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On a simplex homomorphism

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§1. Introduction.

A simplex space can be expressed as a subspace of $C(K)$, which is not necessarily a Banach lattice, but whose dual space is an AL space. In contrast to a lattice homomorphism of a Banach lattice, we define a simplex homomorphism T of a simplex space $E(\subset C(K_1))$ into a simplex space $F(\subset C(K_2))$ [8]. In case of a Markov operator, a Riesz homomorphism defined by F. Jellett [5] is equivalent to a simplex homomorphism. In this paper, we investigate some properties of a simplex homomorphism (Theorem 1). By using a function γ on K_2 and a mapping $k: K_2 \to K_1$, we can express a simplex homomorphism T as

$$Tf(x)=\gamma(x)f\cdot k(x) \qquad \text{for any } f\in E$$

in a similar way to the case of a lattice homomorphism of $C(K)$ [10]. In general, k is not continuous. So we introduce a special topology on K_1 (and K_2) with respect to which k is continuous on $\{x\in K_2; \gamma(x)\neq 0\}$ and obtain a condition which is equivalent to a simplex homomorphism (Theorem 2).

By using the continuity of k with respect to this topology, we shall obtain some results about the spectrum of a simplex homomorphism in a forthcoming paper [9].

§2. Simplex homomorphism.

Let E and F be simplex spaces, i.e. preduals of AL-spaces. Put $X=\{x\in E'; x\geqq 0, \|x\|\leqq 1\}$ and $Y=\{y\in F'; y\geqq 0, \|y\|\leqq 1\}$ endowed with the weak*-topology. Then E [resp. F] can be expressed as a space of functions on $\overline{\partial X}$ [resp. $\overline{\partial Y}$] (the weak*-closure of the set ∂X [resp. ∂Y] of all extreme points of X [resp. Y]), namely $\{f\in C(\overline{\partial X}); f(x)=\int f\,d\mu_x$ for all $x\in\overline{\partial X}$ and $f(0)=0\}$ [resp. $\{g\in C(\overline{\partial Y}); g(y)=\int g\,d\nu_y$ for all $y\in\overline{\partial Y}$ and $g(0)=0\}$] where μ_x [resp. ν_y] is the maximal probability measure on X [resp. Y] with resultant x [resp. y] [3, §28].

We call $T\in\mathfrak{L}(E, F)$ a simplex homomorphism if for any $f, g\in E$ and any $y\in\partial Y$, there exists $h\in E$ such that $h\geqq f$, g and $Th(y)=\max\{Tf(y), Tg(y)\}$ [8]. We recall that a lattice homomorphism T of a Banach lattice E into a

Banach lattice F is an operator satisfying $T(f\vee 0)=Tf\vee 0$ for all $f\in E$. Then we have

THEOREM 1. *Let E, F be simplex spaces. For $T\in\mathfrak{L}(E, F)$, the following assertions are equivalent:*

i) *T is a simplex homomorphism.*

ii) *For any $y\in\partial Y$, we have $T'y=0$ or $\dfrac{T'y}{\|T'y\|}\in\partial X$.*

iii) *T'' is a lattice homomorphism of E'' into F''.*

iv) *For any $f, g\in E$, $p\in F$ satisfying $Tf, Tg\leqq p$ and any $\varepsilon>0$, there exists $h\in E$ such that $f, g\leqq h$ and $Th(y)\leqq p(y)+\varepsilon$ for all $y\in\overline{\partial Y}$.*

v) *For any $y\in\partial Y$, $f, g\in E$ and any $\varepsilon>0$, there exists $h\in E$ such that $h\geqq f, g$ and $Th(y)\leqq\max\{Tf(y), Tg(y)\}+\varepsilon$.*

Before proving this theorem, we show the following lemma.

LEMMA 1. *Let f, g be upper semi-continuous affine functions on X such that $f|\partial X=g|\partial X$.[1] Then $f=g$ on X.*

PROOF. Suppose that there exists $x_0\in X$ such that $f(x_0)>g(x_0)$. Since $g(x)=\inf\{h(x); h\in E, h\geqq g\}$ holds [1, Proposition I. 1. 2], there exists $h\in E$ such that $h\geqq g$ and $f(x_0)>h(x_0)$. Then $f-h$ is upper semi-continuous affine on X, $(f-h)(x_0)>0$ and $(f-h)|\partial X\leqq(f-g)|\partial X=0$. This implies a contradiction, since an upper semi-continuous affine function attains its maximum on ∂X. Therefore $f=g$ on X //

PROOF OF THEOREM 1. i)$\longleftrightarrow$ii) follows from the definition and [6, Proposition 4].

ii)$\to$iii) At first, we show that for any $f\in E$, $T''(f\vee\!\!\!\vee 0)=Tf\vee\!\!\!\vee 0$ holds, where the sign "$\vee\!\!\!\vee$" is used to denote the supremum of two elements in E'' or F''. For $y\in\partial Y$ with $\|T'y\|\neq 0$, we have

$$T''(f\vee\!\!\!\vee 0)(y)=(f\vee\!\!\!\vee 0)(T'y)=\|T'y\|(f\vee\!\!\!\vee 0)\left(\frac{T'y}{\|T'y\|}\right)$$

$$=\|T'y\|\max\left\{f\left(\frac{T'y}{\|T'y\|}\right), 0\right\}=\max\{Tf(y), 0\}=(Tf\vee\!\!\!\vee 0)(y)$$

by the relation $\dfrac{T'y}{\|T'y\|}\in\partial X$ and [6, Proposition 4]. Since $T''(f\vee\!\!\!\vee 0)$ and $(Tf\vee\!\!\!\vee 0)$ are both upper semi-continuous affine functions on Y [7, Lemma] and the relation $T''(f\vee\!\!\!\vee 0)|\partial Y=(Tf\vee\!\!\!\vee 0)|\partial Y$ holds, we have by Lemma 1,

$$T''(f\vee\!\!\!\vee 0)=Tf\vee\!\!\!\vee 0. \qquad (*)$$

Next we show that $[0, T'y]=T'[0, y]$ holds for any $y\in F'$, $y\geqq 0$. Since T' is positive, $T'[0, y]\subset[0, T'y]$ holds. If there exists $z\in[0, T'y]$ such that

1) $f|K$ is the restriction of f to K.

$z \notin T'[0, y]$, then there exists $f \in F$ such that

$$f(z) > c > \sup\{f(w)\,;\, w \in T'[0, y]\}$$

since $T'[0, y]$ is a weak*-closed convex subset of F'. So we have $T''(f \vee 0)(y) = (f \vee 0)(T'y) = \sup\{f(w)\,;\, 0 \leqq w \leqq T'y\} \geqq f(z) > c$. On the other hand, we have $(Tf \vee 0)(y) = \sup\{Tf(w)\,;\, 0 \leqq w \leqq y\} = \sup\{f(T'w)\,;\, 0 \leqq w \leqq y\} < c$. Hence $Tf \vee 0 \neq T''(f \vee 0)$, which is a contradiction to $(*)$. Therefore for any $\phi \in E''$ and any $y \in Y$, we have

$$\begin{aligned} T''(\phi \vee 0)(y) &= (\phi \vee 0)(T'y) = \sup\{\phi(w)\,;\, 0 \leqq w \leqq T'y\} \\ &= \sup\{T''\phi(w)\,;\, 0 \leqq w \leqq y\} = (T''\phi \vee 0)(y)\,. \end{aligned}$$

So we have $T''(\phi \vee 0) = T''\phi \vee 0$ for any $\phi \in E''$.

iii)$\to$iv) By [7, Lemma], we have for any $f, g \in E$

$$(Tf \vee Tg)(y) = \inf\{q(y)\,;\, q \in F,\ q \geqq Tf,\ Tg\}$$

for any $y \in \overline{\partial Y}$. Since T'' is a lattice homomorphism,

$$\begin{aligned} (Tf \vee Tg)(y) &= T''(f \vee g)(y) = (f \vee g)(T'y) \\ &= \inf\{h(T'y)\,;\, h \in E,\ h \geqq f,\ g\}\,. \end{aligned}$$

Let $p \in F$ satisfy $Tf, Tg \leqq p$. Then for any $y \in \overline{\partial Y}$ and any $\varepsilon > 0$, there exists $h_y \in E$ such that $h_y \geqq f, g$ and $Th_y(y) = h_y(T'y) \leqq p(y) + \varepsilon/2$. Since $\overline{\partial X}$ is weak*-compact and p is continuous, we have $h \in E$ such that $h \geqq f, g$ and $h(T'y) \leqq p(y) + \varepsilon$ for all $y \in \overline{\partial Y}$.

iv)$\to$v) For any $f, g \in E$ and any $y \in \partial Y$, we have $\max\{Tf(y), Tg(y)\} = (Tf \vee Tg)(y) = \inf\{q(y)\,;\, q \in F,\ q \geqq Tf,\ Tg\}$. So for any $\varepsilon > 0$, there exists $p \in F$ such that $p \geqq Tf, Tg$ and $p(y) \leqq \max\{Tf(y), Tg(y)\} + \varepsilon$. By iv), there exists $h \in E$ such that $h \geqq f, g$ and $Th(y) \leqq p(y) + \varepsilon$. Therefore

$$Th(y) \leqq \max\{Tf(y), Tg(y)\} + 2\varepsilon\,.$$

v)$\to$i) Fix $f, g \in E$ and $y \in \partial Y$. Then we have

$$(f \vee g)(T'y) = \inf\{h(T'y)\,;\, h \in E,\ h \geqq f,\ g\} = \max\{f(T'y), g(T'y)\}$$

by using v). Let $\phi = f \vee g$. Then ϕ is upper semi-continuous affine on X [7, Lemma]. Put

$$\psi(x) = \begin{cases} \max\{\|f\|, \|g\|\} & \text{for } x \in X \setminus \left(\dfrac{T'y}{\|T\|} \cup \{0\}\right) \\ \max\left\{f\left(\dfrac{T'y}{\|T\|}\right), g\left(\dfrac{T'y}{\|T\|}\right)\right\} & \text{for } x = \dfrac{T'y}{\|T\|} \\ 0 & \text{for } x = 0\,. \end{cases}$$

Then ψ is lower semi-continuous on X and $\psi \geqq \phi$. Put $\check{\psi}(x) = \sup\{h(x)\,;\, h \in E,\ h \leqq \psi\}$ for $x \in X$. Then by [2, Lemma 1.2], we have $\check{\psi}(x) = \inf\{\mu(\psi)\,;\, \mu \in P_x(X)\}$, where $P_x(X)$ is the set of all probability measures on X with resultant x. Since ϕ is upper semi-continuous affine on X, we have $\phi(x) = \mu(\phi)$ for any $\mu \in P_x(X)$ and

$\check{\phi}(x) \geqq \phi(x)$ for any $x \in X$. Therefore $\check{\phi} \geqq \phi$ and $\check{\phi}$ is a lower semi-continuous convex function on X. Hence by [3, Theorem 28.6], there exists $h \in E$ such that $\check{\phi} \geqq h \geqq \phi$. Then $h \geqq f, g$ and $Th(y) = \max\{Tf(y), Tg(y)\}$. //

REMARK. In case of a Markov operator, F. Jellett [5] showed that the above condition ii) is equivalent to the following:

iv′) For any $f, g \in E$, $a \in F$ satisfying $Tf, Tg \leqq a$, there exists $h \in E$ such that $f, g \leqq h$ and $Th \leqq a$.

But when T is not a Markov operator, iv′) is not equivalent to ii) as shown in the following example and the above theorem shows that a slightly modified condition iv) is equivalent to ii).

EXAMPLE. Let $E = \left\{ f \in C([-1, 1]) ;\ f(0) = \frac{1}{2}\{f(1) + f(-1)\} \right\}$ and

$$Tf(x) = \begin{cases} 2|x| f(x) & |x| \leqq 1/2 \\ 2(1 - |x|) f(x) & |x| > 1/2 \end{cases}$$

for all $f \in E$. Then E is a simplex space and T is a simplex homomorphism of E into E.

Let $f_0(x) = x$, $g_0(x) = -x$ for $x \in [-1, 1]$, and

$$p(x) = \begin{cases} 2x^2 & |x| \leqq 1/2 \\ 2(1 - |x|)|x| & |x| > 1/2. \end{cases}$$

Then $f_0, g_0, p \in E$ and $Tf_0, Tg_0 \leqq p$. In this case, there exists no $h \in E$ such that $h \geqq f_0, g_0$ and $Th \leqq p$. But for any $\varepsilon > 0$, there exists $h \in E$ such that $h \geqq f_0, g_0$ and $Th(y) \leqq p(y) + \varepsilon$ for all $y \in [-1, 1]$.

This example shows that though T is a simplex homomorphism, the condition iv′) does not hold, but iv) holds.

§ 3. θ-topology.

Hereafter let E be a simplex space satisfying the condition:

$$\inf\{\|x\| ;\ x \in \overline{\partial X} \setminus \{0\}\} = a > 0 .$$

Then we have

LEMMA 2. *Let K be a closed subset of $\overline{\partial X}$ and B_r be the set $\{x \in E' ;\ \|x\| \leqq r\}$. Then for $r \geqq 1$, $[c, \infty)K \cap B_r$ is a closed set for any c, $0 \leqq c \leqq 1$.*

PROOF. For $r \geqq 1$, put $K_r = \left[c, \frac{r}{a}\right]K$, as $0 < a \leqq 1$. Then we have $[c, \infty)K \cap B_r = K_r \cap B_r$. Since K_r is compact, $K_r \cap B_r$ is compact and so closed. //

The following lemma is easily obtained.

LEMMA 3. *Let K be a closed subset of $\overline{\partial X}$.*

i) *If* $0 \notin K$, *then* $[0, \infty)K \cap \overline{\partial X} \setminus \{0\} = [a, \infty)K \cap \overline{\partial X}$.
ii) *If* $0 \in K$, *then* $[0, \infty)K \cap \overline{\partial X} = [a, \infty)K \cap \overline{\partial X}$.

Hence we define the following mapping θ: For any closed subset K of $\overline{\partial X}$, put $\tilde{\theta}(K) = [a, \infty)K \cap \overline{\partial X}$ and for any relatively open subset U of $\overline{\partial X}$, put $\theta(U) = \overline{\partial X} \setminus \tilde{\theta}(\overline{\partial X} \setminus U)$.

Then we have

LEMMA 4. *Let* $\mathfrak{U}$ *be the set of all relatively open subsets of* $\overline{\partial X}$. *Then*

i) $U \supset \theta(U)$ $\quad (U \in \mathfrak{U})$
ii) $\theta(U)$ *is relatively open.* $\quad (U \in \mathfrak{U})$
iii) $\theta(\theta(U)) = \theta(U)$ $\quad (U \in \mathfrak{U})$
iv) $\theta(U_1 \cap U_2) = \theta(U_1) \cap \theta(U_2)$ $\quad (U_1, U_2 \in \mathfrak{U})$
v) $\theta(\bigcup_\alpha \theta(U_\alpha)) = \bigcup_\alpha \theta(U_\alpha)$ $\quad (U_\alpha \in \mathfrak{U})$
vi) $\theta(\emptyset) = \emptyset$, *where* $\emptyset$ *is an empty set and* $\theta(\overline{\partial X}) = \overline{\partial X}$.

Therefore $\{\theta(U) : U \in \mathfrak{U}\}$ defines a topology on $\overline{\partial X}$. We shall call it θ-topology.

PROPOSITION. $\overline{\partial X}$ *equipped with* θ*-topology is* T_4 *space.*

PROOF. For each disjoint pair of θ-closed subsets of $\overline{\partial X}$, A and B, there are disjoint open subsets of $\overline{\partial X}$, U and V such that $A \subset U$ and $B \subset V$, since weak* topology is normal. By the definition of θ, we have $A \subset \theta(U) \subset U$ and $B \subset \theta(V) \subset V$. Therefore $\theta(U) \cap \theta(V) = \emptyset$. //

Now we have another characterization of a simplex homomorphism.

THEOREM 2. *Let* E, F *be simplex spaces satisfying the conditions:*

$$\inf\{\|x\| : x \in \overline{\partial X} \setminus \{0\}\} > 0 \quad and \quad \inf\{\|y\| ; y \in \overline{\partial Y} \setminus \{0\}\} > 0 .$$

Then for $T \in \mathfrak{L}(E, F)$, *the following are equivalent.*

i) T *is a simplex homomorphism of* E *into* F.
ii) *There exist a function* $\gamma(Y)$ *on* $\overline{\partial Y}$ *and a mapping* $k : \overline{\partial Y} \to \overline{\partial X}$ *satisfying:*
  a) $0 \leqq \gamma(y) \leqq \|T\|$ *for all* $y \in \overline{\partial Y}$.
  b) $k(\partial Y) \subset \partial X$, k *is* θ*-continuous*[2] *on* $\{y \in \overline{\partial Y} ; \gamma(y) \neq 0\}$ *and* $k(y_\alpha) = k(y_{\alpha'})$ *if* $y_\alpha = c_\alpha y_{\alpha'}$ *for some* $c_\alpha > 0$.
  c) $Tf(y) = \gamma(y) f \circ k(y)$ *for any* $f \in E$ *and any* $y \in \overline{\partial Y}$.

PROOF. i)$\to$ii): By ii) of Theorem 1, we see

$$T'(\partial Y) \subset [0, \|T\|]\partial X .$$

2) We call that k is θ-continuous if it is continuous with respect to θ-topology for both X and Y.

Since T' is continuous, we get

$$T'(\overline{\partial Y}) \subset [0, \|T\|]\overline{\partial X}. \tag{**}$$

For $y \in \overline{\partial Y}$, define $\gamma(y) = \inf\{\gamma \in R\,;\ T'y \in \gamma \cdot \overline{\partial X}\}$ and

$$k(y) = \begin{cases} \dfrac{T'y}{\gamma(y)} & \text{if } T'y \neq 0 \\ 0 & \text{if } T'y = 0. \end{cases}$$

Then we have $k(y) \in \overline{\partial X}$, $T'y = \gamma(y) \cdot k(y)$ and $0 \leqq \gamma(y) \leqq \|T\|$ by (**). For $y \in \partial Y$, we have $\gamma(y) = \|T'y\|$ and $k(y) \in \partial X$. Next we show that k is θ-continuous on on $\{y \in \overline{\partial Y}\,;\ \gamma(y) \neq 0\}$. For any θ-open subset U of $\overline{\partial X} \setminus \{0\}$, let $K = \overline{\partial X} \setminus U$. Then $K \ni 0$. By the relation $k^{-1}(K) = T'^{-1}([0, \|T\|]K) \cap \overline{\partial X}$, we have $k^{-1}(K)$ is θ-closed and so $k^{-1}(U)\ (= \overline{\partial Y} \setminus k^{-1}(K))$ is θ-open. Since $\{0\}$ is θ-closed, k is θ-continuous on $\{y \in \overline{\partial Y}\,;\ \gamma(y) \neq 0\}$.

ii)$\to$i): For any $y \in \partial Y$, we have $k(y) \in \partial X$ and $T'y = \gamma(y) \cdot k(y)$ by ii). So the relation $\|k(y)\| = 1$ implies $\|T'y\| = \gamma(y)$. Therefore $T'y = 0$ or $\dfrac{T'y}{\|T'y\|} = \dfrac{T'y}{\gamma(y)}$ $= k(y) \in \partial X$. //

References

[1] E.M. Alfsen, Compact convex sets and boundary integrals, Ergebnisse der Math. **57**, Springer-Veriag, 1971.

[2] E.M. Alfsen and E.G. Effros, Structure in real Banach spaces, Ann. of Math. **96** (1972), 98–128.

[3] G. Choquet, Lectures on Analysis II, Benjamin, New York 1969.

[4] N. Dunford and J. Schwartz, Linear operators, Part I, Interscience 1958.

]5] F. Jellett, Homomorphisms and Inverse limits of Choquet simplexes, Math. Zeitschr. **103** (1968), 219–226.

[6] F. Takeo, Decomposition of a positive operator in a simplex space to its irreducible components, J. Fac. Sci. Univ. Tokyo Sect. IA, **24** (1977), 425–432.

[7] F. Takeo, On the second dual of a simplex space, Natur. Sci. Rep. Ochanomizu Univ. **33** (1982), 27–36.

[8] F. Takeo, On the spectrum of positive operators in simplex spaces, J. Fac. Sci. Univ. Tokyo Sect. IA, **30** (1983) 93–109.

[9] F. Takeo, On the peripheral point spectrum of a simplex homomorphism, in preparation.

[10] M. Wolff, Über das Spektrum von Verbandshomomorphismen in $C(X)$, Math. Ann. **182** (1969), 161–169.