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Remarks on Type 0 Cylindrical Measures

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For a probability measure μ defined on a finite dimensional space R^n, we have another measure as follows:

$$\lambda(\cdot)=\int_{u\in O(n)}\mu(u(\cdot))dm_n(u),$$

where $O(n)$ is the family of all unitary operators on R^n and m_n is the normalized Haar measure defined on $O(n)$. Thus we can make a rotationally invariant measure λ from μ. λ is an average of μ with respect to rotations.

Can we have the same result about cylindrical measures on an infinite dimensional space?

The answer is "yes" for special kinds of cylindrical measures, for example, (strongly) rotationally quasi-invariant ((S)RQI) cylindrical measures (in fact, these coincide with RQI-cylindrical measures [5]).

In the former half part of this paper, we treat the above problem for type 0-cylindrical measures. We present the main result in chapter 1.

The latter half part gives a detailed account of the relation between Shimomura's result ([5]) and GRQI-cylindrical measures ([2]) concerning the case of type 0-cylindrical measures.

1. An average of a cylindrical measure with respect to rotations

Let H be a real separable Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the induced norm $|\cdot|=\sqrt{\langle\cdot,\cdot\rangle}$, $\mathscr{B}(H)$ the Borel σ-algebra of H and $\mathscr{P}(H)$ the family of all finite dimensional orthogonal projections of H. A set $Z\subset H$ is said to be a cylindrical set if there are $P\in\mathscr{P}(H)$ and $B\in\mathscr{B}(R^n)$ such that $Z=\{x\in H : Px\in B\}$, where $n=\dim PH$. Let C_H denote the family of all cylindrical sets of H. A map μ from C_H into $[0,1]$ is called a cylindrical measure if it satisfies the following two conditions: (1) $\mu(H)=1$; (2) Restrict μ to the σ-algebra of cylindrical sets which are generated by a fixed $P\in\mathscr{P}(H)$. Then each such restriction is countably additive.

Let $\mathfrak{S}_H$ be the collection of all closed balls of H and μ be a cylindrical measure on H. We say that μ is of type 0 if for any $\varepsilon>0$ there exists $S\in\mathfrak{S}_H$ such that $\mu(x^{-1}(x(S)))\geqq 1-\varepsilon$ for all $x\in H$.

Let $\{e_n\}_{n=1,2,\dots}$ be a complete orthonormal system (CONS) of H. $\mathfrak{U}_H$ the family of all unitary operators of H and N be the set of all positive integers. Let H_n $(n\in N)$ be the subspace of H generated by $\{e_1,\cdots,e_n\}$. For each $n\in N$, we denote by $\mathfrak{U}_H(n)$ the subset of $\mathfrak{U}_H$ consisting of u satisfying the following two conditions: (1) Restrict u to the space H_n. Then such restriction is a unitary operator of H_n. (2) Restrict u to the space $H_n^{\perp}$. Then it is the identity operator on $H_n^{\perp}$.

It is easy to see that $\mathfrak{U}_H(n)$ is isomorphic to the n-dimensional orthogonal group $O(n)$. Since $O(n)$ is the compact topological group, we have the normalized Haar measure on $\mathfrak{U}_H(n)$, denote it by m_n. Define

$$(*)\qquad \lambda_n(A)=\int_{\mathfrak{U}_H(n)}\mu(u(A))dm_n(u)\qquad\text{for all}\quad A\in C_H.$$

Then λ_n is a cylindrical measure on H ([1], p. 97).

Let $\{\nu_\iota\}_{\iota\in I}$ be a collection of cylindrical measures on H. We say that $\{\nu_\iota\}_{\iota\in I}$ is uniformly of type 0 for $\iota\in I$ if for any $\varepsilon>0$ there exists $S\in\mathfrak{S}_H$ such that $\nu_\iota(x^{-1}(x(S)))\geqq 1-\varepsilon$ for all $x\in H$ and all $\iota\in I$.

LEMMA 1. *If μ is of type 0, then the class $\{\lambda_n\}_{n\in N}$ is uniformly of type 0 for $n\in N$.*

PROOF. By the assumption, for any $\varepsilon>0$ there exists $S\in\mathfrak{S}_H$ such that $\mu(x^{-1}(x(S)))\geqq 1-\varepsilon$ for all $x\in H$. Since $u(x^{-1}(x(S)))=\{u(x)\}^{-1}\{u(x)\}(S)$ for every $u\in\mathfrak{U}_H$, we have $\lambda_n(x^{-1}(x(S)))\geqq 1-\varepsilon$ for all $x\in H$ and all $n\in N$. □

Let T be an injective Hilbert-Schmidt operator of H and (i,H,E) be the abstract Wiener space induced from the measurable norm $p(x)=|Tx|$ $(x\in H)$. Then H and E are two Hilbert spaces and the inclusion map i is a Hilbert-Schmidt operator. We denote by $\|\cdot\|_2$ the Hilbert-Schmidt norm.

LEMMA 2 ([3], p. 215). *Let (i,H,E) be as above, $\alpha=\|i\|_2$ and μ a cylindrical measure on H satisfying that $\mu(x^{-1}(x(B_1)))\geqq 1-\eta$ for all $x\in H$, where B_1 is the unit ball of H and $\eta>0$. Then $i(\mu)$ satisfies that $g(i(\mu))(g(B_2))\geqq 1-\varepsilon$ for every finite dimensional vector space G and for every continuous linear map $g: E\to G$ wnere B_2 is the unit ball of E and $\varepsilon=C\max(\alpha,\eta)$ and C is a universal constant.*

THEOREM 1. *Let μ be a cylindrical measure on H and $\{\lambda_n\}_{n\in N}$ be as in (*). If μ is of type 0, then the sequence $\{\lambda_n\}$ contains a convergent subsequence $\{\lambda_{n_j}\}_{j=1,2,\ldots}$. The limit is denoted by λ. Furthermore, λ is rotationally invariant.*

PROOF. Let (i, H, E) be the abstract Wiener space as above. Using Lemma 1, we have the following result. For every $\eta>0$ there exists a ball $B_1\in\mathfrak{S}_H$ such that $\lambda_n(x^{-1}(x(B_1)))\geqq 1-\eta$ for all $x\in H$ and all $n\in N$. Let $\varepsilon>0$ and set $\eta=\alpha=\varepsilon/2C$. Then the expression $C\max(\alpha,\eta)$ of the preceding lemma is equal to $\varepsilon/2$. Denote by H_{B_1} the vector space H equipped with the gauge of B_1 as norm. On the other hand let E_{B_2} be the vector space E equipped with the gauge of the ball $B_2\subset E$ as norm, where we choose B_2 so that the Hilbert-Schmidt norm of $i : H_{B_1}\to E_{B_2}$ shall be $\leqq\alpha$. This can be achieved by taking the radius of B_2 sufficiently large. It follows then from Lemma 2 that $g(i(\lambda_n))(g(B_2))\geqq 1-\varepsilon/2$ for every finite dimensional vector space G and for every continuous linear map $g : E\to G$ and for all $n\in N$. Therefore, for every $n\in N$, $i(\lambda_n)$ is extensible to a Radon probability measure on E, denote it by $\tilde{\lambda}_n$. It follows from the above statement that for any $\varepsilon>0$ there exists $B_2\in\mathfrak{S}_E$ such that $\tilde{\lambda}_n(B_2)\geqq 1-\varepsilon/2$ for all $n\in N$. Hence we can say that for any $\varepsilon>0$ there exists a compact set $K\subset E$ such that $\tilde{\lambda}_n(K)\geqq 1-\varepsilon$ for all $n\in N$ ([3]). Then the set $\{\tilde{\lambda}_n\}_{n\in N}$ is relatively compact with respect to the narrow topology ([4]). Thus we have a convergent subsequence $\{\tilde{\lambda}_{n_j}\}_{j=1,2,\ldots}$ and its limit $\tilde{\lambda}$.

Using the same arguement in [1], we can get the cylindrical measure λ on H such that $\tilde{\lambda}$ is the extension of $i(\lambda)$. Obviously the sequence $\{\lambda_{n_j}\}_{j=1,2,\ldots}$ converges cylindrically to λ. The same consideration of Theorem 1 in [1] shows that λ is rotationally invariant. □

COROLLARY. *Let $\{\nu_\iota\}_{\iota\in I}$ be a family of cylindrical measures on H, (i, H, E) be the abstract Wiener space as in the above sense and $\tilde{\nu}_\iota$ is the extension of $i(\nu_\iota)$ if it exists. Assume that $\{\nu_\iota\}_{\iota\in I}$ is uniformly of type 0 with respect to $\iota\in I$. Then the set $\{\tilde{\nu}_\iota\}_{\iota\in I}$ is relatively compact with respect to the narrow topology.*

This proof is quite immediate and may be omitted.

We need more processes for the sake of regarding λ in Theorem 1 as an average of μ with respect to rotations. The following two problems are open.

PROBLEM 1. *Does every convergent subsequence of $\{\lambda_n\}_{n\in N}$ get the same or eqivalent limit ?*

PROBLEM 2. *Turn the CONS* $\{e_n\}_{n\in N}$ *into another one* $\{f_n\}_{n\in N}$. *Then, how does the limits change?*

These problems are solved for the special class of cylindrical measures.

DEFINITION 1. For every $\varepsilon>0$, there exists a $\delta>0$ such that $\mu(A)<\delta$ $(A\in\mathcal{C}_H)$ implies $\mu(u(A))<\varepsilon$ for all $u\in\mathfrak{U}_H$. Then μ is called *a (strongly) rotationally quasi-invariant ((S)RQI) cylindrical measure.*

For any $\varepsilon>0$ and any $u\in\mathfrak{U}_H$, there exists a $\delta>0$ such that $\mu(A)<\delta$ $(A\in\mathcal{C}_H)$ implies $\mu(u(A))<\varepsilon$. Then μ is simply called *an RQI-cylindrical measure.*

Let $FD(H)$ be the family of all finite dimensional subspaces of H. For $G\in FD(H)$, let $\mathcal{C}_H(G^\perp)$ be the family of cylindrical sets of which bases are on $G^\perp$ and $\mathfrak{U}_H(G^\perp)$ the subset of $\mathfrak{U}_H$ consisting of u satisfying the following condition: Restrict u to the space $G^\perp$. Then such restriction is a unitary operator of $G^\perp$.

DEFINITION 2. A cylindrical measure μ on H is called *a generalized rotationally quasi-invariant (GRQI) cylindrical measure* if for every $\varepsilon>0$, there exist a $\delta>0$ and $G\in FD(H)$ such that $\mu(u(A))<\varepsilon$ for all $u\in\mathfrak{U}_H(G^\perp)$ and for every $A\in\mathcal{C}_H(G^\perp)$ for which $\mu(A)<\delta$.

Let λ, ν be two cylindrical measures on H. We say that ν is [*weakly*] *cylindrically absolutely continuous* with respect to λ, in symbols $\nu\ll_c\lambda$ [resp. $\nu\ll'_c\lambda$], if for every $\varepsilon>0$, there exists a $\delta>0$ [resp. and $G\in FD(H)$] such that $\lambda(A)<\delta$ $(A\in\mathcal{C}_H)$ [resp. $(A\in\mathcal{C}_H(G^\perp))$] implies $\nu(A)<\varepsilon$.

λ and ν for which both $\nu\ll_c\lambda$ [resp. $\nu\ll'_c\lambda$] and $\lambda\ll_c\nu$ [resp. $\lambda\ll'_c\nu$] are said to be [*weakly*] *cylindrically equivalent,* in symbols $\lambda\sim_c\nu$ [resp. $\lambda\sim'_c\nu$].

THEOREM 2. *Let* μ *be an (S)RQI-cylindrical measure. Then all limits in the above problems are equivalent.*

PROOF. If μ is an (S)RQI-cylindrical measure, then Theorem 1 in [1] shows that $\mu\sim_c\lambda$ for every limit λ. ▯

2. Remarks on GRQI-cylindrical measures

Recently, H. Shimomura ([5]) showed the result that every RQI-cylindrical measure is an (S)RQI-cylindrical measure. Arranging the idea in his main lemma for the case of cylindrical measures, we have the following lemma.

Let $\{e_n\}_{n\in N}$ be a CONS of H, $\mathfrak{U}_H(\infty)=\bigcup_{n=1}^{\infty}\mathfrak{U}_H(n)$ and $\mathfrak{U}_H(n)^{\perp}=\{u\in\mathfrak{U}_H(\infty)$: The restriction of u to H_n is the identity$\}$.

LEMMA 3. *Let μ be an RQI-cylindrical measure. Then, for any $\varepsilon>0$, there exist a $\delta>0$ and $n_0\in N$ such that $\mu(A)<\delta$ $(A\in\mathcal{C}_H(H_{n_0}^{\perp}))$ implies $\mu(u(A))<\varepsilon$ for all $u\in\mathfrak{U}_H(n_0)^{\perp}$.*

PROOF. Assume that $\varepsilon_0>0$ and there exist $u_1\in\mathfrak{U}_H(1)^{\perp}$ and $A_1\in\mathcal{C}_H(H_1^{\perp})$ such that $\mu(A_1)<1$ and $\mu(u_1(A_1))\geqq\varepsilon_0$. From the definition of $\mathfrak{U}_H(1)^{\perp}$, u_1 belongs to $\mathfrak{U}_H(n_2)$ for some n_2. Without loss of generality we can suppose that $n_2>1\equiv n_1$. Next assume that there exist $u_2\in\mathfrak{U}_H(n_2)^{\perp}$ and $A_2\in\mathcal{C}_H(H_{n_2}^{\perp})$ such that $\mu(A_2)<1/2$ and $\mu(u_2(A_2))\geqq\varepsilon_0$.

We repeat this procedure. Then as it is easily seen, a sequence $n_1<n_2<\cdots\cdots<n_k<\cdots\cdots$, $u_k\in\mathfrak{U}_H(n_k)^{\perp}\cap\mathfrak{U}_H(n_{k+1})$ and $A_k\in\mathcal{C}_H(H_{n_k}^{\perp})$ are defined inductively such that $\mu(A_k)<1/k$ and $\mu(u_k(A_k))\geqq\varepsilon_0$ for all $k\in N$.

Since $\mathfrak{U}_H(n_k)^{\perp}\cap\mathfrak{U}_H(n_{k+1})$ is regarded as the orthogonal group on $R^{n_{k+1}-n_k}$, it is compact in the natural topology. Hence the direct product $K\equiv\prod_{k=1}^{\infty}\mathfrak{U}_H(n_k)^{\perp}\cap\mathfrak{U}_H(n_{k+1})$ is again a compact group. Clearly we have $K\subset\mathfrak{U}_H$.

Since μ is an RQI-cylindrical measure, we can say that for any $\varepsilon>0$ there exists a $\delta>0$ such that $\mu(A)<\delta$ $(A\in\mathcal{C}_H)$ implies $\mu(u(A))<\varepsilon$ for all $u\in K$.

Take $\varepsilon=\varepsilon_0$. Then we have $k\in N$ such that $1/k<\delta$. As above mentioned, we have $A_k\in\mathcal{C}_H(H_{n_k}^{\perp})\subset\mathcal{C}_H$ and $u_k\in\mathfrak{U}_H(n_k)^{\perp}\cap\mathfrak{U}_H(n_{k+1})\subset K$ such that $\mu(A_k)<1/k<\delta$ and $\mu(u_k(A_k))\geqq\varepsilon_0$.

This contradicts the above statement. Therefore it follows that for any $\varepsilon>0$, there exist a $\delta>0$ and $n_0\in N$ such that $\mu(A)<\delta$ $(A\in\mathcal{C}_H(H_{n_0}^{\perp}))$ implies $\mu(u(A))<\varepsilon$ for all $u\in\mathfrak{U}_H(n_0)^{\perp}$. □

On the other hand we have the following result about GRQI-cylindrical measures.

LEMMA 4. *Suppose that μ is a type 0-cylindrical measure on H and satisfying that there exists a CONS $\{e_n\}_{n\in N}$ of H and for any $\varepsilon>0$ there exist a $\delta>0$ and $n_0\in N$ such that $\mu(A)<\delta$ $(A\in\mathcal{C}_H(H_{n_0}^{\perp}))$ implies $\mu(u(A))<\varepsilon$ for all $u\in\mathfrak{U}_H(n_0)^{\perp}$. Then μ is a GRQI-cylindrical measure.*

PROOF. By the assumption, we have a non-decreasing sequence of positive integers $\{k_n\}_{n=1,2,\ldots}$ and a non-increasing sequence of positive numbers $\{\delta_n\}_{n=1,2,\ldots}$ satisfying that for every $n\in N$, $\mu(A)<\delta_n$ implies $\mu(u(A))<1/n$ for all $u\in\mathfrak{U}_H(k_n)^{\perp}$ if $A\in\mathcal{C}_H(H_{k_n}^{\perp})$.

Let $H_{k,m}$ $(k, m \in N \cup \{0\})$ be the subspace of H generated by $\{e_{k+1}, \cdots, e_m\}$ and $P_{k,m}$ be the normalized Haar measure of $\mathfrak{U}_H(k)^{\perp} \cap \mathfrak{U}_H(m)$. Define $\lambda_{k,m}(A) = \int_{\mathfrak{U}_H(k)^{\perp} \cap \mathfrak{U}_H(m)} \mu(u(A)) dP_{k,m}(u)$ for every $A \in \mathcal{C}_H$. Also define a map ϕ_k as follows:

$$\phi_k : x = \sum_{i=1}^{\infty} \langle x, e_{k+i} \rangle e_{k+i} \in H_k^{\perp} \mapsto \bar{x} = \sum_{i=1}^{\infty} \langle x, e_{k+i} \rangle e_i \in H.$$

Let us consider the restriction of $\lambda_{k,m}$ on $H_k^{\perp}$ and denote it by $\bar{\lambda}_{k,m}$. Put $\lambda_n = \bar{\lambda}_{k_n, k_n+2^n} \circ \phi_{k_n}^{-1}$ for $n = 1, 2, \cdots$.

Since μ is of type 0, we can use the same arguement of Theorem 1 in [2]. Therefore, we have a convergent subsequence $\{\lambda_{n_j}\}$ and its limit λ such that $\mu \sim_c' \lambda$. It follows again from Theorem 1 in [2] that μ is a GRQI-cylindrical measure. □

By Lemmas 3 and 4, for the case of type 0-cylindrical measures, we have the following result without using the latter half part of [5].

THEOREM 3. *Let μ be a type 0-cylindrical measure on H. If μ is an RQI-cylindrical measure, then μ is a GRQI-cylindrical measure.*

PROOF. It follows from Lemma 3 that for any $\varepsilon > 0$, there exist a $\delta > 0$ and $n_0 \in N$ such that $\mu(A) < \delta$ $(A \in \mathcal{C}_H(H_{n_0}^{\perp}))$ implies $\mu(u(A)) < \varepsilon$ for all $u \in \mathfrak{U}_H(n_0)^{\perp}$. By the assumption, μ is of type 0. Therefore, Lemma 4 says that μ is a GRQI-cylindrical measure. □

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