{1}{3}-\frac{1}{2}+\cdots+\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}+\cdots$$

by τ_n for each $n\in\boldsymbol{N}$, we have $0<\tau_n<2$. This is an immediate consequence of the following inequalities:

$$-\frac{1}{2n}+\frac{1}{4n+1}+\frac{1}{4n+3}<0<\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}.$$

This being premised, we shall prove first that if $v\in R$, $0<v<5$, and $V=[0, v]$, then $|\psi(R\cap V;\delta)|<2$ for every $\delta>0$. Keeping v and δ fixed, let us write for short $A=\psi(R\cap V;\delta)$. We may plainly assume that $A\neq 0$. Let us denote by q the largest of the integers $n>0$ such that $r_n\leqq v$ and $r_n-r_{n-1}=a_n>\delta$, the existence of q being ensured by $A\neq 0$. The integer q is expressible, by means of a $k\in\boldsymbol{N}$, in one of the forms

$$q=2k-1, \quad \text{or } q=4k-2, \quad \text{or } q=4k.$$

Since $a_1>a_3>a_2>\cdots>a_{4n-3}>a_{4n-1}>a_{2n}>\cdots$, we find without difficulty that according to the above three forms of q, the quantity A is expressed respectively as follows:

(1) If $q=2k-1$, then $A=\tau_{2k-1}$;

(2) If $q=4k-2$, then $\quad A=\tau_{4k-3}-\frac{1}{2}\sum_{m=0}^{k-1}\frac{1}{k+m}\geqq\tau_{4k-3}-\frac{1}{2}$;

(3) If $q=4k$, then $\quad A=\tau_{4k-1}-\frac{1}{2}\sum_{m=0}^{k}\frac{1}{k+m}>\tau_{4k-1}-1$.

This, combined with the relation $0<\tau_n<2$ obtained already, leads at once to $-1<A<2$. The announced inequality $|A|<2$ is thus verified.

It is now easy to prove the validity of $|\psi(R\cap W;\delta)|<4$ for every δ and every interval $W=[u,v]$ such that $u\in R$, $v\in R$, and $u<v<5$. In fact, if $u>0$, then we evidently have

$$\psi(R\cap W;\delta)=\psi(R\cap V;\delta)-\psi(R\cap U;\delta),$$

where we write for brevity $U=[0,u]$ and $V=[0,v]$. Hence it follows at once, from what we established already, that $|\psi(R\cap W;\delta)|<4$. But this holds for $u=0$ also, since $\psi(R\cap W;\delta)=\psi(R\cap V;\delta)$ in this case.

We pass on finally to the remaining case in which $u\in R$, $u<5$, and $W=[u,5]$. Writing $A=\psi(R\cap W;\delta)$, we shall derive the appraisal $|A|<4$ for each W and each $\delta>0$. We may assume that $A\neq 0$. If $u=0$, there is an $n\in \boldsymbol{N}$ for which $A=\psi(R;\delta)=\tau_n$, so that we have $0<A<2$. We may therefore suppose u positive. Writing $U=[0,u]$, we can express A in the form $A=\psi(R;\delta)-\psi(R\cap U;\delta)$. But we have

$$0\leqq\psi(R;\delta)<2 \quad\text{and}\quad |\psi(R\cap U;\delta)|<2$$

by what was already proved. We thus obtain $|A|<4$.

To conclude, we have $|\psi(R\cap W;\delta)|<4$ for every $\delta>0$ and every closed interval whose end points belong to R.

We shall say that a function is *incrementally continuous*, or IC for short, on a closed set S, if the function is continuous on S and varies incrementally on every closed set T contained in S. When this is the case, the function is evidently IC on every such set T. In other words, as a property of a function, the incremental continuity on a closed set is hereditary with respect to this set. Again, *if two functions are incrementally continuous on a closed set, the same is true of any linear combination, with constant cofficients, of these functions.*

We do not know whether *the continuity of the function on the set S is superfluous in the above definition of the incremental continuity.*

EXAMPLE 3. We shall show that the function $\varphi(x)$ of Example 1 is not IC on the set $Q=\{1,q_0,q_1,\cdots\}$. For this purpose, consider the set R consisting of 1 and of the numbers q_{4m} and q_{4m+3}, where $m\in\boldsymbol{M}$. Since

$q_n - q_{n-1} = \{n(n+1)\}^{-1}$ for $n \in \boldsymbol{N}$, the three points $q_{4m} < q_{4m+3} < q_{4m+4}$ of R determine two open intervals of respective length

$$q_{4m+3} - q_{4m} = \frac{1}{4m+1} - \frac{1}{4m+4} = \frac{3}{4(m+1)(4m+1)},$$

$$q_{4m+4} - q_{4m+3} = \frac{1}{(4m+4)(4m+5)} = \frac{1}{4(m+1)(4m+5)}.$$

In what follows, we shall write for short

$$f(m) = q_{4m+4} - q_{4m+3} \quad \text{and} \quad g(m) = q_{4m+3} - q_{4m}.$$

Clearly, both $f(m)$ and $g(m)$ are decreasing functions of $m \in \boldsymbol{M}$.

Let us now associate with each $k \in \boldsymbol{N}$ a number δ_k subject to the condition $f(2k) > \delta_k > f(2k+1)$. If k is fixed, the inequality $f(m) > \delta_k$ has exactly $2k+1$ solutions $m = 0, 1, \cdots, 2k$. On the other hand, we have $g(m) > \delta_k$ for $m = 0, 1, \cdots, 3k$ at least, since

$$g(3k) = \frac{3}{4(3k+1)(12k+1)} > \frac{1}{4(2k+1)(8k+5)} = f(2k) > \delta_k.$$

This being so, we go on to appraise the quantity $\varphi(R; \delta_k)$ from below for $k \in \boldsymbol{N}$. Noting the relation

$$\sigma_{4m+3} - \sigma_{4m} = \frac{1}{4m+1} - \frac{1}{4m+2} + \frac{1}{4m+3} > \frac{1}{4m+2},$$

we find successively that

$$\begin{aligned}\varphi(R; \delta_k) &\geqq \sum_{m=0}^{2k} (\sigma_{4m+4} - \sigma_{4m+3}) + \sum_{m=0}^{3k} (\sigma_{4m+3} - \sigma_{4m}) \\ &= \sum_{m=0}^{2k} (\sigma_{4m+4} - \sigma_{4m}) + \sum_{m=2k+1}^{3k} (\sigma_{4m+3} - \sigma_{4m}) \\ &> \sigma_{8k+4} + \sum_{m=2k+1}^{3k} \frac{1}{4m+2} > \sigma_{8k} + \frac{1}{14}.\end{aligned}$$

Suppose, if possible, that the function $\varphi(x)$ is IC on the set Q. Then we have $\varphi(R \cap W; \delta) \longrightarrow \varphi(W)$ as $\delta \to 0$, whenever W is a closed interval whose end points belong to R. Choosing W to be $[0, 1]$, so that $R \cap W = R$ and $\varphi(W) = \varphi(1) = \log 2$, we find that

$$\varphi(W) = \lim_{\delta \to 0} \varphi(R; \delta) = \lim_{k \to \infty} \varphi(R; \delta_k) \geqq \lim_{k \to \infty} \left(\sigma_{8k} + \frac{1}{14}\right) = \log 2 + \frac{1}{14},$$

which is a contradiction. This shows that the function $\varphi(x)$ is not IC on the set Q.

We now widen slightly the notion of condition (B) as defined on p. 4 of [1]. Given a closed set S, we denote by H a generic open interval (if existent) contiguous to S, and by W an arbitrary closed interval (if existent) whose end points belong to S. A function $\varphi(x)$ will be said to fulfil the *condition* (B) on S, if

$$\sum_{H}|\varphi(H)|<+\infty \quad \text{and if} \quad \sum_{H\subset W}\varphi(H)=\varphi(W)$$

for every W, where in each series a possible void sum means zero. We find easily that this property of $\varphi(x)$ is hereditary with respect to the set S.

In the special case in which the set S is compact and nonconnected, the new condition (B) plainly reduces to the old one, so that there is no fear of ambiguity or confusion.

Each of the following three theorems admits a proof similar to that of the corresponding result of the paper [1].

THEOREM 1. *Every function which is* BV *and varies incrementally, on a closed set, fulfils the condition* (B) *on this set.*

THEOREM 2. *Every function which fulfils the condition* (B) *on a closed set, is both* AC *and* IC, *on this set.*

THEOREM 3. *Every function which is both* GBV *and* IC, *on a closed set, is* GAC *on this set.*

THEOREM 4. *Every function* $\varphi(x)$ *which is* IC *on a closed set* S *and* AD *at all points of a set* $E\subset S$, *maps* E *onto a null set.*

PROOF. The function $\varphi(x)$ is GAC on the set E by a well-known theorem (see Saks [5], p. 239). Hence E is the union of a sequence of bounded sets on each of which $\varphi(x)$ is AC. Let M be any one of these sets. It is enough to show that $|\varphi[R]|=0$, where R denotes the closure of M. The function $\varphi(x)$, which is clearly AC on R, must be BV on R, since R is a bounded set. By hypothesis, on the other hand, $\varphi(x)$ varies incrementally on R. Consequently $\varphi(x)$ satisfies the condition (B) on R, on account of Theorem 1. It thus follows from Theorem 5 of [1] that if R is not connected, we have $|\varphi[R]|=0$. But this holds for connected R also, since $\varphi(x)$ is then a constant on R by the above definition of the condition (B). This completes the proof.

By an argument similar to the above, we can show that Theorem 4 has the following analogue for Dirichlet continuous functions: *Every function which is Dirichlet continuous on a compact nonconnected set* Q *and* AD *at all points of a set* $E\subset Q$, *maps* E *onto a null set.* This result includes Theorem 11 of [3] whose proof was unnecessarily prolix (occupying about two pages).

We do not know whether *every function which is* IC *on a closed set, maps this set onto a null set and is* AD *at almost all points of this set.* The main result that we have at present on this assertion is the following theorem, whose proof is the same as that for Theorem 12 of [3], except that the above Theorem 4 is used in place of Theorem 11 of [3].

THEOREM 5. *In order that a function which is* IC *on a compact set* Q, *should carry* Q *onto a null set, it is necessary and sufficient that the function be* AC *superposable on* Q.

Starting with the unit interval $E_0=[0,1]$, let us now construct a descending infinite sequence of figures, $\langle E_0, E_1, \cdots\cdot\rangle$, by the inductive rule $E_{m+1}=E_m(3)$, where $m\in \boldsymbol{M}$ and where $E_m(3)$ means the 3-sized ramification of the figure E_m (see [1], p. 22). The intersection $\Gamma=E_0\cap E_1\cap\cdots\cdot$ is then a perfect set of measure zero, and E_0 is the minimal closed interval containing Γ.

THEOREM 6. *There exists a function that is* IC *on the set* Γ *without being* GBV *on any portion of* Γ.

PROOF. Let ξ be any point of Γ. For each $m\in \boldsymbol{M}$, there is among the component intervals of the figure E_m exactly one, say K_m, to which the point ξ belongs. The intersection $K_m\cap E_{m+1}$, which coincides with the ramification $K_m(3)$, is the union of three disjoint intervals, say $A<B<C$ in their natural ordering, and the interval K_{m+1} is one of them. We now define $w_{m+1}(\xi)$ to be 1 or 0, according as $K_{m+1}=B$ or not, respectively. Using the binary sequence $\langle w_n(\xi); n\in \boldsymbol{N}\rangle$ thus obtained, we construct on the set Γ a function $\Omega(\xi)$, writing by definition

$$\Omega(\xi)=\sum_{n=1}^{\infty}\frac{w_n(\xi)}{n}\left(\frac{1}{3}\right)^n.$$

We then extend the domain of definition of this function from Γ to $\boldsymbol{R}$, in such a manner that the extended function, still denoted by Ω, vanishes outside $E_0=[0,1]$ and is linear on each closed interval contiguous to Γ.

We proceed to show that the function $\Omega(x)$, thus defined over $\boldsymbol{R}$, conforms to the statement of the theorem.

In the first place, $\Omega(x)$ is a continuous function such that

$$\mathrm{O}(\Omega;K)<\frac{2}{m+1}\left(\frac{1}{3}\right)^{m+1},$$

where $m\in \boldsymbol{M}$ and K is any component of the figure E_m. In fact, as is readily seen, Lemma 6 of [3] and its proof hold good, *mutatis mutandis*, for our function $\Omega(x)$ also. Moreover, inspecting Lemma 7 of [3] and its proof, we find that $\Omega(x)$ is not BV upon the set $\Gamma \cap K$ for any component K of E_m, where $m \in \boldsymbol{M}$. From this it follows by a routine inference that $\Omega(x)$ is not GBV on $\Gamma \cap K$ for any K, or what comes to the same thing, that $\Omega(x)$ is not GBV on any portion of Γ.

We shall now ascertain that the function $\Omega(x)$ is incrementally continuous on the set Γ. For this purpose, suppose that the end points of a closed interval W belong to a closed set Q contained in the intersection $\Gamma \cap W$. We shall prove first that $\Omega(Q;\delta) \longrightarrow \Omega(W)$ as $\delta \rightarrow 0$.

More precisely, it will be shown that $|\Omega(W)-\Omega(Q;\delta)|<4n^{-1}$, if $n \in \boldsymbol{N}$ and $5^{-n} \leqq \delta < 5^{1-n}$. Let H denote a generic open interval contiguous to the set Q and let k be the number of the intervals H with $|H|>\delta$. We shall distinguish three cases, according as $k \geqq 2$, $k=1$, or $k=0$.

Consider the case in which $k \geqq 2$ and arrange all the H with $|H|>\delta$ in a sequence $H_1 < \cdots\cdot < H_k$ in their natural ordering. For $i=1, \cdots\cdot, k$ we write $H_i=(p_i, q_i)$ and we choose from among the components of the figure E_{n-1} that one, say M_i, to which the point p_i belongs. Then the point q_{i-1} necessarily belongs to this interval M_i for each $i=2, \cdots\cdot, k$. To see this, select from among the components of E_{n-1} that one, say L_{i-1}, to which q_{i-1} belongs. Since $q_{i-1} \leqq p_i$, we must have either $L_i=M_i$ or $L_i<M_i$. To show the truth of $L_i=M_i$, let us suppose, if possible, that $L_i<M_i$, so that $q_{i-1}<p_i$. From the construction of the sequence $E_0 \supset E_1 \supset \cdots\cdot$ we find easily that the distance between any two different components of E_{n-1} is at least 5^{1-n}. It follows that the interval $[q_{i-1}, p_i]$ must contain at least one open interval H^* with $|H^*| \geqq 5^{1-n}$ and contiguous to the set Q. We then have $H_{i-1}<H^*<H_i$, and this relation, together with the hypothesis $5^{-n} \leqq \delta < 5^{1-n}$, contradicts the definition of the sequence $H_1 < \cdots\cdot < H_k$. We thus have $L_i=M_i$, so that $q_{i-1} \in M_i$.

If we write $W=[p, q]$, then $\{p, q\} \subset Q \subset \Gamma \cap W$ by hypothesis and we thus have $p \leqq p_1 < q_1 \leqq \cdots\cdot \leqq p_k < q_k \leqq q$, where for each $i=2, \cdots\cdot, k$ the set $\{q_{i-1}, p_i\}$ is contained in one and the same component of the figure E_{n-1}, as shown just now. But this is true of the sets $\{p, p_1\}$ and $\{q_k, q\}$ also, the verification being the same as above.

Now we have $\mathrm{O}(\Omega; K)<2n^{-1}3^{-n}$ for each compont K of E_{n-1}, as stated already. It follows successively that

$$\Omega(W)-\Omega(Q;\delta)=\Omega(W)-\Omega(H_1)-\cdots\cdot-\Omega(H_k)$$
$$=\sum_{i=2}^{k}\{\Omega(p_i)-\Omega(q_{i-1})\}+\{\Omega(p_1)-\Omega(p)\}+\{\Omega(q)-\Omega(q_k)\},$$
$$|\Omega(W)-\Omega(Q;\delta)|<2(k+1)n^{-1}3^{-n}.$$

On the other hand, since E_{n-1} is the union of exactly 3^{n-1} components of equal length 5^{1-n}, there are exactly $3^{n-1}-1$ open intervals contiguous to

E_{n-1} and further each component of E_{n-1} can contain at most five of the intervals H such that $|H|>\delta\geqq 5^{-n}$. Hence $k+1\leqq 5\cdot 3^{n-1}+3^{n-1}=6\cdot 3^{n-1}$, and it follows that $|\Omega(W)-\Omega(Q;\delta)|<4n^{-1}$.

Passing to the case in which $k=1$, denote by $H_1=(p_1, q_1)$ the uniquely determined interval H with $|H|>\delta$. Then

$$\Omega(W)-\Omega(Q;\delta)=\{\Omega(p_1)-\Omega(p)\}+\{\Omega(q)-\Omega(q_1)\},$$

where $W=[p, q]$. Since each of the sets $\{p, p_1\}$ and $\{q_1, q\}$ is contained in a component of E_{n-1}, we have

$$|\Omega(W)-\Omega(Q;\delta)|<4n^{-1}3^{-n}<2n^{-1}.$$

Suppose finally that $k=0$. Then $\Omega(Q;\delta)=0$ and the interval W is contained in a component of E_{n-1}. Consequently

$$|\Omega(W)-\Omega(Q;\delta)|=|\Omega(W)|<2n^{-1}3^{-n}<n^{-1}.$$

This completes the proof of $|\Omega(W)-\Omega(Q;\delta)|<4n^{-1}$, where $5^{-n}\leqq\delta<5^{1-n}$.

The closed interval W and the closed set $Q\subset\Gamma\cap W$ have been kept fixed hitherto. We now make them vary arbitrarily, assuming of course that the end points of W belong to Q. If $0<\delta<1$, there exists an $n\in\boldsymbol{N}$ such that $5^{-n}\leqq\delta<5^{1-n}$. We therefore have $|\Omega(W)-\Omega(Q;\delta)|<4n^{-1}\leqq 4$. On the other hand, if $\delta\geqq 1$, then $\Omega(Q;\delta)=0$; so that

$$|\Omega(W)-\Omega(Q;\delta)|=|\Omega(W)|\leqq \mathrm{O}(\Omega;E_0)<2/3.$$

Thus $|\Omega(Q;\delta)|\leqq 4+|\Omega(W)|<5$ for every choice of W, Q, and δ.

The theorem is thus established.

A function will be called *generalized incrementally continuous*, or simply GIC, on a linear set E, if the function is continuous on E and if E is expressible as the union of a sequence of closed sets on each of which the function is IC.

When this is the case, E must be a sigma-closed set and the function is GIC on every sigma-closed set contained in E. The GIC property of a function is hereditary in this sense. Again, *every linear combination of two functions which are* GIC *on a set, is itself* GIC *on this set.*

If E is a countable set, every function is GIC on E. For instance, the function $\varphi(x)$ of Example 1 is GIC on the set $Q=\{1, q_0, q_1, \cdots\}$ which is countable, though this function is not IC on Q.

We can establish the following theorem in almost the same way as in Theorem 11 of [1].

THEOREM 7. *In order that a function which is continuous over a nonvoid closed set S, be* GIC *on S, it is necessary and sufficient that every* *nonvoid closed subset of* S *contain a portion on whose closure the function is* IC.

A function $\varphi(x)$ will be termed *incrementally regular,* or simply IR, on a closed set S, if the function is continuous on S and if there corresponds to each closed set $Q\subset S$ a function $\psi_Q(x)$ fulfilling the conditions:

(i) the function $\psi_Q(x)$ varies incrementally on the set Q,

(ii) the function $\psi_Q(x)$ is AD to zero at almost all points of Q,

(iii) the function $\varphi(x)-\psi_Q(x)$ is AC on Q.

This property of $\varphi(x)$ is plainly hereditary with respect to the set S. Further, any function $\varphi(x)$ with this property is AD at almost all points of S, as readily seen by specializing the set Q to S itself. On the other hand, *any function which is* AC *on a closed set, is* IR *on this set.* Again, *every function which is* IC *on a closed set* S, *is* IR *on* S, *provided that the funtion is* AD *to zero at almost all points of* S (especially, provided that S is a null set).

We do not know whether *the continuity of* $\varphi(x)$ *on* S *is superfluous in the above definition of the incremental regularity.*

We shall say that a function is *generalized incrementally regular,* or briefly GIR, on a set E, if the function is continuous on E and if E is expressible as the union of a sequence of closed sets on each of which the function is IR.

When this is the case, E must be a sigma-closed set and the function is GIR on every sigma-closed set contained in E. The GIR property of a function is hereditary in this sense. Again, we find easily that *a function which is* GIR *on a set, is necessarily* AD *at almost all points of this set.* On the other hand, *every function which is* GAC *on a sigma-closed set, is* GIR *on this set.* Finally, *each function which is* GIC *on a set* E, *is* GIR *on* E, *provided that the function is* AD *to zero at almost all points of* E.

The following two theorems are immediate.

THEOREM 8. *Every linear combination of two functions which are* IR [*or* GIR] *on a closed set* [*or a set*], *is itself so on this set.*

THEOREM 9. *In order that a function which is continuous on a nonvoid closed set* S *be* GIR *on* S, *it is necessary and sufficient that every nonvoid closed subset of* S *contain a portion on whose closure the function is* IR.

Of the following three theorems, the second one may be established in the same way as in Theorem 6 of [3], while each of the other two admits a proof similar to that of the corresponding theorem of [4].

THEOREM 10. *A function which is both* GBV *and* GIR, *on a set, is necessarily* GAC *on this set.*

THEOREM 11. *Suppose that a function $F(x)$ is* AC *on a closed set S and a function $G(x)$ varies incrementally on S. If $F(x)$ is* AD *to $F'_{\mathrm{ap}}(x) \geqq 0$ at almost every point of S and if the function $\xi(x)=F(x)+G(x)$ has a nonnegative increment over each closed interval contiguous to S, then $\xi(x)$ is monotone nondecreasing over S.*

THEOREM 12. *Given two functions $\varphi(x)$ and $\psi(x)$ which are* GIR *on a closed interval I, if we have $\varphi'_{\mathrm{ap}}(x) \leqq \psi'_{\mathrm{ap}}(x)$ at almost every point of I at which both $\varphi(x)$ and $\psi(x)$ are* AD, *then the function $\psi(x)-\varphi(x)$ is* AC *and nondecreasing, on the interval I.*

Consequently, if two functions are GIR *on a closed interval I and if they are approximately equiderivable almost everywhere on I, then the functions differ over I only by an additive constant.*

THEOREM 13. *Suppose that a function $F(x)$ is* GAC *on a set E and that a function $G(x)$ is* GIC *on E. If in addition the function $G(x)$ is* AD *to zero at almost all points of E, then the function $\varphi(x)=F(x)+G(x)$ is* GIR *on E.*

The proof is easy and may be omitted.

Let us consider the special case of this theorem in which the set E is a closed interval I. Then the function $G(x)$ is GIR on I, and Theorem 12 requires that $G(x)$ is a constant, say C, over I. Consequently we have $\varphi(x)=F(x)+C$ on I, so that the function $\varphi(x)$ must be GAC on I.

Thus a function which is not GAC on I, even though it is GIR on I, is not expressible on I in the form $F(x)+G(x)$, where $F(x)$ is GAC on I and $G(x)$ is GIC on I as well as AD to 0 almost everywhere on I. The function $\Omega(x)$ constructed in the proof of Theorem 6 is an example of such a function, the interval I being $[0, 1]$.

§ 2. The incremental integration.

We are now in a position to state the descriptive definition of the incremental integration. A function $f(x)$ will be termed *incrementally integrable* on a closed interval I, if there exists a function $\varphi(x)$ which is GIR on I and which has $f(x)$ for its approximate derivative almost everywhere on I. Any such function $\varphi(x)$ is then called *indefinite incremental integral* of $f(x)$ on I. By the *definite incremental integral* of $f(x)$ over I we shall mean the increment $\varphi(I)$ of its indefinite integral $\varphi(x)$. This number $\varphi(I)$, which is uniquely determined by the function $f(x)$ and the interval I on account of Theorem 12, will be denoted by $(\mathfrak{J})\int_I f(x)\,dx$.

All the properties, except Theorem 19, of the powerwise integral that are stated on pp. 16–17 of [1] are shared also by the incremental integral, as may immediately be seen. On the other hand, the function $\Omega(x)$ constructed in the proof of Theorem 6 shows that the incremental integration is strictly wider than the Denjoy integration.

We shall now proceed to obtain the integration by parts theorem and the second mean value theorem for the incremental integral. For this purpose, we establish first the following basic theorem. We remark that a function is called *of bounded variation*, if it is BV on every closed interval (see Saks [5], p. 59).

THEOREM 14. *Given a function $M(x)$ of bounded variation and given a function $C(x)$ continuous on a closed interval $I=[a,b]$, let $K(x)$ be any function such that*

$$K(x)-K(a)=\int_a^x M(t)\,dC(t) \qquad \textit{for} \quad x\in I,$$

where the integral is a Riemann-Stieltjes one. If the function $C(x)$ varies incrementally on a closed set $Q\subset I$ and is linear on each closed interval contiguous to Q, then the function $K(x)$ varies incrementally on Q, too.

PROOF. We may assume the set Q nonconnected, for otherwise $C(x)$ is a constant on Q and hence so must also be the function $K(x)$. Plainly, we may restrict ourselves to the case where the function $M(x)$ is nondecreasing over $\boldsymbol{R}$ and where I is the minimal closed interval containing Q. We suppose further that $C(a)=0$, as we clearly may.

In what follows, let H denote a generic open interval contiguous to Q. Writing $\theta(H)=C(H)\cdot|H|^{-1}$, we associate with each $\delta>0$ a function $p_\delta(x)$ defined as follows:

(a) $p_\delta(x)=\theta(H)$, if x belongs to an H with $|H|>\delta$;

(b) $p_\delta(x)=0$, if x belongs to no such H.

Using the function $p_\delta(x)$ which is evidently summable over $\boldsymbol{R}$, we define further a function $P_\delta(x)$ by

$$P_\delta(x)=\int_a^x p_\delta(t)\,dt \qquad \text{for} \quad x\in\boldsymbol{R}.$$

Let W denote hereafter a generic closed interval whose end points belong to Q. Then $P_\delta(W)=C(Q\cap W;\delta)$ for each W and each $\delta>0$, where $C(Q\cap W;\delta)$ denotes the sum of $C(H)$ for all $H\subset Q\cap W$ with $|H|>\delta$, as in §1. In point of fact, we have

$$P_\delta(W)=\sum_{H\subset W}{}'\theta(H)\cdot|H|=\sum_{H\subset W}{}'C(H)=C(Q\cap W;\delta),$$

where the prime indicates the additional condition $|H|>\delta$.

Let ξ be a point of the interval I. We shall show that if ξ is kept fixed, $P_\delta(\xi)$ tends to $C(\xi)$ as $\delta\to 0$. Since $P_\delta(a)=0=C(a)$, we may suppose $a<\xi\leqq b$. By hypothesis the function $C(x)$ is linear on the closure of each H, and the same is true of the function $P_\delta(x)$ by its construction. We may therefore assume further that $\xi\in Q$. Writing $L=[a,\xi]$ for brevity, we find that $P_\delta(\xi)=P_\delta(L)=C(Q\cap L;\delta)$. But this last quantity tends to $C(L)=C(\xi)-C(a)=C(\xi)$ as $\delta\to 0$, since $C(x)$ varies incrementally on Q by hypothesis.

The same hypothesis ensures the existence of a number $C_0>0$ independent of both W and δ, such that $|C(Q\cap W;\delta)|\leqq C_0$ for every W and every δ. Then, specializing W to the above interval $L=[a,\xi]$, we obtain $|P_\delta(\xi)|\leqq C_0$. It follows that $|P_\delta(x)|\leqq C_0$ for every $x\in I$.

Writing $W=[r,s]$, where $r\in Q$, $s\in Q$, and $r<s$, let us consider the function $N(x)$ coinciding on W with $M(x)$ and such that

$$N(x)=M(r)\ \text{for}\ x<r,\qquad N(x)=M(s)\ \text{for}\ x>s.$$

Plainly, $N(x)$ is a nondecreasing function continuous from the left at $x=r$ and from the right at $x=s$. Integrating by parts the right-hand side of $K(W)=\int_W M(x)\,dC(x)$, we find that

$$K(W)=\Big[M(x)C(x)\Big]_r^s-\int_W C(x)\,dM(x),$$

where the Riemann-Stieltjes integral on the right can be transformed into a Lebesgue-Stieltjes one, as follows:

$$\int_W C(x)\,dM(x)=\int_W C(x)\,dN(x)=(\mathfrak{S})\int_W C(x)\,dN(x)=\int_W C(x)\,dN^*(x).$$

Hence
$$K(W)=\Big[M(x)C(x)\Big]_r^s-\int_W C(x)\,dN^*(x).$$

If we replace the function $C(x)$ by $P_\delta(x)$, we obtain similarly

$$\int_W M(x)\,dP_\delta(x)=\Big[M(x)P_\delta(x)\Big]_r^s-\int_W P_\delta(x)\,dN^*(x).$$

As already shown, however, we have the relations

$$|P_\delta(x)|\leqq C_0\quad\text{and}\quad\lim_{\delta\to 0}P_\delta(x)=C(x)$$

for $x\in I$. Hence Lebesgue's Convergence Theorem gives

$$\lim_{\delta\to 0}\int_W M(x)\,dP_\delta(x)=K(W).$$

Applying Lemma 3 of [4] to the integral on the left, we get

$$\int_W M(x)\,dP_\delta(x)=\int_W M(x)p_\delta(x)\,dx,$$

where $p_\delta(x)=0$ for $x\in Q$. Consequently

$$\int_W M(x)\,dP_\delta(x)=\sum_{H\subset W}{}'\int_H M(x)p_\delta(x)\,dx=\sum_{H\subset W}{}'\theta(H)\int_H M(x)\,dx,$$

where the prime indicates, as before, the additional condition $|H|>\delta$. But the function $C(x)$ is linear on $[u, v]$, if we write $H=(u, v)$. Hence

$$K(H)=\int_u^v M(x)\,dC(x)=\theta(H)\int_H M(x)\,dx,$$

$$K(Q\cap W;\delta)=\sum_{H\subset W}{}'K(H)=\int_W M(x)\,dP_\delta(x).$$

Combining the above results, we find that

$$K(Q\cap W;\delta)\longrightarrow K(W)\quad\text{as}\quad\delta\to 0.$$

It remains to show the existence of a number $K_0>0$ independent of both W and δ, such that $|K(Q\cap W;\delta)|<K_0$ for every W and every δ. Writing again $W=[r, s]$ and transforming the above integral expression for $K(Q\cap W;\delta)$, we obtain

$$K(Q\cap W;\delta)=\Big[M(x)P_\delta(x)\Big]_r^s-\int_W P_\delta(x)\,dM(x).$$

Since the function $M(x)$ is nondecreasing, $|M(x)|$ has an upper bound $M_0>0$ on the interval I. On the other hand, we have $|P_\delta(x)|\leqq C_0$ for $x\in I$, as already shown. We can therefore choose as K_0 the number $4C_0M_0>0$. This completes the proof.

The reader will have noticed the important role played by the condition (ii) on p. 131 in the above proof.

Each of the following two theorems admits a proof quite similar to that of the corresponding theorem of [4].

THEOREM 15. *If $M(x)$ is a function of bounded variation and if $f(x)$ is a function incrementally integrable on a closed interval $I=[a, b]$, then the function $M(x)f(x)$ shares this latter property with $f(x)$, and moreover denoting by $F(x)$ any idefinite incremental integral of $f(x)$ on I, we have*

$$(\mathfrak{I})\int_I M(x)f(x)\,dx=\Big[M(x)F(x)\Big]_a^b-\int_I F(x)\,dM(x).$$

THEOREM 16. *If $M(x)$ is a nondecreasing function and if $f(x)$ is a function incrementally integrable on a closed interval $I=[a, b]$, there necessarily exists a point $\xi \in I$ such that*

$$\int_a^b M(x)f(x)\,dx = M(a)\int_a^\xi f(x)\,dx + M(b)\int_\xi^b f(x)\,dx,$$

where each integral is an incremental one.

AMENDMENTS. (i) The estimation at the end of §1 of [2] ought to be substituted by the following one which is simpler:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} > \int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}.$$

(ii) In the proof of Lemma 11 of [3], we used Lemma 9 to obtain the appraisal $d \leqq 3^p(p-1)\omega^p(D)$ for the quantity

$$d = |\varphi([\alpha', \beta'])|^p - |\varphi([\alpha', \alpha])|^p - |\varphi([\alpha, \beta])|^p - |\varphi([\beta, \beta'])|^p.$$

But the evident appraisal $d \leqq |\varphi([\alpha', \beta'])|^p \leqq \omega^p(D)$ serves our purpose quite as well, the first inequality of Lemma 11 being then replaced by

$$\Lambda(\varphi; p; Q) - \Lambda(\varphi; p; Q^*) \leqq \sum_D \omega^p(D).$$

Thus Lemma 9 was superfluous in [3].

References

[1] K. Iseki: On the Powerwise Integration. Nat. Sic. Rep. Ochanomizu Univ., **35** (1984), 1-46.

[2] K. Iseki: On the Dirichlet Continuity of Functions. Nat. Sci. Rep. Ochanomizu Univ., **36** (1985), 1-13.

[3] K. Iseki: On the Powerwise Integration in the Wide Sense. Nat. Sci. Rep. Ochanomizu Univ., **36** (1985), 15-39.

[4] K. Iseki: On an Integration Called Dirichlet Totalization. Nat. Sci. Rep. Ochanomizu Univ., **36** (1985), 115-130.

[5] S. Saks: Theory of the Integral. Warszawa-Lwów 1937.