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# An Attempt to Generalize the Denjoy Integration

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This is a continuation of our papers [1] to [3]. We shall be concerned with introducing, for functions of a real variable, two new kinds of integration each of which includes that of Denjoy. At present, however, we do not know whether they really extend the Denjoy integration. There are, moreover, a number of other problems which we have failed as yet in solving.

We shall make free use of the terminology, notation, and results of the papers mentioned above. Thus a *function*, by itself, will mean a finite real-valued one defined on the whole real line, unless stated otherwise. Again, by a *sequence* we shall understand a nonvoid countable one finite or infinite. A sequence will usually be indexed with natural numbers in this paper; however, even such an expression as "for all values of $n$" where $n$ is the index, will not necessarily mean that $n$ ranges over the whole set $N$, unless the sequence is expressly restricted to being infinite.

The diameter of a linear set $X$ (i.e. any subset of the real line) will be denoted by $\mathrm{d}(X)$. In particular, $\mathrm{d}(X)$ means 0 or $+\infty$ according as $X$ is void or unbounded, respectively. The oscillation $\mathrm{O}(\varphi; S)$ of a function $\varphi$ on a linear set $S$ is defined to be the diameter $\mathrm{d}(\varphi[S])$.

We shall call a function $\varphi$ to *dwindle* on a set $S$, if for each $\varepsilon>0$ this set can be covered by a sequence of open intervals $\langle H_1, H_2, \cdots\rangle$ such that $\sum \mathrm{d}(\varphi[H_n \cap S])<\varepsilon$, where the summation extends over all values of $n$ (under consideration). Evidently, a function which dwindles on a set necessarily does so on every subset of the same set.

THEOREM 1. *Any function $\varphi$ which dwindles on a set $S$, is continuous on $S$ and we have $|\varphi[S]|=0$.*

PROOF. We need only prove $|\varphi[S]|=0$. With the same notation as above, we have successively

$$S=\bigcup(H_n\cap S),\quad \varphi[S]=\bigcup \varphi[H_n\cap S],$$

$$|\varphi[S]|\leqq\sum|\varphi[H_n\cap S]|\leqq\sum \mathrm{d}(\varphi[H_n\cap S])<\varepsilon,$$

as $|X|\leqq \mathrm{d}(X)$ for every linear set $X$. Hence the result.

In order to avoid any possible misunderstanding, we mention here explicitly that an *interval*, by itself, may be finite (closed, open, or half open) as well as infinite, while neither the void set nor the singletonic sets are counted among intervals. By a *nonterminate interval* we shall understand any interval which is an open set. Every open interval is of course nonterminate, but not *vice versa*.

The union of the first $n$ constituents of a sequence of sets will be called its *nth partial union*. Needless to say, if the sequence is finite and of length $k$, this number $n$ can only range over $1, 2, \cdots, k$.

LEMMA. *If the union* $[\varDelta]$ *of a sequence* $\varDelta$ *of nonterminate intervals is an interval, it is always possible to extract, from among the constituents of* $\varDelta$, *a distinct sequence* $\varDelta^*$ (*i.e. one without repetitions*) *whose union coincides with* $[\varDelta]$ *and each of whose partial unions is an interval.*

This is a very slight extension of the lemma given in [1] on p. 253 *et seq.* The proof is a strictly verbal repetition of that for the mentioned lemma and may be omitted. It is worth notice that the sequence $\varDelta^*$ is not necessarily a subsequence of the original sequence $\varDelta$.

THEOREM 2. *In order that a function* $\varphi$ *should dwindle on a set* $S$, *it is necessary that given any* $\varepsilon>0$ *we can cover this set* $S$ *with a non-overlapping sequence of closed intervals* $\langle I_1, I_2, \cdots\rangle$ *such that* $\sum \mathrm{d}(\varphi[I_n \cap S])<\varepsilon$.

*This condition is also sufficient, provided the fudction* $\varphi$ *is continuous on* $S$.

REMARK. The continuity assumption on the function cannot be deleted off the sufficiency part of the theorem, as seen by considering the simple case in which the set $S$ is the real line and the function $\varphi(x)$ is equal to 0 or 1 according as $x<0$ or $x\geqq 0$ respectively.

PROOF. (a) Sufficiency. If $\varphi$ is continuous on $S$, then for each $n$ we can enclose the intersection $I_n \cap S$ in an open interval $H_n$ such that

$$\mathrm{d}(\varphi[H_n \cap S])<\mathrm{d}(\varphi[I_n \cap S])+2^{-n}\varepsilon .$$

The sequence $\langle H_1, H_2, \cdots\rangle$ covers $S$ and we have

$$\sum_n \mathrm{d}(\varphi[H_n \cap S])<\sum_n \mathrm{d}(\varphi[I_n \cap S])+\sum_n 2^{-n}\varepsilon<2\varepsilon ,$$

which completes the sufficiency proof.

(b) Necessity. The proof will proceed virtually in the same way as for the Theorem on p. 254 of [1]. Given any $\delta>0$, we can cover the set $S$ with a sequence of open intervals $\Theta=\langle H_1, H_2, \cdots\rangle$ such that $\sum \mathrm{d}(\varphi[H_m \cap S])<\delta$, where $m$ ranges over all its values.

Noting that the union $[\Theta]$ is a nonvoid open set, let us consider any component interval, say $D$, of this union, so that $D$ is a nonterminate interval. Then those intervals $H_m$ which are contained in $D$ form together a subsequence of $\Theta$, which we shall denote by $\varDelta$. Since plainly $[\varDelta]=D$, the above Lemma applies to this sequence $\varDelta$. Thus $D$ is the union of a distinct sequence $\varDelta^*=\langle G_1, G_2, \cdots\rangle$ whose constituent intervals $G_i$ belong to $\varDelta$ and all whose partial unions $P_1, P_2, \cdots$ are intervals.

Let us write $A_1=P_1$ and $A_i=P_i\setminus P_{i-1}$ for $i>1$ as long as $P_i$ exists, so that $\langle A_1, A_2, \cdots\rangle$ is a disjoint sequence of sets and has $D$ for its union. Each set $A_i$ is either (i) void, or (ii) a subinterval of the open interval $G_i$, or (iii) the union of two disjoint subintervals of $G_i$, say $B_i$ and $C_i$. In case (i) we simply strike $A_i$ off the sequence $\langle A_1, A_2, \cdots\rangle$; in case (ii) we replace $A_i$ by its closure $\bar{A}_i$; in case (iii) we replace $A_i$ by the pair of the closures $\bar{B}_i$ and $\bar{C}_i$. Applying this procedure to $\langle A_1, A_2, \cdots\rangle$ for $i=1, 2, \cdots$ as long as $A_i$ exists, we obtain a new sequence $\varPsi_D=\langle K_1, K_2, \cdots\rangle$, which is a non-overlapping sequence of closed intervals. We find at once that the union $[\varPsi_D]$ is $D$ and that

$$\sum_j \mathrm{d}(\varphi[K_j\cap S])\leqq 2\sum_i \mathrm{d}(\varphi[G_i\cap S]).$$

Writing $I$ generically for a closed interval which appears in the sequence $\varPsi_D$ for some $D$, we arrange all the intervals $I$ in a distinct sequence $\langle I_1, I_2, \cdots\rangle$. This sequence conforms to the assertion if $2\delta<\varepsilon$, on account of the appraisal

$$\sum_n \mathrm{d}(\varphi[I_n\cap S])=\sum_D\sum_j \mathrm{d}(\varphi[K_j\cap S])\leqq 2\sum_m \mathrm{d}(\varphi[H_m\cap S])<2\delta.$$

THEOREM 3. *If a function $\varphi$ dwindles on a subset $S$ of a closed interval $I$, then given an arbitrary $\varepsilon>0$ we can cover the set $S$ with a non-overlapping sequence $\langle I_1, I_2, \cdots\rangle$ of closed intervals contained in $I$, in such a manner that*

$$\sum \mathrm{d}(\varphi[I_n\cap S])<\varepsilon.$$

PROOF. We may plainly assume that the set $S$ contains a point, say $c$, which is interior to the interval $I$. We take an arbitrary $\delta>0$, and we shall keep both $c$ and $\delta$ fixed during the proof.

By the foregoing theorem the set $S$ admits a covering by a non-overlapping sequence of closed intervals $\langle J_1, J_2, \cdots\rangle$ such that $\sum \mathrm{d}(\varphi[J_i\cap S])<\delta$. From this sequence we shall derive another non-overlapping sequence $\langle K_1, K_2, \cdots\rangle$ of closed intervals by means of the following process which we shall temporarily call *left hand modification.*

Writing $I=[a, b]$, so that $a<c<b$, we shall distinguish two cases according as $a\notin S$ or $a\in S$.

Let us first treat the case $a\notin S$. Consider any interval $J_i$ of the sequence $\langle J_1, J_2, \cdots\rangle$. If $J_i\subset(-\infty, a]$, we delete $J_i$ off this sequence. On account of the presence of the point $c$, not all the $J_i$ are thus deleted. On the other hand, if $J_i\not\subset(-\infty, a]$, then the intersection $J_i\cap[a, +\infty)$ must be a closed interval, and we replace $J_i$ by this interval. We thus obtain from $\langle J_1, J_2, \cdots\rangle$ a new non-overlapping sequence of closed intervals, which is the required $\langle K_1, K_2, \cdots\rangle$.

We now pass on to the remaining case $a\in S$. Since the function $\varphi$ is continuous on $S$, we can take a point $p$ such that $a<p<c$ and $\mathrm{O}(\varphi; S\cap[a, p])<\delta$. Consider any interval $J_i$ of the sequence $\langle J_1, J_2, \cdots\rangle$. If $J_i\subset(-\infty, p]$, we strike $J_i$ off this sequence. By the presence of the point $c$, not all the $J_i$ are thus struck off. If $J_i \not\subset (-\infty, p]$, then $J_i \frown [p, +\infty)$ is a closed interval, and we replace $J_i$ by this interval. We denote by $\langle K_2, K_3, \cdots\rangle$ the new sequence thus derived from $\langle J_1, J_2, \cdots\rangle$. Then $\langle K_1, K_2, \cdots\rangle$, where $K_1=[a, p]$, is the required sequence.

Making a scrutiny into the above construction, we ascertain without difficulty that the new sequence has the following properties: (a) $\langle K_1, K_2, \cdots\rangle$ is a non-overlapping sequence of closed intervals contained in $[a, +\infty)$; (b) this sequence covers the set $S$; (c) we have $\sum \mathrm{d}(\varphi[K_j \frown S])<2\delta$, where $j$ ranges over all its values.

To the sequence $\langle K_1, K_2, \cdots\rangle$ we now apply a *right hand modification*, the meaning of which it would be needless to expound. The resulting sequence, for which we write $\langle I_1, I_2, \cdots\rangle$, must have three properties corresponding to (a)$\sim$(c) above. We thus conclude finally that $\langle I_1, I_2, \cdots\rangle$ is a non-overlapping sequence of closed subintervals of $I$ and that it covers $S$ so as to satisfy $\sum \mathrm{d}(\varphi[I_n \frown S])<3\delta$. This completes the proof, since $\delta$ is arbitrary.

THEOREM 4. *If a function $\varphi$ dwindles on a set $S$ of finite outer measure, then we can make correspond to each $\varepsilon>0$ a non-overlapping sequence $\langle I_1, I_2, \cdots\rangle$ of closed intervals with lengths $|I_n|<\varepsilon$, so as to cover $S$ and to fulfil simultaneously*

$$\sum_n |I_n|<|S|+\varepsilon \quad \textit{and} \quad \sum_n \mathrm{d}(\varphi[I_n \frown S])<\varepsilon .$$

PROOF. By hypothesis, we can cover $S$ with a non-overlapping sequence $\langle K_1, K_2, \cdots\rangle$ of closed intervals with lengths $|K_i|<\varepsilon$ in such a way that $\sum|K_i|<|S|+\varepsilon$. Since the function $\varphi$ dwindles for each $i$ on the intersection $K_i \frown S$, the foregoing theorem shows that each $K_i \frown S$ can be covered by a non-overlapping sequence $\langle L_1^{(i)}, L_2^{(i)}, \cdots\rangle$ consisting of closed subintervals of $K_i$ and fulfilling the inequality

$$\mathrm{O}(\varphi;\, L_1^{(i)} \frown S)+\mathrm{O}(\varphi;\, L_2^{(i)} \frown S)+\cdots<2^{-i}\varepsilon .$$

If we consider all the intervals $L_j^{(i)}$ and arrange them in a simple sequence $\langle I_1, I_2, \cdots\rangle$, we see readily that this sequence is non-overlapping and covers the set $S$. Moreover, $|I_n|<\varepsilon$ for every $n$ and

$$\sum_n |I_n|=\sum_i \sum_j |L_j^{(i)}| \leqq \sum_i |K_i|<|S|+\varepsilon ,$$

$$\sum_n \mathrm{O}(\varphi;\, I_n \frown S)=\sum_i \sum_j \mathrm{O}(\varphi;\, L_j^{(i)} \frown S)<\sum_i 2^{-i}\varepsilon=\varepsilon .$$

THEOREM 5. *If a function $\varphi$ is absolutely continuous on a set $S$, then to each $\varepsilon>0$ there corresponds a $\delta>0$ such that, for every non-overlapping sequence of closed intervals $\langle I_1, I_2, \cdots\rangle$ fulfilling $\sum|I_n|<\delta$, we have $\sum_n \mathrm{d}(\varphi[I_n \frown S])<\varepsilon$.*

*In particular, therefore, a function which is absolutely continuous on a given set of measure zero, necessarily dwindles on the same set.*

PROOF. By the definition of absolute continuity, given any $\eta>0$ there exists a $\delta>0$ such that for every non-overlapping sequence $\langle J_1, J_2, \cdots\rangle$ of closed intervals whose endpoints belong to $S$, the inequality

$$\sum_k |J_k| < \delta \quad \text{implies} \quad \sum_k |\varphi(J_k)| < \eta\,,$$

where $\varphi(J_k)$ means the increment over $J_k$ of the function $\varphi$. We shall show in the following lines that this number $\delta$ certainly conforms to our assertion if we take $\eta = 2^{-1}\varepsilon$.

Let $\langle I_1, I_2, \cdots\rangle$ be any sequence considered in the theorem. In order to verify the displayed inequality of the theorem, we may suppose without loss of generality that, for every $n$, the set $I_n \cap S$ contains at least two points. Noting that the function $\varphi$ is bounded on every bounded set, we make correspond to each $n$, as we clearly can, a closed interval $K_n$ whose endpoints belong to $I_n \cap S$ and which fulfils the inequality

$$\mathrm{d}(\varphi[I_n \cap S]) < |\varphi(K_n)| + 2^{-n}\eta\,.$$

Since $\sum |K_n| \leqq \sum |I_n| < \delta$, we then have

$$\sum_n \mathrm{d}(\varphi[I_n \cap S]) < \sum_n |\varphi(K_n)| + \sum_n 2^{-n}\eta < \eta + \eta = \varepsilon\,,$$

which completes the proof.

THEOREM 6. *If a function $\varphi$ is absolutely continuous on a null sel $S$ (i.e. a set of measure zero) and if a function $\psi$ dwindles on this set, then their sum $\varphi + \psi$ dwindles on $S$.*

PROOF. Given any $\varepsilon > 0$, let us take a $\delta > 0$ agreeable to the assertion of Theorem 5. Writing $\eta = \min(\varepsilon, \delta)$ and applying Theorem 4 to the function $\psi$, we find that the set $S$ can be covered by a non-overlapping sequence of closed intervals $\langle I_1, I_2, \cdots\rangle$ fulfilling simultaneously

$$\sum_n |I_n| < \eta \quad \text{and} \quad \sum_n \mathrm{d}(\psi[I_n \cap S]) < \eta\,.$$

It then follows from Theorem 5 that $\sum_n \mathrm{d}(\varphi[I_n \cap S]) < \varepsilon$.

On the other hand, if we write $\chi = \varphi + \psi$, the inequality

$$\mathrm{d}(\chi[L]) \leqq \mathrm{d}(\varphi[L]) + \mathrm{d}(\psi[L])$$

holds for any linear set $L$ whatsover. In point of fact, for every pair of points $u$, $v$ of $L$ we have the relation

$$|\chi(u) - \chi(v)| \leqq |\varphi(u) - \varphi(v)| + |\psi(u) - \psi(v)| \leqq \mathrm{d}(\varphi[L]) + \mathrm{d}(\psi[L])\,,$$

whence the result follows at once. Consequently

$$\mathrm{d}(\chi[I_n \cap S]) \leqq \mathrm{d}(\varphi[I_n \cap S]) + \mathrm{d}(\psi[I_n \cap S])$$

for all $n$ under consideration, and hence

$$\sum_n \mathrm{d}(\chi[I_n \cap S]) \leqq \sum_n \mathrm{d}(\varphi[I_n \cap S]) + \sum_n \mathrm{d}(\psi[I_n \cap S]) < 2\varepsilon\,.$$

This completes the proof, since the function $\chi$ is continuous on $S$ together with $\varphi$ and $\psi$.

THEOREM 7. *A function $\varphi$ which dwindles on every null subset of a set $E$, is necessarily continuous on $E$.*

PROOF. Let $c$ be a fixed point of the set $E$ supposed nonvoid. It suffices to show that for every point sequence $\langle x_n; n \in \boldsymbol{N} \rangle$ taken from $E$ and converging to $c$, the sequence $\langle \varphi(x_n); n \in \boldsymbol{N} \rangle$ converges to $\varphi(c)$. By Theorem 1, the function $\varphi$ is continuous on the null set $S=\{c, x_1, x_2, \cdots\cdots\}$. Hence the result.

A function which fulfils the hypothesis of the above theorem, will be called *continuous* (M) on the set $E$. Such a function maps every null subset of $E$ onto a null set, by Theorem 1, and hence fulfils the condition (N) on $E$. Further, such a function is continuous (M) on every subset of $E$.

From now on, if a theorem not quoted from elsewhere is stated without a proof, it will be meant that the proof is easy and left to the reader.

THEOREM 8. *Every function which is absolutely continuous on a set, is continuous* (M) *on the same set.*

THEOREM 9. *The sum of two functions one of which is absolutely continuous on a set and the other of which is continuous* (M) *on the same set, is necessarily continuous* (M) *on this set.*

A function which is continuous on a set $E$ and which fulfils the condition (N) on this set, will be termed *continuous* (N) on $E$. Continuity (M) on a set implies continuity (N) on the same set.

We shall call a function $\varphi$ to be *generalized continuous* (M), or simply GCM, on a set $E$, if it is continuous on $E$ and if $E$ is the union of a sequence of sets on each of which $\varphi$ is continuous (M).

THEOREM 10. *A function which is generalized continuous* (M) *on a set, is necessarily continuous* (N) *on this set.*

THEOREM 11. *Every function which is generalized absolutely continuous on a set, is generalized continuous* (M) *on the same set.*

THEOREM 12. *The sum of two functions one of which is generalized absolutely continuous on a set and the other of which is generalized continuous* (M) *on the same set, is necessarily generalized continuous* (M) *on this set.*

We dealt in our paper [1] with functions which we called generalized highly continuous (or GHC) on a set. We now go on to prove that these functions are generalized continuous (M) on the same set. For this purpose, it is convenient to begin by recapitulating a few relevant definitions.

A function $\varphi$ will be called *strongly semiabsolutely continuous* (or SSC) on a linear set $E$, if there exists in the interval $(0, 1]$ a number $\alpha$ (depending on both the function and the set) with the following properties: (i) The set $E$ is of

$\alpha$-dimensional volume zero; (ii) given any $\varepsilon>0$ there is an $\eta>0$ such that for every non-overlapping sequence $\langle I_1, I_2, \cdots\cdot\rangle$ of closed intervals whose extremities belong to $E$, the inequality $|I_1|^\alpha+|I_2|^\alpha+\cdots\cdot<\eta$ implies $|\varphi(I_1)|+|\varphi(I_2)|+\cdots\cdot<\varepsilon$. On account of this property (ii), a function which is SSC on a set is necessarily continuous on this set.

We shall term a function $\varphi$ to be *highly continuous* (or HC) on a set $E$, if on this set it is either AC (absolutely continuous) or SSC. Again, a function $\varphi$ will be called *generalized highly continuous* (or GHC) on a set $E$, if it is continuous on $E$ and if $E$ is expressible as the union of a sequence of sets on each of which $\varphi$ is HC.

The following lemma is a special case of the Theorem given on p. 254 of [1].

LEMMA. *If $E$ is a set of $\alpha$-dimensional volume zero, where $\alpha$ is some positive number, then for any $\delta>0$ we can cover the set $E$ with a non-overlapping sequence of closed intervals $\langle K_1, K_2, \cdots\cdot\rangle$ such that $\sum|K_n|^\alpha<\delta$.*

THEOREM 13. *Every function which is generalized highly continuous on a set, is generalized continuous* (M) *on the same set.*

PROOF. In view of Theorem 8, we need only show that a function $\varphi$ which is SSC on a set $E$, is continuous (M) on this set.

Let $\alpha$ be the number which appears in the above definition of an SSC function. The set $E$, which is of $\alpha$-dimensional volume zero, must have measure zero. Accordingly it suffices to verify that the function $\varphi$ dwindles on $E$. Let us write $\delta=\min(\varepsilon, \eta)$, where $\varepsilon$ and $\eta$ are the numbers which appear in the property (ii) above. By the Lemma stated just now, we can cover the set $E$ with a non-overlapping sequence of closed intervals $\langle K_1, K_2, \cdots\cdot\rangle$ such that $\sum|K_n|^\alpha<\delta$. It is enough to deduce $\sum \mathrm{d}(\varphi[K_n\cap E])<2\varepsilon$, where we may, without loss of generality, suppose that every $K_n\cap E$ contains at least two points.

For each $n$ under consideration, we can choose a closed interval $I_n$ whose endpoints belong to $K_n\cap E$ and which fulfils the inequality

$$\mathrm{d}(\varphi[K_n\cap E])<|\varphi(I_n)|+2^{-n}\delta\,.$$

Then $\langle I_1, I_2, \cdots\cdot\rangle$ is a non-overlapping sequence of closed intervals with endpoints belonging to the set $E$, and we have the relation $\sum|I_n|^\alpha\leqq\sum|K_n|^\alpha<\delta\leqq\eta$. Hence, the above property (ii) gives $\sum|\varphi(I_n)|<\varepsilon$. It follows that

$$\sum_n \mathrm{d}(\varphi[K_n\cap E])<\sum_n|\varphi(I_n)|+\sum_n 2^{-n}\delta<\varepsilon+\delta\leqq 2\varepsilon\,, \qquad \text{q. e. d.}$$

REMARK. This theorem includes Theorem 11; in fact, by the definition of a GHC function, a function which is GAC (generalized absolutely continuous) on a set is always GHC on the same set. On the other hand, we constructed in the paper [2] a class of functions which are GHC, without being GAC, on the unit interval [0, 1]. These functions must, however, be generalized continuous (M) on [0, 1] on account of the present theorem.

Let $E$ be any linear set. A function will be called *approximately derivable* (N) [or (M)] on the set $E$, if it is continuous (N) [or generalized continuous (M)] on $E$ and approximately derivable at almost all points of $E$. We shall often write simply ADN and ADM for these properties, respectively.

Every function which is GHC on a measurable set must be ADM on this set, on account of the preceding theorem and the Theorem on p. 253 of [1]. Again, it is evident that a function which is ADM on a set $E$, is necessarily ADN on $E$. However, the converse is false as shown by the following theorem.

THEOREM 14. *Let $U$ be the unit closed interval* $[0, 1]$.
(i) *There are functions which are* ADM, *but not* GAC, *on $U$.*
(ii) *There are functions which are* ADN, *but not* ADM, *on $U$.*

PROOF. *Re* (i): As already stated in the above Remark, there certainly exist functions which are GHC, and hence ADM, without being GAC, on $U$.

*Re* (ii): S. Mazurkiewicz constructed in his paper [4] a function $f(x)$ which is defined and continuous (N) on a nonvoid compact null set $Q \subset U$ and further such that the function $f(x)+\alpha x$, where $\alpha$ is any positive constant, maps the set $Q$ onto a nonnull set. We can easily extend this function $f(x)$ to one defined and ADN on the real line. However, on account of Theorem 12, this extended function cannot be ADM on $U$. By adding an arbitrary constant to the extended $f(x)$, we obtain an infinity of functions which are ADN, but not ADM, on $U$.

THEOREM 15. *If $D$ is a nonvoid open set contained in a closed interval $K$ and if a function $\varphi$ is absolutely continuous on the compact set $Q=K\setminus D$, then the linear modification $\lambda$ of this function with respect to $Q$ is absolutely continuous on the whole interval $K$.*

PROOF. During the proof, we shall understand by an *interval of the first* [or *second*] *kind* any closed interval whose endpoints belong to the set $Q$ [or whose interior is contained in the set $D$]. Clearly, every interval of the first or second kind is a subinterval of $K$. Of course, it is possible that an interval is of the first and the second kind simultaneously. Again, any non-overlapping sequence of intervals of the first [or second] kind will simply be called *sequence of the first* [or *second*] *kind.*

Let us arrange the component open intervals of $D$ in a distinct sequence $\langle I_1, I_2, \cdots\cdots \rangle$. Since the function $\varphi$ is AC on $Q$, given any $\varepsilon>0$ there is a $\delta>0$ such that for every sequence $\langle J_1, J_2, \cdots\cdots \rangle$ of the first kind the inequality

$$|J_1|+|J_2|+\cdots\cdots<\delta \quad \text{implies} \quad |\varphi(J_1)|+|\varphi(J_2)|+\cdots\cdots<\varepsilon.$$

We proceed to show that $\sum|\varphi(I_n)|<+\infty$, where $n$ ranges over all its values. For this purpose, we may suppose that $\sum|\varphi(I_n)|$ is an infinite series. As $\sum|I_n|=|D|<+\infty$, there is a $k\in \boldsymbol{N}$ for which we have $|I_k|+|I_{k+1}|+\cdots\cdots<\delta$. Accordingly, by considering the closure sequence $\langle \bar{I}_1, \bar{I}_2, \cdots\cdots \rangle$ which is of the first kind, we have $|\varphi(I_k)|+|\varphi(I_{k+1})|+\cdots\cdots<\varepsilon$. Hence $\sum|\varphi(I_n)|<+\infty$.

By definition, the linear modification $\lambda$ coincides on the set $Q$ with the function $\varphi$ and further is linear on the closure of each interval $I_n$. Combining this fact with $\sum|\lambda(I_n)|=\sum|\varphi(I_n)|<+\infty$, we find easily that if $\langle K_1, K_2, \cdots\rangle$ is any sequence of the second kind, then $\sum|\lambda(K_i)|$ can be made arbitrarily small by making $\sum|K_i|$ sufficiently small.

Consider now any non-overlapping sequence $\langle L_1, L_2, \cdots\rangle$ of closed subintervals of $K$. The proof will be complete if we verify that the sum $\sum|\lambda(L_j)|$ can be made arbitrarily small by making $\sum|L_j|$ sufficiently small. But we know that this actually comes about if $\langle L_1, L_2, \cdots\rangle$ is of the first or second kind. And the general case is ascribable to these special cases, since every $L_j$ which is neither of the first nor of the second kind, is expressible as the union of at most three non-overlapping intervals each of which is of the first or second kind.

We quote the following five Theorems 16~20 from pp. 233, 220, 285, 225, 227 of Saks [5] respectively. However, Theorems 16 and 17 are only partial quotations.

THEOREM 16. *In order that a function which is continuous on a nonvoid closed set $E$, be* GAC *on $E$, it is necessary and sufficient that every nonvoid closed subset of $E$ contain a portion on which the function is* AC.

THEOREM 17. *If two functions of a real variable coincide on a measurable set $E$, their approximate derivates coincide respectively at almost every point of $E$.*

THEOREM 18. *In order that a continuous function $\varphi$ be absolutely continuous on a closed interval $K$, it is necessary and sufficient that the function $\varphi$ fulfil on $K$ the condition* (N) *and the condition*

$$\int_P \varphi'(x)dx<+\infty$$

*simultaneously, where $P$ denotes the set of the points of $K$ at which the function $\varphi$ has a finite positive derivative.*

THEOREM 19. *A fcnction which is* GAC *on a set necessarily fulfils the condition* (N) *on this set.*

THEOREM 20. *In order that a function $\varphi$ which is both continuous and* BV *on a compact set $E$, be* AC *on $E$, it is necessary and sufficient that $\varphi$ fulfil on this set the condition* (N) *of Luzin.*

THEOREM 21. *Suppose that a function $\varphi$ is continuous* (N) *on a closed interval $I$ and that a function $\psi$ is* GAC *on $I$. If we have*

$$\varphi'_{\mathrm{ap}}(x)\leqq\psi'_{\mathrm{ap}}(x) \quad [or\ \varphi'_{\mathrm{ap}}(x)\geqq\psi'_{\mathrm{ap}}(x)]$$

*for almost every point $x\in I$ at which both $\varphi$ and $\psi$ are* AD, *then the difference $\psi-\varphi$ is* AC *and non-decreasing [or non-increasing], on the interval $I$, so that the function $\varphi$ is* GAC *on $I$ also.*

REMARK. The symbol AD is short for "approximately derivable." It is worth notice that the function $\varphi$ is not assumed to be ADN on $I$. This theorem includes Theorem 8 of [3] as well as the following Theorem 22 concerning linearly ordered series.

PROOF. We may restrict ourselves to the half of the assertion in which we assume the inequality $\varphi'_{\mathrm{ap}}(x) \leqq \psi'_{\mathrm{ap}}(x)$. We shall have recourse to the same technique as used in the proof for Theorem 7 of [3].

During the proof, we shall understand by an *admissible interval* any closed or open interval $J \subset I$ with the following two properties: (i) *the function* $\varphi$ *is* GAC *on* $J$; (ii) *every closed interval* $C \subset J$ *fulfils the inequality* $\varphi(C) \leqq \psi(C)$. As we readily see, in order that a closed interval be admissible, it is necessary and sufficient that its interior be so.

Let $S$ be the union of all the admissible open intervals, so that $S$ is an open set contained in $I$. Supposing for the time being that $S$ is nonvoid, we decompose $S$ into its component open intervals, which we shall denote generically by $H$. We are going to show that each $H$ is an admissible interval. For this purpose, it is clearly sufficient to verify that if $A$ is a closed interval contained in $H$, the function $\varphi$ is GAC on $A$ and we have $\varphi(A) \leqq \psi(A)$.

There exists, for such an interval $A$, a $\delta > 0$ such that any two points $p$ and $q$ of $A$ both belong to one of the admissible open intervals, provided only that $|p-q| < \delta$. Indeed, if the contrary were true, we could extract from $A$ two infinite sequences of points, say $\langle p_n \rangle$ and $\langle q_n \rangle$, such that $\lim |p_n - q_n| = 0$ and that, for each $n \in \boldsymbol{N}$, no admissible open interval would contain both $p_n$ and $q_n$. We may, without loss of generality, suppose that both these sequences converge to a common limit, say $\xi$, belonging to $A$. Since $\xi \in H$, there would exist an admissible open interval containing $\xi$. Both $p_n$ and $q_n$ would belong to this interval for $n$ sufficiently large, and we should thus arrive at a contradiction.

Let us express the interval $A$ considered just now as the union of a finite non-overlapping sequence $\langle A_1, \cdots, A_k \rangle$ of closed intervals. It follows immediately from the above that, if the lengths of $A_1, \cdots, A_k$ are simultaneously sufficiently small, each of these $k$ intervals is contained in some admissible open interval, so that the function $\varphi$ is GAC on every $A_i$ and we have the inequality $\varphi(A_i) \leqq \psi(A_i)$ for every $A_i$. It ensues that $\varphi$ is GAC on $A$ and that

$$\varphi(A) = \sum \varphi(A_i) \leqq \sum \psi(A_i) = \psi(A).$$

We have thus confirmed that every component interval $H$ of the union $S$ of the admissible open intervals is itself one of them, provided that $S$ is nonvoid.

We now do away with this last assumption on $S$ and we proceed to show that $S$ coincides in reality with the interior of the interval $I$. Let us write $E = I \setminus S$ for this purpose, so that $E$ is a nonvoid closed subset of $I$. (If $S$ happens to be nonvoid, the component intervals of $S$ are no other than the open intervals contiguous to this set $E$.) Assuming that $E$ contains at least one interior point of $I$, we shall derive a contradiction in what follows.

Let us show in the first place that every point $p \in E$ interior to $I$ must be an accumulation point for $E$. Indeed, if this were false, there would exist a pair of admissible open intervals $J_1$ and $J_2$ abutting at $p$. The union $J=J_1\cup J_2\cup\{p\}$ would then be an admissible open interval, as we can easily verify. It would follow that $p\in J\subset S$, which is certainly absurd.

This being so, we can take, in the interior of $I$, a closed interval $K_0$ whose endpoints belong to $E$ and whose intersection with $E$ is a nonvoid perfect set. Since the function $\psi$ is GAC on $I$, and hence on $E$ also, we deduce from Theorem 16 that this intersection $E\cap K_0$ contains a portion on which $\psi$ is AC. We then can choose in $K_0$ a closed interval $K$ whose endpoints belong to $E$ and whose intersection with $E$ is a nonvoid perfect set, say $Q$, contained in the portion just mentioned. The function $\psi$ is AC on this set $Q$.

We shall first treat the case in which $Q\neq K$, so that the open set $D=K\setminus Q$ is nonvoid. Each component interval $J$ of this set $D$ is contiguous to $Q$ and hence to $E$ also. Thus $J$ is a component of the set $S$. But we proved already that all the components of $S$ are admissible. It follows that we have $\varphi(J)\leqq\psi(J)$ for every component $J$ of $D$.

Since the function $\psi$ is AC on $Q$, we find by Theorem 15 that the linear modification $\Psi$ of $\psi$ with respect to the set $Q$, is AC on the interval $K$. Hence $\Psi$ is derivable at almost all points of $K$. For definiteness, let us define $\Psi'(x)$ to mean 0 at every point $x\in\boldsymbol{R}$ at which $\Psi$ is not derivable. Then $\Psi'(x)$ is summable on $K$.

Let $\Phi$ be the linear modification of the function $\varphi$ with respect to the set $Q$. We find at once that $\Phi$ is continuous (N) on $K$. Both $\Phi$ and $\Psi$ are derivable at all points of the open set $D$, since they are linear on every component $J$ of $D$. We have moreover $\Phi'(x)\leqq\Psi'(x)$ for all $x\in D$, as $\Phi(J)=\varphi(J)\leqq\psi(J)=\Psi(J)$.

Let us consider now the set, say $M$, of all the points of $K$ at which the function $\Phi$ is derivable. By what was stated just now, this set $M$ contains $D$. We proceed to examine the set $M\setminus D$, which is easily found to be a measurable subset of $Q$. Since the functions $\Phi$ and $\varphi$ coincide on $Q$, we see by Theorem 17 that their approximate derivates coincide respectively at almost every point of $Q$. A similar statement can of course be made for the pair of functions $\Psi$ and $\psi$. On the other hand, $\Psi(x)$ is derivable to $\Psi'(x)$ at almost every point of $K$, as already mentioned. By hypothesis, furthermore, we have $\varphi'_{\mathrm{ap}}(x)\leqq\psi'_{\mathrm{ap}}(x)$ at almost every point $x\in I$ at which both $\varphi(x)$ and $\psi(x)$ are AD. Combining all these items, we conclude that $\Phi'(x)\leqq\Psi'(x)$ at almost all points $x$ of $M\setminus D$. But we know that the same inequality holds for all $x\in D$. Consequently we have $\Phi'(x)\leqq\Psi'(x)$ for almost all points $x$ of the set $M=D\cup(M\setminus D)$.

We are now in a position to apply Theorem 18 to the function $\Phi$ and the interval $K$. Indeed, denoting by $P$ the set of the points of $K$ at which $\Phi$ has a finite positive derivative and taking notice of the summability of $\Psi'(x)$ on $K$, we find that the condition of the mentioned theorem is fulfilled as follows:

$$\int_P\Phi'(x)dx\leqq\int_P\Psi'(x)dx\leqq\int_K|\Psi'(x)|\,dx<+\infty\,.$$

It follows that the function $\Phi$ is AC on $K$. In consequence, $\Phi$ is derivable at almost all points of $K$, so that we have $\Phi'(x) \leqq \Psi'(x)$ almost everywhere on $K$. Let us, for definiteness, define $\Phi'(x)$ to be 0 at every point $x \in \boldsymbol{R}$ at which $\Phi$ is not derivable. Then $\Phi'(x)$ is summable on $K$.

As already mentioned, every component $J$ of the open set $D$ is an admissible interval. The function $\varphi$ is therefore GAC on every $J$, and hence on the set $D$, too. On the other hand, $\varphi$ coincides with $\Phi$ on the set $Q$, on which $\Phi$ is AC. It follows that $\varphi$ is GAC on the whole interval $K$.

We now want to show that $K$ is admissible. For this purpose, we need only prove the inequality $\varphi(C) \leqq \psi(C)$ for every closed interval $C \subset K$. A closed interval $A$ will, for a while, be called *manageable*, if it is contained in the closure of a component $J$ of $D$, or if both its endpoints belong to the set $Q$. (A manageable interval may happen to fulfil these two conditions simultaneously, but this does not matter.) We see at once that any closed interval $C \subset K$ can be expressed as the union of at most three non-overlapping intervals which are manageable. Accordingly our task in hand is reduced to verifying $\varphi(A) \leqq \psi(A)$ for every manageable interval $A$.

This inequality is obvious if $A \subset \bar{J}$, where $J$ is a component of $D$, since the closure $\bar{J}$ is admissible together with $J$. Supposing, therefore, that the endpoints of $A$ belong to the set $Q$, we have

$$\psi(A)-\varphi(A)=\Psi(A)-\Phi(A)=$$

$$=\int_A \Psi'(x)dx-\int_A \Phi'(x)dx=\int_A \{\Psi'(x)-\Phi'(x)\}\,dx \geqq 0\,,$$

since we have $\Phi'(x) \leqq \Psi'(x)$ almost everywhere on $K$, as already mentioned.

The interval $K$ is thus admissible. The set $S$ therefore contains the interior of $K$. However, this contradicts the fact that $K$ contains the nonvoid perfect set $Q$ which is disjoint with $S$.

Up to now, we have been dealing with the case $Q \neq K$. But the treatment of the remaining case $Q=K$ is far simpler, and we shall equally arrive at a contradiction. We leave the details to the reader.

To conclude, we have proved by *reductio ad absurdum* that the interior of $I$, and hence $I$ itself, is an admissible interval. In other words, the function $\varphi$ is GAC on $I$ and we have the inequality $\varphi(C) \leqq \psi(C)$ for every closed interval $C \subset I$. The difference $f(x)=\psi(x)-\varphi(x)$ is therefore non-decreasing on $I$. But this function $f$ is GAC on $I$, together with $\varphi$ and $\psi$. Theorem 19 then shows that $f$ is continuous (N) on $I$. It follows finally by Theorem 20 that $f$ is AC on $I$.

THEOREM 22. *Given two linearly ordered series $\sum f(\omega)$ and $\sum g(\omega)$, where $\omega$ ranges over a countable indexing set $\Omega$ furnished with a linear ordering, suppose that the former series is Luzin convergent and the latter Denjoy convergent. If $f(\omega) \leqq g(\omega)$ [or $f(\omega) \geqq g(\omega)$] for all $\omega \in \Omega$, then $\sum f(\omega)$ is Denjoy convergent also and we have*

$$\sum f(\omega) \leqq \sum g(\omega) \quad [or \ \sum f(\omega) \geqq g(\omega)]\,.$$

REMARK. In connection with this theorem, let it be mentioned that we do not know whether the following assertion is true: *With the same notation as above, if both* $\sum f(\omega)$ *and* $\sum g(\omega)$ *are Luzin convergent and if* $f(\omega) \leqq g(\omega)$ *for all* $\omega \in \Omega$, *then necessarily* $\sum f(\omega) \leqq \sum g(\omega)$.

Now let $I$ be a closed interval. We shall call two functions $\varphi$ and $\psi$ to be *approximately equiderivable* on $I$, if both of them are ADN on $I$ and if we have $\varphi'_{\mathrm{ap}}(x) = \psi'_{\mathrm{ap}}(x)$ for almost every point $x \in I$ at which both $\varphi$ and $\psi$ are AD. With this wording, Theorem 8 of [3] admits the following brief expression: *If two functions are approximately equiderivable on a closed interval I and if one of them is* GAC *on I, then they differ on I only by an additive constant.*

The class of all the functions that are ADN [or ADM] on a closed interval $I$, will be called the ADN [or ADM] *class* on $I$. Similarly for the GAC and GHC classes on $I$. Let $\mathfrak{S}$ be a nonvoid subclass of the ADN class on $I$. We shall call a function $\varphi$ to be *stable on I within the class* $\mathfrak{S}$, if it belongs to $\mathfrak{S}$ and if every member of $\mathfrak{S}$ with which $\varphi$ is approximately equiderivable on $I$ differs over $I$ from $\varphi$ only by an additive constant. When $\mathfrak{S}$ coincides, in particular, with the ADN [or ADM] class on $I$, a function stable on $I$ within $\mathfrak{S}$ will usually be called *function stable* (N) [or (M)] for short.

Again, in the above expressions, we may omit the reference to the interval $I$ if there is no fear of misunderstanding or confusion. For example, Theorem 8 of [3] can now be expressed as follows: *Every function which is* GAC *on a closed interval is stable* (N) *on this interval,* or more briefly, *every* GAC *function is stable* (N).

Let $\mathfrak{S}$ and $\mathfrak{T}$ be two nonvoid subclasses of the ADN class. $\mathfrak{S}$ will be termed *stable within* $\mathfrak{T}$, if $\mathfrak{S} \subset \mathfrak{T}$ and if all members of $\mathfrak{S}$ are stable within $\mathfrak{T}$. If especially $\mathfrak{S} = \mathfrak{T}$ here, we shall usually call $\mathfrak{S}$ *self-stable*. On the other hand, if in the above definition $\mathfrak{T}$ is the ADN [or ADM] class, $\mathfrak{S}$ will usually be called *stable* (N) [or (M)]. In other words, $\mathfrak{S}$ is stable (N) [or (M)] if and only if all its members are so. The GAC class is stable (N), while Theorem (ii) on p. 256 of [1] shows that the GHC class is self-stable.

A nonvoid class of functions will be termed *additive*, or *closed under addition*, if the sum of its two members (identical or not) always belongs to it. For example, the GAC and the GHC class are additive. However, the ADN class is not additive, as shown incidentally in the proof for the assertion (ii) of Theorem 14.

We do not know whether the following assertions are true. (The reference to the underlying interval has been omitted.)

(i) *The* ADN *class is self-stable.* (ii) *The* ADM *class is stable* (N). (iii) *The* ADM *class is self-stable.* (iv) *The* GHC *class is stable* (N). (v) *The* GHC *class is stable within the* ADM *class.* (vi) *The* ADM *class is closed under addition.*

We now quote the following result from p. 299 of Saks [5].

THEOREM 23. *The extreme approximate derivates of any finite measurable function of one real variable are themselves measurable functions.*

We shall call a function $f$ to be *integrable* (N) [or (M)] on a closed interval $I$, if there exists a function $\varphi$ stable (N) [or (M)] on $I$, such that $f(x)$ coincides with the approximate derivative $\varphi'_{\mathrm{ap}}(x)$ at almost every point $x \in I$ at which the function $\varphi$ is AD. When this is the case, the increment $\varphi(I)$, which is plainly uniquely determined, will be called the *integral* (N) [or (M)] of the function $f$ on the interval $I$. These integrals (N) and (M) will be denoted by $\mathrm{N}(f; I)$ and $\mathrm{M}(f; I)$ respectively. We shall also use the expressions *Luzin integrable* [or *integral*] in place of "integrable [or integral] (N)" respectively.

We find at once by Theorem 23 that *a function which is integrable* (N) *or* (M) *on a closed interval $I$ is necessarily measurable on $I$.*

Of the following three theorems, the first one is evident, the second follows directly from the stability (N) of the GAC class, and the third is ensured by Theorem 21.

THEOREM 24. *If a function $f$ is Luzin integrable as well as integrable* (M), *on a closed interval $I$, then the Luzin integral coincides with the integral* (M), *i.e. we have* $\mathrm{N}(f; I) = \mathrm{M}(f; I)$.

THEOREM 25. *Every function $f$ which is Denjoy integrable on a closed interval $I$, is Luzin integrable as well as integrable* (M), *on the same interval, and the three kinds of integral coincide one with another, i.e.*

$$\mathrm{N}(f; I) = \mathrm{M}(f; I) = \mathrm{D}(f; I),$$

*where* $\mathrm{D}(f; I)$ *stands for the Denjoy integral.*

THEOREM 26. *Given two functions $f$, $g$ and a closed interval $I$, suppose that, on this interval, $f$ is Lusin integrable and $g$ is Denjoy integrable. If we have the inequality $f(x) \leqq g(x)$ [or $f(x) \geqq g(x)$] almost everywhere on $I$, then the function $f$ is necessarily Denjoy integrable on $I$.*

REMARK. We do not know whether the following assertion is true:

*If two functions $f$, $g$ are integrable* (N) [*or* (M)] *on a closed interval $I$ and if we have $f(x) \leqq g(x)$ almost everywhere on $I$, then*

$$\mathrm{N}(f; I) \leqq \mathrm{N}(g; I) \quad [\textit{or } \mathrm{M}(f; I) \leqq \mathrm{M}(g; I)].$$

*In other words, integration* (N) [*or* (M)] *is a monotone non-decreasing functional of the integrand function.*

THEOREM 27. *Given two functions $f$, $g$ and a closed interval $I$, suppose that, on this interval, $f$ is integrable* (M) *and $g$ is Denjoy integrable. Then the sum $h = f + g$ is also integrable* (M) *on $I$ and we have*

$$\mathrm{M}(h; I) = \mathrm{M}(f; I) + \mathrm{D}(g; I).$$

REMARK. We do not know whether, in this theorem, the integrability (M) of the function $f$ can be replaced by Luzin integrability. Much less do we know whether the following is true:

*The class of the functions which are integrable* (N) [*or* (M)] *is additive, and the integration* (N) [*or* (M)] *is an additive functional on this class.*

This assertion is, as we see without difficulty, equivalent to the assertion that *the class of the functions which are stable* (N) [*or* (M)] *is additive.*

PROOF. By hypothesis we can select two functions $\varphi$, $\psi$ the former of which is stable (M) on $I$ and the latter is GAC on $I$, so as to fulfil simultaneously

$$f(x)=\varphi'_{\mathrm{ap}}(x) \quad \text{and} \quad g(x)=\psi'_{\mathrm{ap}}(x)$$

for almost all $x\in I$. By theorem 12, the sum $\chi(x)=\varphi(x)+\psi(x)$ is generalized continuous (M) on $I$ and we have

$$\chi'_{\mathrm{ap}}(x)=\varphi'_{\mathrm{ap}}(x)+\psi'_{\mathrm{ap}}(x)=f(x)+g(x)=h(x)$$

for almost all $x\in I$, so that $\chi$ is ADM on $I$.

We proceed to show that this function $\chi$ is stable (M) on $I$. For this purpose, let $\theta$ be any function which is, on the interval $I$, ADM and approximately equiderivable with $\chi$, so that $\theta'_{\mathrm{ap}}(x)=\chi'_{\mathrm{ap}}(x)$ for almost all the points $x\in I$. Writing $\rho(x)=\theta(x)-\psi(x)$ we find, by Theorem 12, that $\rho$ is generalized continuous (M) on $I$. We have further

$$\rho'_{\mathrm{ap}}(x)=\theta'_{\mathrm{ap}}(x)-\psi'_{\mathrm{ap}}(x)=\chi'_{\mathrm{ap}}(x)-\psi'_{\mathrm{ap}}(x)=\varphi'_{\mathrm{ap}}(x)$$

for almost all $x\in I$. Consequently $\rho$ is ADM and approximately equiderivable with $\varphi$, on $I$. Since $\varphi$ is stable (M) by hypothesis, the difference $\rho-\varphi$ must be a constant, say $c$, over $I$. We then have

$$\theta(x)-\chi(x)=\rho(x)-\varphi(x)=c \quad (x\in I),$$

which shows that $\chi$ is stable (M) on $I$.

It follows that the function $h$ is integrable (M) on $I$ and that

$$\mathrm{M}(h; I)=\chi(I)=\varphi(I)+\psi(I)=\mathrm{M}(f; I)+\mathrm{D}(g; I),$$

which completes the proof.

We do not know whether the following assertions are true:

(i) *Luzin integration amounts to the same as integration* (M). (ii) *Lusin integration is no more than that of Denjoy.* (iii) *Integration* (M) *is no more than that of Denjoy.*

## References

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