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On a Torsion-free Class with Separable Types

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§ 1. Introduction.

Throughout this paper, all groups are abelian and notations and terminology mostly follow [3]. A torsion-free class is defined in [1] as a family of finite rank torsion-free groups which is closed with respect to finite direct sums, pure subgroups, and torsion-free homomorphic images. We have to introduce "separable type" which will play an important role in this paper. A type is said to be *separable* if in a hight-sequence which belongs to this type, there does not appear symbol ∞.

Our aim in this paper is to investigate the torsion-free class which is generated by all rank 1 torsion-free groups of separable type.

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§ 2. *T*-groups.

If a group G satisfies that $\mathrm{Hom}(G, Z(p^\infty))$ is a torsion group for every prime p, we call G a *T-group*.

PROPOSITION 1. *The family of all T-groups is closed with respect to a finite direct sum, a homomorphic image, a subgroup, and an extension by a T-group.*

PROOF. First part follows from $\mathrm{Hom}(\bigoplus_{i=1}^{n} G_i, Z(p^\infty)) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}(G_i, Z(p^\infty))$. Let $0 \to K \to G \to H \to 0$ be an exact sequence. Then

$$0 \to \mathrm{Hom}(H, Z(p^\infty)) \to \mathrm{Hom}(G, Z(p^\infty)) \to \mathrm{Hom}(K, Z(p^\infty)) \to 0$$

is also an exact sequence since $Z(p^\infty)$ is injective. Consequently, $\mathrm{Hom}(G, Z(p^\infty))$ is a torsion group if and only if $\mathrm{Hom}(H, Z(p^\infty))$ and $\mathrm{Hom}(K, Z(p^\infty))$ are torsion groups. Thus the rest of our proposition has been proved.

LEMMA 1. *Let G be a p-group. Then G is a T-group if and only if G is bounded.*

PROOF. Suppose G has an unbounded basic subgroup. Then we can choose a basic subgroup B such that $G/B \cong \oplus Z(p^\infty) \neq 0$. Since $\mathrm{Hom}(Z(p^\infty), Z(p^\infty))$ is the additive group of p-adic integers, G is not a T-group. The sufficiency is obvious.

PROPOSITION 2. *Let G be a torsion group. Then G is a T-group if and only if the p-primary part G_p of G is bounded for every prime p.*

PROOF. This follows immediately from Lemma 1 and from the fact $\mathrm{Hom}(G, Z(p^\infty)) \cong \prod_q \mathrm{Hom}(G_q, Z(p^\infty)) = \mathrm{Hom}(G_p, Z(p^\infty))$.

COROLLARY. *The notation of T-group is invariant under a quasi-isomorphism, where the definition of quasi-isomorphism follows* [1].

PROOF. Let G be a T-group and suppose G and G' are quasi-isomorphic. There exists a homomorphism $f: G \to G'$ such that $G'/f(G)$ is bounded. By Propositions 1 and 2, G' is a T-group.

PROPOSITION 3. *Let G be a rank 1 torsion-free group. G is a T-group if and only if type(G) is separable.*

PROOF. Suppose type(G) is not separable. Then G is p-divisible for some prime p. Let x be a non-zero elements of G and let f be a non-trivial homomorphism from a free group $\langle x \rangle$ to $Z(p^\infty)$. Since $Z(p^\infty)$ is injective, f can be extended to $f': G \to Z(p^\infty)$. $f'(G)$ is p-divisible because G is p-divisible. Since non-trivial p-divisible subgroup of $Z(p^\infty)$ is $Z(p^\infty)$ itself, it follows that $f'(G) = Z(p^\infty)$. Hence f' is a torsion-free element of $\mathrm{Hom}(G, Z(p^\infty))$.

Conversely, suppose type(G) is separable. For any prime p, there exists an element x of G such that $h_p(x) = 0$. Let y be an arbitrary element of G. There is a relation $sy = tx$ where s and t are integers with $(s, t) = 1$. Besides, $(s, p) = 1$ follows from $h_p(x) = 0$. For any $f \in \mathrm{Hom}(G, Z(p^\infty))$, $p^n f(x) = 0$ for some integer n. Hence $sp^n f(y) = tp^n f(x) = 0$. Invoking $(s, p) = 1$, we know $p^n f(y) = 0$ for each element y of G. Therefore $\mathrm{Hom}(G, Z(p^\infty))$ is a torsion group.

LEMMA 2. *Let G be a torsion-free T-group. Then, G is of finite rank.*

PROOF. It is sufficient to show that G is not a T-group in case G is a free group of countable rank. Suppose $G = \oplus \langle x_i \rangle$ where $\langle x_i \rangle \cong Z$ for $i = 1, 2, 3, \cdots$. There is an element a_i in $Z(p^\infty)$ whose order is p^i. Define a homomorphism $f_i \in \mathrm{Hom}(\langle x_i \rangle, Z(p^\infty))$ by $f_i(x_i) = a_i$. Then $f = (f_1, f_2, \cdots)$ belongs to $\mathrm{Hom}(G, Z(p^\infty))$ but its order is not finite.

PROPOSITION 4. *A torsion-free group is a T-group if and only if it is a subgroup of a finite direct sum of rank* 1 *T-groups.*

PROOF. Let G be a torsion-free T-group. By Lemma 2, G can be embedded in a finite direct sum of Q's. Let $H_1, \cdots, H_n$ be the projections of G in each Q. By Proposition 1, $H_1, \cdots, H_n$ are rank 1 T-groups. Hence G is isomorphic to a subgroup of $H_1 \oplus \cdots \oplus H_n$. Sufficiency follows immediately from Proposition 1.

§ 3. The torsion-free class which is generated by all rank 1 T-groups.

We will call the members of the least torsion-free class which contains all rank 1 torsion-free groups "R-groups" following [1].

PROPOSITION 5. *The family of all torsion-free T-groups is a torsion-free class.*

PROOF. This is an immediate consequence of Proposition 1 and Lemma 2.

PROPOSITION 6. *The torsion-free class generated by all rank* 1 *torsion-free groups of separable type coincides with the family of R-groups with separable types.*

PROOF. Let G be the member of the torsion-free class generated by all rank 1 torsion-free groups of separable type. Clearly G is an R-group. By Propositions 1 and 3, G is a T-group. For any $x \neq 0$ in G, the pure subgroup $\langle x \rangle_*$ of G generated by x is again a T-group. Hence type(x) is separable.

Conversely let G be an R-group with separable types. We can write $G = A_1 + \cdots + A_n$ where A_i is a pure rank 1 subgroup of G by Theorem 4.4 in [1]. Hence type(A_i) is separable. This completes the proof of Proposition 6.

PROPOSITION 7. *A torsion-free T-group is not necessarily an R-group.*

PROOF. We will construct a rank 2 torsion-free T-group with infinite type set, since an R-group must have a finite type set by Theorem 4.6 in [1].

Let A be a subgroup of Q such that $1 \in A$ and hight sequence of 1 in A is $(1, 1, 1, \cdots)$. Let $t_i = (1, i) \in A \oplus A$, $i = 1, 2, 3, \cdots$ and let P be the set of all primes. Write $P = \cup P_i$ as the disjoint union of infinitely many infinite sets. $h^A(1) = (1, 1, 1, \cdots)$ implies $(1/p)t_i \in A \oplus A$ for any prime p. Define G to be the subgroup of $A \oplus A$ generated by all $(1/p_i)t_i$ where $p_i \in$

P_i and $i=1, 2, 3, \cdots$. G is clearly a T-group. Now we will show that G has infinitely many different types.

Suppose type(t_i)=type(t_j) for some $i \neq j$. That is $h_p(t_i)=h_p(t_j)$ for almost all primens p. Fix j, then for any $p \in P_j$, $h_p(t_j) \geqq 1$. Hence for infinitely many p which belong to P_j, $h_p(t_i)=h_p(t_j) \geqq 1$. That is

$$\frac{1}{p}(1, i)=r\frac{1}{p}(1, j)+\sum_{k=1}^{n} r_k \frac{1}{q_k}(1, i_k),$$

$$(1, i)=r(1, j)+p\sum_{k=1}^{n} r_k \frac{1}{q_k}(1, i_k).$$

Hence $1-r=p\sum_{k=1}^{n}\frac{r_k}{q_k} \in Z$. Since each q_k is different from p, $\sum_{k=1}^{n}\frac{r_k}{q_k} \in Z$. And since q_k's are different from each other, $r_k/q_k \in Z$ for $k=1, 2, \cdots, n$. Therefore $1 \equiv r (\mathrm{mod}\, p)$ and $i \equiv rj (\mathrm{mod}\, p)$ for infinitely many p. It follows that $i=j$. This contradicts the assumption $i \neq j$. Hence G has infinitely many different types.

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