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A Recursive Formula of the Ramsey Number $r(3, n)$

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§1. Introduction.

The Ramsey number $r(m, n)$ is defined to be the smallest integer p such that every graph of order p contains a complete subgraph on m vertices or an independent set of n vertices. Obviously, $r(m, n)=r(n, m)$, $r(1, n)=1$ and $r(2, n)=n$, but very few exact Ramsey numbers are known for $m\geqq 3$ and $n\geqq 3$. The following table shows all Ramsey numbers $r(3, n)$ known to data, together with the best known upper and lower bounds (see [2]).

Table

n	3	4	5	6	7	8	9	10
$r(3, n)$	6	9	14	18	23	28–29	36	39–44

Using J. G. Kalbfleisch's upper bounds for $11\leqq n\leqq 14$ (see [3]), the author and S. Tachibana [4] showed that the Ramsey number $r(3, n)$ satisfies

$$(1) \qquad r(3, n)\leqq\binom{n}{2}-c \qquad \text{for all } n\geqq n_0$$

if the inequality (1) holds for both $n=n_0$ and $n=n_0+1$, where c is a non-negative integer. Hence by the table we see

$$(2) \qquad r(3, n)\leqq\binom{n}{2} \qquad \text{for } n\geqq 9.$$

For general $m\geqq 2$ and $n\geqq 2$ the following inequality is known

$$(3) \qquad r(m, n)\leqq r(m, n-1)+r(m-1, n)$$

(for example, see [1]). This yields together with $r(2, n)=n$

$$(4) \qquad r(3, n)\leqq r(3, n-1)+n\,.$$

As easily checked from the table, for $6\leqq n\leqq 10$ the inequality (4) is replaced by a stronger inequality

$$(5) \qquad r(3, n)\leqq r(3, n-1)+n-1\,.$$

Hence it seems very likely that the inequality (5) would be true for all $n\geqq 6$.

In this paper, we prove a slight modification of (5) as

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(6) $$r(3, n) \leqq r(3, n-2)+2n-3 \quad \text{for all } n \geqq 6.$$

It should be noted that $r(3, n) \leqq r(3, n-1)+n-1$ and $r(3, n-1) \leqq r(3, n-2)+n-2$ imply (6).

§ 2. Lemma and Theorem.

For a simple graph G we denote by $V(G)$, $d(v)$ and $N(v)$ the vertex set, the degree of $v \in V(G)$ and the neighborhood of v respectively. Let $\mathrm{diam}(G)$ denote the diameter of G, $|X|$ the cardinality of X and I_n an independent set of n vertices.

It is easy to see that if G has no triangle, then $N(v)$ is an independent set of $d(v)$ vertices for all $v \in V(G)$.

LEMMA. *If a graph G of order $p \geqq 3$ contains no triangle, then there is a graph G' of the same order p such that G' contains no triangle, $G \subset G'$ and* $\mathrm{diam}(G')=2$.

PROOF. If $\mathrm{diam}(G) \geqq 3$, then there are two vertices u and v with the distance at least three. Let G_0 be the graph $G+uv$ obtained by joining u and v by an edge. Then G_0 contains no triangle and $G \subset G_0$. If $\mathrm{diam}(G_0) \geqq 3$, we repeat the operation that joins, by an edge, two vertices with distance at least three in G_0. By continueing the same operation we have a graph G' such that $\mathrm{diam}(G')=2$. Obviously, $G \subset G'$ and G' contains no triangle.

In Lemma, it should be noted that every independent set in G' is also an independent one in G.

THEOREM. *For the Ramsey number $r(3, n)$, the recursive formula*

(*) $$r(3, n) \leqq r(3, n-2)+2n-3 \quad \textit{for all } n \geqq 6$$

holds.

PROOF. By the table we see that the inequality (*) is true for $6 \leqq n \leqq 9$. Then let $n \geqq 10$ and $p=r(3, n-2)+2n-3$, and let G be any graph of order p which has no triangle. In order to prove the theorem it suffices to show that G has an I_n. We can assume $\mathrm{diam}(G)=2$. For, if necessary, we may consider the graph G' satisfing all the conditions in Lemma. Note that if G' has an I_n, so does G. Furthermore, we can assume

(7) $$d(v) \leqq n-1 \quad \text{for all } v \in V(G),$$

because if there is a vertex $v \in V(G)$ with $d(v) \geqq n$, then $N(v)$ contains an I_n.

Case 1. Suppose $r(3, n-1) \leqq r(3, n-2)+n-3$.

Then the inequality (*) holds by (4).

Case 2. Suppose $r(3, n-1)=r(3, n-2)+n-2$.

Then it follows from the iequality (2) together with $p=r(3, n-2)+2n-3$

$$p=r(3, n-1)+n-1\leqq\frac{1}{2}n(n-1). \tag{8}$$

If there is a vertex $v\in V(G)$ with $d(v)\leqq n-2$, then $T=V(G)-N(v)\cup\{v\}$ satisfies $|T|\geqq p-(n-1)=r(3, n-1)$. Hence T contains an I_{n-1} and $I_{n-1}\cup\{v\}$ is an I_n. Therefore we can assume that G is $(n-1)$-regular by (7).

Now, suppose that $|N(u)\cap N(v)|\leqq 2$ holds for every non-adjacent vertices u and v. Let v be any fixed vertex and let $T=V(G)-N(v)\cup\{v\}$ and $T_i=\{u\in T: |N(u)\cap N(v)|=i\}$ for $i=1, 2$. Then, by (8), $t_i=|T_i|$ $(i=1, 2)$ satisfy

$$t_1+t_2=|T|=p-n\leqq\frac{1}{2}n(n-3).$$

On the other hand, counting the edges joining $N(v)$ to T, we have $t_1+2t_2=(n-1)(n-2)$. These yield $t_1\leqq -2$, which contradicts $t_1\geqq 0$. Thus there exists a pair of non-adjacent vertices u and v satisfying $|N(u)\cap N(v)|\geqq 3$. Let $T^*=V(G)-N(u)\cup N(v)\cup\{u, v\}$. Since $|T^*|\geqq p-(2n-3)=r(3, n-2)$, T^* contains an I_{n-2} and $I_{n-2}\cup\{u, v\}$ is an I_n.

Case 3. Suppose $r(3, n-1)=r(3, n-2)+n-1$.

By (2) we have

$$p=r(3, n-2)+2n-3=r(3, n-1)+n-2\leqq\frac{1}{2}(n+1)(n-2). \tag{9}$$

If there is a vertex $v\in V(G)$ with $d(v)\leqq n-3$, then $T=V(G)-N(v)\cup\{v\}$ satisfies $|T|\geqq p-(n-2)=r(3, n-1)$. Hence T contains an I_{n-1} and $I_{n-1}\cup\{v\}$ is an I_n.

Thus we can assume

$$d(v)=n-1 \quad\text{or}\quad n-2 \qquad \text{for all } v\in V(G). \tag{10}$$

First, we assume that there is a vertex $v\in V(G)$ with $d(v)=n-2$. We shall show that there exists a vertex u which is non-adjacent to v and satisfies $|N(u)\cap N(v)|\geqq 2$. If $|N(u)\cap N(v)|\leqq 1$ for every vertex u non-adjacent to v, then $T=V(G)-N(v)\cup\{v\}$ satisfies $|T|=p-(n-1)\leqq\frac{1}{2}n(n-3)$ by (2) and $|T|\geqq(n-2)(n-3)$ by counting the edges joining $N(v)$ to T. Hence $3\leqq n\leqq 4$, which contradicts $n\geqq 10$. Thus we have a vertex u non-adjacent to v which satisfies $|N(u)\cap N(v)|\geqq 2$. Let $T^*=V(G)-N(u)\cup N(v)\cup\{u, v\}$, then by (10) $|T^*|\geqq p-(2n-3)=r(3, n-2)$, which implies that T^* contains an I_{n-2} and $I_{n-2}\cup\{u, v\}$ is an I_n.

Secondly, we assume $d(v)=n-1$ for all $v\in V(G)$. Then G is $(n-1)$-regular. Thus similarly to the proof of Case 2 one can show that G contains an I_n. So we shall omit it.

Thus the proof has been completed.

REMARK. Table 1 in [4] contains a mistake which states $34\leqq r(4, 6)\leqq 36$. It seems that the best bounds known up to date is $34\leqq r(4, 6)\leqq 44$, [2].

References

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