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On the Existence of Condenser-type Measures with Respect to Function-Kernels

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§ 1. Introduction.

Let X be a locally compact Hausdorff space with a countable base, G be a continuous function-kernel on X and N be a lower semicontinuous function-kernel which is locally bounded outside the diagonal set. Further, assume that each non-empty open set is not G-negligible and not N-negligible. In this note we shall ask necessary and sufficient conditions in order that for any pair $\langle K, F\rangle$ of disjoint compact sets there exists a (G, N)-condenser-type measure α of each $\lambda \in M_K^+$ of which $N\lambda$ is locally bounded on K, i.e., there exists a measure $\alpha=\mu_0-\mu_1$ such that

$$\mathrm{supp}(\mu_0)\subset K, \qquad \mathrm{supp}(\mu_1)\subset F,$$

$$G\alpha=N\lambda \quad G\text{-n.e. on } K, \qquad G\alpha=0 \quad G\text{-n.e. on } F,$$

$$0\leqq G\alpha\leqq N\lambda \quad \text{on } X.$$

In case where $N=1$ (the constant kernel) this problem has been solved in [6], i.e. G satisfies the condenser principle if and only if, G is non-degenerate and it satisfies the domination and the classical maximum principles.

In § 3 we shall show that for any pair $\langle K, F\rangle$ of a non-G-negligible compact set K and a compact set F with $A(K)\cap F=\emptyset$, there exists a (G, N)-condenser-type measure of each $\lambda\in M_K^+$ of which $N\lambda$ is bounded on K if and only if, G satisfies the domination principle and the relative domination principle with respect to N.

It is well-known that in Dirichlet spaces there exists a condenser measure for any pair $\langle \bar{U}, \bar{V}\rangle$ of a relatively compact open set U and an open set V with $\bar{U}\cap \bar{V}=\emptyset$. Further, C. Berg has proved in [1] the existence of the condenser measure for any pair $\langle \bar{U}, \bar{V}\rangle$ with respect to the kernel $\kappa=\int\mu_t dt$, where $(\mu_t)_{t>0}$ be a transient convolution semigroup on a locally compact abelian group.

In § 4 we shall ask similar problems for any pair $\langle K, F\rangle$ of a non-G-negligible compact set K and a closed set F with $A(K)\cap F=\emptyset$.

§ 2. Various domination pinciples.

Let X be a locally compact Hausdorff space with a countable base. A lower semicontinuous function-kernel on X is a non-negative lower semicontinuous function defined on $X \times X$ which is strictly positive on the diagonal set Δ of $X \times X$. A continuous function-kernel is a lower semicontinuous kernel which is continuous in the extended sense in $X \times X$ and finite-valued outside Δ. We use G and N for denoting lower semicontinuous function-kernels and introduce various domination principles.

DEFINITION 1. We say that G satisfies the relative domination principle with respect to N, if for $\mu, \nu \in M_K^+$ with $\int G\mu d\mu < +\infty$

$$G\mu \leqq N\nu \quad \text{on} \quad \mathrm{supp}(\mu) \qquad \text{implies} \quad G\mu \leqq N\nu \quad \text{in} \quad X.$$

We use the symbol $G \prec N$ for this principle.

"$G \prec G$" is said to be simply the domination principle.

"$G \prec 1$" is said to be the classical maximum principle.

DEFINITION 2. We say that G satisfies the transitive domination principle with respect to N, if for $\mu, \nu \in M_K^+$ with $\int G\mu d\mu < +\infty$

$$G\mu \leqq G\nu \quad \text{on} \quad \mathrm{supp}(\mu) \qquad \text{implies} \quad N\mu \leqq N\nu \quad \text{in} \quad X.$$

We use the symbol $G \sqsubset N$ for this principle.

DEFINITION 3. We say that G satisfies the relative balayage principle with respect to N, if for every compact set K and every $\mu \in M_K^+$ such that $N\mu \neq +\infty$ on K there exists $\nu \in M_K^+$ such that

$$\mathrm{supp}(\nu) \subset K$$

$$G\nu \leqq N\mu \quad \text{in} \quad X$$

$$G\nu = N\mu \quad G\text{-n.e.} \quad \text{on} \quad K.$$

Here "$G\nu = N\mu$ G-n.e. on K" means that the set $\{x \in K;\ G\nu(x) \neq N\mu(x)\}$ is a G-negligible set and a Borel measurable subset A of X is called G-negligible if any measure $\lambda \in M_K^+$ with $\mathrm{supp}(\lambda) \subset A$ and $\int G\lambda d\lambda < +\infty$ is equal to 0. Such a measure ν is called a relatively balayaged measure of μ onto K with respect to (G, N). If G satisfies the relative balayage principle with respect to G, we say that G satisfies the balayage principle and a relatively balayaged measure with respect to (G, G) is called a G-balayaged measure, simply a balayaged measure. If G satisfies the relative balayage principle with respect to the constant kernel 1, we say that G satisfies the equilibrium principle.

DEFINITION 4. We say that G satisfies the continuity principle if $G\mu (\mu \in M_K^+)$ is finite and continuous everywhere whenever it is finite and continuous on supp(μ).

We denote by $\mathcal{L}(G)$ (resp. $\mathcal{F}(G)$) the set of measures $\mu \in M_K^+$ such that $G\mu$ are locally bounded (resp. $G\mu$ are finite and continuous).

Hereafter we shall assume that G is a continuous function-kernel on X and N is a lower semicontinuous function-kernel on X which is locally bounded outside the diagonal set Δ of $X \times X$.

PROPOSITION 1. *Assume that each non-empty open set is not G-negligible and not N-negligible. If* $G \,[\, N$, *then* $\mu \in \mathcal{L}(G)$ *implies* $\mu \in \mathcal{L}(N)$.

PROOF. Let μ be a measure in $\mathcal{L}(G)$. Obviously the potential $N\mu$ is locally bouuded outside supp(μ). Let x_0 be an arbitrary point in supp(μ). Since $G(x_0, x_0) > 0$, we can find a compact neighborhood K of x_0 and a constant c such that $cG(x, y) \geqq 1$ on $K \times K$. Denote by μ_0 the restriction of μ to K. Since $G\mu_0$ is also locally bounded, there are a compact neighborhood K_1 of x_0 and a constant b such that $K_1 \subset K$ and $G\mu_0 \leqq b$ on K_1. First, assume that x_0 is an isolated point. Then $\{x_0\}$ is not G-negligible and not N-negligible by the assumption and we have $G(x_0, x_0) < +\infty$ and $N(x_0, x_0) < +\infty$. Denote by μ_1 the restriction of μ_0 to K_1. Then $G\mu_1 \leqq bcG\varepsilon_{x_0}$ on K_1 and hence $N\mu_1 \leqq bcN\varepsilon_{x_0}$ on X by $G \,[\, N$. Since $N\varepsilon_{x_0}$ and $N(\mu - \mu_1)$ are bounded on K_1, $N\mu$ is also bounded on K_1. Secondly, we consider the case where x_0 is not an isolated point. Take $x_1 \in K_1 \setminus \{x_0\}$ and find a compact neighborhood K_2 of x_0 satisfying $K_2 \subset K_1$ and $x_1 \notin K_2$. If we denote by the restriction of μ_0 to K_2, we have $G\mu_2 \leqq bcG\varepsilon_{x_1}$ on K_2 and hence $N\mu_2 \leqq bcN\varepsilon_{x_1}$ on X. Since $N\varepsilon_{x_1}$ and $N(\mu - \mu_2)$ are bounded on K_2, $N\mu$ is bounded on K_2. Thus we obtain that $N\mu$ is locally bounded.

PROPOSIGION 2. *Assume that each non-empty open set is not G-negligible and not N-negligible. If G satisfies the continuity principle and* $G \,[\, N$, *then* $\check{G}$ *satisfies the relative balayage principle with respect to* $\check{N}$. *Here* $\check{G}$ *(resp.* $\check{N}$*) is the adjoint kernel of G (resp. N).*

PROOF. Let K be a non-G-negligible compact set and μ be a measure in M_K^+ with $N\mu \not\equiv +\infty$ on K. Put

$$\mathcal{S} := \{u = G\sigma - G\tau\,;\ \sigma \in M_K^+,\ \ \tau \in \mathcal{F}(G),\ \mathrm{supp}(\tau) \subset K\}$$

and define for each $f \in C(K)$

$$p(f) := \inf\left\{\int N\sigma d\mu - \int N\tau d\mu\,;\ u = G\sigma - G\tau \in \mathcal{S},\ u \geqq f \ \text{ on } \ K\right\}.$$

We remark that $N\tau$ is locally bounded by Proposition 1 and $\int N\tau d\mu < +\infty$. By the assumption we can find, for each $f \in C(K)$, $\sigma_0 \in \mathcal{F}(G)$ satisfying $|f| \leqq G\sigma_0$ on K. Then $p(f) \leqq \int N\sigma_0 d\mu < +\infty$. Take $u = G\sigma - G\tau \in \mathcal{S}$ with $u \geqq f$ on K and supp$(\tau) \subset K$. Then $G\tau \leqq G\sigma + G\sigma_0$ on K and $G \,[\, N$, we have $N\tau \leqq N\sigma + N\sigma_0$. Consequently $p(f) \geqq -\int N\sigma_0 d\mu > -\infty$. Since the mapping $f \mapsto p(f)$ is sublinear on $C(K)$, there exists a linear functional ν on $C(K)$ such that

$$\nu(f) \leqq p(f) \quad \text{for all} \quad f \in C(K).$$

If $f \leqq 0$, it follows that $\nu(f) \leqq p(f) \leqq 0$. Hence ν is a positive measure on K. For each $\sigma \in M_K^+$, we have

$$\int G\sigma d\nu = \sup\left\{\int f d\nu ;\ 0 \leqq f \leqq G\sigma \text{ on } K,\ f \in C(K)\right\}$$

$$\leqq \sup\{p(f) ;\ 0 \leqq f \leqq G\sigma \text{ on } K,\ f \in C(K)\} \leqq \int N\sigma d\mu .$$

Especially, $\int G\varepsilon_x d\nu \leqq \int N\varepsilon_x d\mu$ and hence $\check{G}\nu(x) \leqq \check{N}\mu(x)$ for each $x \in X$. For each $\tau \in \mathcal{F}(G)$ with $\operatorname{supp}(\tau) \subset K$, we have

$$\int -G\tau d\nu \leqq p(-G\tau) \leqq \int -N\tau d\mu .$$

Thus $\int \check{G}\nu d\tau = \int \check{N}\nu d\tau$ for each $\tau \in \mathcal{F}(G)$ with $\operatorname{supp}(\tau) \subset K$. Since G satisfies the continuity principle by the assumption, it follows that $\check{G}\nu = \check{N}\mu$ G-n.e. on K.

§ 3. Condenser-type theorems for disjoint compact sets.

Hereafter we assume that G is a continuous function-kernel and N is a lower semicontinuous function-kernel which is locally bounded outside Δ. Further, assume that each non-empty open set is not G-negligible and not N-negligible.

For each $x \in X$ and for a closed set K, we define

$$A(x) := \{y \in X ;\ b > 0,\ G(z, y) = bG(z, x) \quad \text{for all} \quad z \in X\}$$

and

$$A(K) := \bigcup_{x \in K} A(x).$$

We shall consider the necessary and sufficient conditions in order that (G, N) has the following property:

(b) Let K be a non-G-negligible compact set, F be a compact set with $A(K) \cap F = \emptyset$ and λ be a measure in M_K^+ of which $N\lambda$ is bounded on K. Then there exists a (G, N)-condenser-type measure $\alpha = \mu_0 - \mu_1$ of λ onto $\langle K, F\rangle$, i.e., there exist $\mu_0 \in \mathcal{L}(G)$ and $\mu_1 \in \mathcal{L}(G)$ such that

b_1) $\operatorname{supp}(\mu_0) \subset K$, $\operatorname{supp}(\mu_1) \subset F$,

b_2) $G\alpha = N\lambda$ G-n.e. on K, $G\alpha = 0$ G-n.e. on F,

b_3) $0 \leqq G\alpha \leqq N\lambda$ on X.

THEOREM 1. *Let N, G be continuous function-kernels. Then the following assertions* (1), (2) *are equivalent*;

(1) *(G, N) has the property* (b).

(2) *$G \prec G$ and $G \prec N$.*

PROOF. (1) → (2): Let K be a non-G-negligible set and λ be a measure in M_K^+ such that $N\lambda$ is bounded on K. Then there exists a (G, N)-condedser-type measure $\mu_0-\mu_1$ of λ onto $\langle K, \emptyset\rangle$ by the assumption. Since $\mu_1=0$, μ_0 is a relatively balayaged measure of λ onto K with respect to (G, N). Using this, we can easily show that G satisties the elementary relative domination principle (cf. Proof of Lemma 2 in [3]). By Theorem 1 in [3] we have $G\prec N$. Next, we shall show that $G\prec G$. Assume that $\check{G}\mu\leqq\check{G}\varepsilon_x$ on $\mathrm{supp}(\mu)$, where $\mu\in M_K^+$ with $\int G\mu d\mu<+\infty$ and $x\in C(\mathrm{supp}(\mu))$. If there is a point $x_1\in X\backslash\mathrm{supp}(\mu)$ such that $G\mu(x_1)>\check{G}\varepsilon_x(x_1)$, we can find a compact neighborhood K of x_1 such that $\check{G}\mu(y)>\check{G}\varepsilon_x(y)$ for all $y\in K$. Remark that $A(K)\cap\mathrm{supp}(\mu)=\emptyset$. By the assumption there exists a (G, N)-condenser-type measure $\mu_0-\mu_1$ of ε_z for $z\in\mathrm{supp}(\mu)$ onto $\langle K, \mathrm{supp}(\mu)\rangle$. It is obvious that $\mu_0\neq 0$. We obtain

$$0=\int(G\mu_0-G\mu_1)d\mu=\int\check{G}\mu d\mu_0-\int\check{G}\mu d\mu_1>\int\check{G}\varepsilon_x d\mu_0-\int\check{G}\varepsilon_x d\mu_1$$
$$=G\mu_0(x)-G\mu_1(x)\geqq 0\,.$$

This is a contradiction. Thus we have $\check{G}\mu\leqq\check{G}\varepsilon_x$ on X. Since $G\prec N$ implies that G satisfits the continuity principle (cf. [3]). Consequently $\check{G}$ satisfies satisfies the domination principle (cf. [5, Theorem II. 3]) and so G does (cf. [2]).

(2) → (1): Since $G\prec G$, G satisfies the continuity principle. Let K be a non-G-negligible compact set, F be a compact set with $A(K)\cap F=\emptyset$ and λ be a measure in M_K^+ such that $N\lambda$ is bounded on K. The assumption $G\prec N$ implies $\check{G}\sqsubset\check{N}$. By Proposition 2, G satisfies the relative balayage principle with respect to N. If ν_0 is a relatively balayaged measure of λ onto K with respect to (G, N), $G\nu_0$ is bounded on K. Since there is $\alpha_0\in\mathcal{F}(G)$ satisfying $G\nu_0\leqq G\alpha_0$ on X, $G\nu_0$ is locally bounded. Further we can choose successively a sequence $\{\nu_n\}$ of measures in $\mathcal{L}(G)$ with the following properties:

(i) $\mathrm{supp}(\nu_{2m})\subset K$, $\mathrm{supp}(\nu_{2m+1})\subset F$

(ii) ν_{2m+1} is a G-balayaged measure of ν_{2m} onto F,

(iii) ν_{2m+2} is a G-balayaged measure of ν_{2m+1} onto K.

Then $\{G\nu_n\}$ is decreasing and $\lim_{m\to\infty}G\nu_{2m}=\lim_{m\to\infty}G\nu_{2m+1}$. Since $G\prec G$, we have $\check{G}\prec\check{G}$. Let β be an arbitrary measure in $\mathcal{F}(\check{G})$. Then we can find $\sigma\in\mathcal{L}(\check{G})$ and $\tau\in\mathcal{L}(\check{G})$ satisfying

$$\check{G}\sigma-\check{G}\tau=\check{G}\beta\ G\text{-n.e. on } K,\quad \check{G}\sigma-\check{G}\tau=0\ G\text{-n.e. on } F,$$
$$0\leqq\check{G}\sigma-\check{G}\tau\leqq\check{G}\beta\ G\text{-n.e. on } X\ (\text{cf. [8]}).$$

Remarking that $\mathrm{supp}(\nu_{2m})\subset K$ and $\mathrm{supp}(\nu_{2m+1})\subset F$, we have

$$\int\lim_{m\to\infty}G\nu_{2m}d\beta=\lim_{m\to\infty}\int G\nu_{2m}d\beta=\lim_{m\to\infty}\int\check{G}\beta d\nu_{2m}=\lim_{m\to\infty}\int(\check{G}\sigma-\check{G}\tau)d\nu_{2m}$$
$$=\int(\lim_{m\to\infty}G\nu_{2m})d\sigma-\int(\lim_{m\to\infty}G\nu_{2m})d\tau$$
$$=\int(\lim_{m\to\infty}G\nu_{2m+1})d\sigma-\int(\lim_{m\to\infty}G\nu_{2m+1})d\tau$$

$$=\lim_{m\to\infty}\int(\check{G}\sigma-\check{G}\tau)d\nu_{2m+1}=0.$$

Consequently $\lim_{m\to\infty} G\nu_{2m}=0$ G-n.e. on X and hence $\lim_{n\to\infty} G\nu_n=0$ G-n.e. on X. Thus we see that the alternative series $\sum_{n=0}^{\infty}(-1)^n G\nu_n$ converges G-n.e. on X. Put

$$g:=\sum_{m=0}^{\infty}(G\nu_{2m}-G\nu_{2m+1}).$$

Since $G\nu_{2m}-G\nu_{2m+1}=0$ G-n.e. on F and $G\nu_{2m+1}-G\nu_{2m+2}=0$ G-n.e. on K, we have

(3.1) $$g=G\nu_0=N\lambda \ G\text{-n.e. on } K, \quad g=0 \ G\text{-n.e. on } F,$$
$$0\leqq g\leqq G\nu_0\leqq N\lambda \ \text{ on } X.$$

Further we can choose γ, $\delta\in\mathcal{L}(\check{G})$ satisfying

$$\check{G}\gamma-\check{G}\delta\geqq 1 \ G\text{-n.e. on } K, \quad \check{G}\gamma-\check{G}\delta=0 \ G\text{-n.e. on } F,$$
$$0\leqq\check{G}\gamma-\check{G}\delta \ G\text{-n.e. on } X.$$

For any $p\in\boldsymbol{N}$, it follows that

$$\sum_{m=0}^{p}\int d\nu_{2m}\leqq\sum_{m=0}^{p}\int(\check{G}\gamma-\check{G}\delta)d\nu_{2m}-\sum_{m=0}^{p}\int(\check{G}\gamma-\check{G}\delta)d\nu_{2m+1}$$
$$=\sum_{m=0}^{p}\int(G\nu_{2m}-G\nu_{2m+1})d\gamma-\sum_{m=0}^{p}\int(G\nu_{2m}-G\nu_{2m+1})d\delta$$
$$\leqq\int g\,d\gamma=\int G\nu_0\,d\gamma<+\infty$$

and hence $\sum_{m=0}^{\infty}\int d\nu_{2m}<+\infty$. Put $\mu_0:=\sum_{m=0}^{\infty}\nu_{2m}$. Then μ_0 is a positive measure supported by K. If we choose $\beta_0\in\mathcal{L}(\check{G})$ satisfying $\check{G}\beta_0>1$ on F, we have

$$\sum_{m=0}^{\infty}\int d\nu_{2m+1}\leqq\sum_{m=0}^{\infty}\int\check{G}\beta_0 d\nu_{2m+1}=\sum_{m=1}^{\infty}\int G\nu_{2m+1}d\beta_0$$
$$\leqq\sum_{m=0}^{\infty}\int G\nu_{2m}d\beta_0=\sum_{m=0}^{\infty}\int\check{G}\beta_0 d\nu_{2m}<+\infty.$$

Consequently $\mu_1:=\sum_{m=0}^{\infty}\nu_{2m+1}$ is a positive measure supported by F. Obviously it follows that $\sum_{m=0}^{\infty}G\nu_{2m}=G\mu_0$, $\sum_{m=0}^{\infty}G\nu_{2m+1}=G\mu_1$ and $g=G\mu_0-G\mu_1$ on X. Since $G\mu_0\leqq G\mu_1-G\nu_0$ and $G\mu_1$ is continuous on K, $G\mu_0$ is bounded on K and hence it is locally bounded on X. Using $G\mu_1\leqq G\mu_0$, $G\mu_1$ is also locally bounded on X. Therefore $\mu_0-\mu_1$ is a (G, N)-condenser-type measure of λ onto $\langle K, F\rangle$.

Putting $N=G$ in Theorem 1, we obtain the following corollary.

COROLLARY 1. *G satisfies the domination principle if and only if (G, G) has the property (b).*

DEFINITION 5. We say that G satisfies the condenser principle if for each non-G-negligible compact set K and each compact set F with $K\cap F=\emptyset$, there exists a measure $\alpha=\mu_0-\mu_1(\mu_0\in\mathcal{L}(G),\ \mu_1\in\mathcal{L}(G))$ such that

$$\mathrm{supp}(\mu_0)\subset K \quad\text{and}\quad \mathrm{supp}(\mu_1)\subset F,$$
$$G\alpha=1\ G\text{-n.e. on } K,\quad G\alpha=0\ G\text{-n.e. on } F,$$
$$0\leqq G\alpha\leqq 1 \quad\text{on } X.$$

If $A(x)=\{x\}$ for any $x\in X$, we say that G is non-degenerate. Putting $N=1$ (the constant kernel) in Theorem 1, we have easily the following well-known result (cf. [6])

Corollary 2. *Let G be non-degenerate. Then G satisfies the condenser principle if and only if $G\prec G$ and $G\prec 1$.*

Corollary 3. *If $G\prec G$, $G\prec N$ and G is non-degenerate, then for any pair of a non-G-negligible compact set K and a compact set F with $K\cap F=0$, there exists uniquely a measure $\alpha=\mu_0-\mu_1(\mu_0,\ \mu_1\in\mathcal{L}(G))$ satisfying $b_1)$, $b_2)$, $b_3)$ in Theorem 1.*

Proof. Let K be a non-G-negligible compact set and F be a compact set with $K\cap F=\emptyset$. Since G is non-degenerate, we have $A(K)\cap F=\emptyset$. By Theorem 1 there exists a measure $\alpha=\mu_0-\mu_1(\mu_0,\ \mu_1\in\mathcal{L}(G))$ satisfying $\mathrm{b}_1)$, $\mathrm{b}_2)$, $\mathrm{b}_3)$. Let $\beta=\nu_0-\nu_1(\nu_0,\ \nu_1\in\mathcal{L}(G))$ be another measure satisfying $\mathrm{b}_1)$, $\mathrm{b}_2)$, $\mathrm{b}_3)$. Since $G\prec G$ and G is non-degenerate, $\check{G}\prec\check{G}$ and $\check{G}$ is also non-degenerate. By Corollary 1 there exists, for each $\gamma\in\mathcal{L}(\check{G})$, a measure $\delta=\sigma-\tau(\sigma,\ \tau\in\mathcal{L}(\check{G}))$ such that

$$\check{G}\delta=\check{G}\gamma\ G\text{-n.e. on } K,\quad \check{G}\delta=0\ G\text{-n.e. on } F,$$
$$0\leqq\check{G}\delta\leqq\check{G}\gamma \quad\text{on } X.$$

Then we have

$$\begin{aligned}\int\check{G}\gamma\,d\mu_0&=\int(\check{G}\sigma-\check{G}\tau)d\mu_0-\int(\check{G}\sigma-\check{G}\tau)d\mu_1\\&=\int(G\mu_0-G\mu_1)d\sigma-\int(G\mu_0-G\mu_1)d\tau\\&=\int(G\nu_0-G\nu_1)d\sigma-\int(G\nu_0-G\nu_1)d\tau\\&=\int(\check{G}\sigma-\check{G}\tau)d\nu_0=\int\check{G}\gamma\,d\nu_0\,.\end{aligned}$$

Let f be a non-negative continuous function on X with compact support. Since G is non-degenerate, we can find sequences $\{\sigma_n\}\,\{\tau_n\}$ in $\mathcal{L}(\check{G})$ satisfying

$$0\leqq\check{G}\sigma_n-\check{G}\tau_n\leqq\check{G}\lambda \quad\text{on } X \text{ for some } \lambda\in F(\check{G}),$$
$$\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)=f\ G\text{-n.e. on } X.$$

Consequently we have $\mu_0=\nu_0$. If F is non-G-negligible, we have $\mu_1=\nu_1$ analogously. If F is G-negligeble, it follows that $\mu_1=\nu_1=0$. Thus we have $\mu_0=\nu_0$ and $\mu_1=\nu_1$.

§ 4. (G, N)-condedser-type theorems for a pair $\langle K, F\rangle$ of a compact set K and a closed set F.

We say that (G, N) has the property (b') if the following property (b') is satisfied:

(b') Let K be a non-G-negligible compact set and F be a closed set with $A(K)\cap F=\emptyset$. Then for each $\lambda\in\mathcal{L}(N)$ there exists a measure $\alpha=\mu_0-\mu_1$ satisfying

b_1') $\operatorname{supp}(\mu_0)\subset K$, $\operatorname{supp}(\mu_1)\subset F$,

b_2') $\int G\mu_1 d\beta<+\infty$ for each $\beta\in\mathcal{L}(\check{G})$,

b_3') $\mu_0(B)=\mu_1(B)=0$ for each G-negligible set B,

b_4') $G\alpha=N\lambda$ G-n.e. on K, $G\alpha=0$ G-n.e. on F,

b_5') $0\leqq G\alpha\leqq N\lambda$ on X if F is compact and $0\leqq G\alpha\leqq N\lambda$ G-n.e. on X if F is not compact.

In this section we ask necessary and sufficient conditions in order that (G, N) has property (b').

THEOREM 2. *Assume that $G\prec G$ and $\mathcal{L}(N)\neq\{0\}$. Then (G, N) has the property (b') if and only if (G, N) has the following three properties (c), (d), (e):*

(c) Let F be a non-G-negligible compact set. Then for every $\mu\in L(N)$ there exists a measure $\nu\in M_K^+$ such that

$$\operatorname{supp}(\nu)\subset K,\quad G\nu=N\mu\quad G\text{-n.e. on } K,\quad G\nu\leqq N\mu\quad \text{on } X.$$

(d) Any $\mu\in\mathcal{L}(G)$ is balayable onto any closed set F, i.e., there exists a positive measure ν such that

$$G\nu=G\mu\quad G\text{-n.e. on } F,\qquad G\nu\leqq G\mu\quad \text{on } X.$$

(e) Let K be a non-G-negligible compact set and F be a closed set with $A(K)\cap F=\emptyset$. Then there exist sequences $\{\sigma_n\}$, $\{\tau_n\}$ (σ_n, $\tau_n\in L(\check{G})$) satisfying

e_1) $\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)=0$ *G-n.e. on F*, $\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)\geqq 1$ G-n.e. *on K*,

e_2) $0\leqq\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)\leqq\check{G}\eta$ *G-n.e. on X for some* $\eta\in\mathcal{L}(\check{G})$,

e_3) $\check{G}\sigma_n\leqq\check{G}\sigma_{n+1}$ *and* $\check{G}\tau_n\leqq\check{G}\tau_{n+1}$ *for each* $n\in\boldsymbol{N}$,

e_4) $\liminf_{n\to\infty}\int\check{N}\sigma_n d\beta<+\infty$ *and* $\liminf_{n\to\infty}\int\check{N}\tau_n d\beta<+\infty$ *for each* $\beta\in\mathcal{L}(N)$.

PROOF. Suppose that (G, N) has the property (b').

(c): Let K be a non-G-negligible set and μ be a measure in $\mathcal{L}(N)$. Then there exists a (G, N)-condedser-type measure $\alpha=\mu_0-\mu_1$ of μ onto $\langle K, \emptyset\rangle$. Since $\mu_1=0$, we have

$$G\mu_0 = N\mu \ G\text{-n.e. on } K, \qquad G\mu_0 \leqq N\mu \ \text{ on } X.$$

(d): We remark that it is sufficient to prove that for each closed set F

(4.1) $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)=0$ G-n.e. on F implies $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)=0$

G-n.e. on X, where $\{\sigma_n\}$, $\{\tau_n\}\subset\mathcal{L}(\check{G})$ satisfying

$$\operatorname{supp}(\sigma_n)\subset F,\quad \operatorname{supp}(\tau_n)\subset F,$$

$$0\leqq \check{G}\sigma_n - \check{G}\tau_n \leqq \check{G}\gamma \ G\text{-n.e. on } F \text{ for some } \gamma\in\mathcal{L}(\check{G}),$$

$$\check{G}\sigma_{n+1} - \check{G}\tau_{n+1} \leqq \check{G}\sigma_n - \check{G}\tau_n \ G\text{-n.e. on } F \quad \text{(cf. [8, Theorem 1'])}$$

Assume that $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)\neq 0$ G-n.e. on X. Since $0\leqq \check{G}\sigma_n - \check{G}\tau_n$ on X, there exists a non-G-negligible compact set contained in $\{x\in CF;\ \lim_{n\to\infty}(\check{G}\sigma_n(x) - \check{G}\tau_n(x)) >0\}$. We remark that $A(K)\cap F=\emptyset$. By the assumption there exists, for $\lambda\in\mathcal{L}(N)(\lambda\neq 0)$, a measure $\mu_0-\mu_1$ satisfying $b_1')\sim b_5')$. Since $\lambda\neq 0$, we have $\mu_0\neq 0$. Consequently

$$\begin{aligned}0&=\lim_{n\to\infty}\left\{\int(G\mu_0-G\mu_1)d\sigma_n-\int(G\mu_0-G\mu_1)d\tau_n\right\}\\&=\int\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)\,d\mu_0-\int\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)d\mu_1\\&=\int\lim_{n\to\infty}(\check{G}\sigma_n-\check{G}\tau_n)d\mu_0>0\,.\end{aligned}$$

This is a contradication. Thus $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)=0$ G-n.e. on F implies $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)=0$ G-n.e. on X.

(e): Let $\{K_n\}$ be an increasing sequence of compact sets with $\bigcup_n K_n = X$. Suppose that K is a non-G-negligible compact set and F be a closed set with $A(K)\cap F=\emptyset$. Put $F_n := F\cap K_n$. We can find $\lambda_0\in\mathcal{F}(\check{G})$ satisfying $\check{G}\lambda_0\geqq 1$ on K. Since $\check{G}\prec\check{G}$, there exists, by Corollary 1, $\delta_n=\alpha_n-\beta_n$ such that

$$\operatorname{supp}(\alpha_n)\subset K,\qquad \operatorname{supp}(\beta_n)\subset F_n\,,$$

$$\check{G}\delta_n=\check{G}\lambda_0 \ G\text{-n.e. on } K,\qquad \check{G}\delta_n=0 \ G\text{-n.e. on } F_n\,,$$

$$0\leqq\check{G}\delta_n\leqq\check{G}\lambda_0 \ \text{ on } X.$$

Since $\check{G}\alpha_n - \check{G}\beta_n \leqq \check{G}\alpha_{n-1} - \check{G}\beta_{n-1}$ on $K\cup F_n$, $\check{G}\alpha_n - \check{G}\beta_n \leqq \check{G}\alpha_{n-1} - \check{G}\beta_{n-1}$ on X. We remark that $\check{G}\alpha_n\leqq\check{G}\alpha_{n+1}$ on X. In fact, there exists, for each $\eta\in\mathcal{L}(G)$, a measure $\sigma_n-\tau_n(\sigma_n,\ \tau_n\in\mathcal{L}(G))$ satisfying

$$\operatorname{supp}(\sigma_n)\subset K,\qquad \operatorname{supp}(\tau_n)\subset F_n\,,$$

$$G\sigma_n - G\tau_n = G\eta \ G\text{-n.e. on } K,\quad G\sigma_n - G\tau_n=0 \ G\text{-n.e. on } F_n\,,$$

$$0\leqq G\sigma_n - G\tau_n \leqq G\eta\,.$$

Then we have

$$\int \check{G}\alpha_n d\eta = \int G\eta d\alpha_n$$

$$= \int (G\sigma_n - G\tau_n) d\alpha_n - \int (G\sigma_n - G\tau_n) d\beta_n$$

$$= \int (\check{G}\alpha_n - \check{G}\beta_n) d\sigma_n - \int (\check{G}\alpha_n - \check{G}\beta_n) d\tau_n$$

$$\leqq \int (\check{G}\alpha_{n+1} - \check{G}\beta_{n+1}) d\sigma_n - \int (\check{G}\alpha_{n+1} - \check{G}\beta_{n+1}) d\tau_n$$

$$= \int (G\sigma_n - G\tau_n) d\alpha_{n+1} - \int (G\sigma_n - G\tau_n) d\beta_{n+1}$$

$$\leqq \int G\eta d\alpha_{n+1} = \int \check{G}\alpha_{n+1} d\eta .$$

Consequently $\check{G}\alpha_n \leqq \check{G}\alpha_{n+1}$ G-n.e. on X and hence $\check{G}\alpha_n \leqq \check{G}\alpha_{n+1}$. Similarily we have $\check{G}\beta_n \leqq \check{G}\beta_{n+1}$ on X. Let λ be an arbitrary measure in $\mathcal{L}(N)$. By the assumption we can find $\alpha = \mu_0 - \mu_1$ satisfying $b_1') \sim b_5')$. Then we have

$$+\infty > \int \check{G}\lambda_0 d\mu_0 \geqq \int \lim_{n\to\infty} (\check{G}\alpha_n - \check{G}\beta_n) d\mu_0 - \int \lim_{n\to\infty} (\check{G}\alpha_n - \check{G}\beta_n) d\mu_1$$

$$= \lim_{n\to\infty} \left\{ \int (G\mu_0 - G\mu_1) d\alpha_n - \int (G\mu_0 - G\mu_1) d\beta_n \right\}$$

$$= \lim_{n\to\infty} \int N\lambda d\alpha_n = \lim_{n\to\infty} \int \check{N}\alpha_n d\lambda .$$

Since $\check{G}\beta_n \leqq \check{G}\alpha_n$ on X and α_n, $\beta_n \in \mathcal{L}(\check{G})$, we have, using (c),

$$\int \check{N}\beta_n d\lambda \leqq \int \check{N}\alpha_n d\lambda \qquad \text{for all} \quad \lambda \in \mathcal{L}(\check{N}) .$$

Consequently $\liminf_{n\to\infty} \int \check{N}\beta_n d\lambda \leqq \lim_{n\to\infty} \int \check{N}\alpha_n d\lambda < +\infty$.

Conversely, suppose that (G, N) has the properties (c), (d), (e). Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \emptyset$ and λ be a measure in $\mathcal{L}(N)$. We denote by ν_0 a relatively balayaged measure of λ onto K with respect to (G, N). Since G has the property (d), we can choose a sequence $\{\nu_n\}$ of positive measures satisfying (i)$\sim$(iii) in Theorem 1. Obviously $\{G\nu_n\}$ is decreasing. We shall show that $\lim G\nu_n = 0$ G-n.e. on X. Let β be an arbitrary measure in $\mathcal{L}(\check{G})$ and b be a positive real number satisfying $b\check{G}\beta \leqq 1$ on K. Using the property (e), for $u := \lim_{n\to\infty} \check{G}\sigma_n$, $v := \lim_{n\to\infty} \check{G}\tau_n$ in (e) we have

$$\int u d\nu_n = \lim_{p\to\infty} \int \check{G}\sigma_p d\nu_n = \lim_{p\to\infty} \int G\nu_n d\sigma_p$$

$$= \liminf_{p\to\infty} \int N\lambda d\sigma_p = \liminf_{p\to\infty} \int \check{N}\sigma_p d\lambda < +\infty .$$

Since the sequence $\left\{\int u\,d\nu_n\right\}$ is decreasing, it follows that

$$\lim_{m\to\infty}\int u\,d\nu_{2m}=\lim_{m\to\infty}\int u\,d\nu_{2m+1}<+\infty\,.$$

Analogously

$$\lim_{m\to\infty}\int v\,d\nu_{2m}=\lim_{m\to\infty}\int v\,d\nu_{2m+1}<+\infty\,.$$

Using these relations, we obtain

$$\begin{aligned} b\int\lim_{m\to\infty} G\nu_{2m}\,d\beta &=\lim_{m\to\infty}\int bG\nu_{2m}\,d\beta=\lim_{m\to\infty}\int b\check{G}\beta\,d\nu_{2m} \\ &\leqq\lim_{m\to\infty}\int\lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{2m} \\ &=\lim_{m\to\infty}\int u\,d\nu_{2m}-\lim_{m\to\infty}\int v\,d\nu_{2m} \\ &=\lim_{m\to\infty}\int u\,d\nu_{2m+1}-\lim_{m\to\infty}\int v\,d\nu_{2m+1} \\ &=\lim_{m\to\infty}\int\lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{2m+1}=0\,. \end{aligned}$$

Consequently $\int\lim_{m\to\infty} G\nu_{2m}\,d\beta=0$ for all $\beta\in\mathcal{L}(\check{G})$. Thus we have $\lim_{m\to\infty} G\nu_{2m}=0$ G-n.e. on X. Thus we see the alternative series series $\sum_{n=0}^{\infty}(-1)^n G\nu_n$ converges G-n.e. on X. Put

$$g:=\sum_{m=0}^{\infty}(G\nu_{2m}-G\nu_{2m+1})\,.$$

Then g has the property (3.1). Next we shall show that $\sum_{m=0}^{\infty}\nu_{2m}$ is a positive measure. For any $q\in\boldsymbol{N}$ we have

$$\begin{aligned} \sum_{m=0}^{q}\int d\nu_{2m} &\leqq\sum_{m=0}^{q}\int\lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{2m}-\sum_{m=0}^{q}\int\lim_{p\to\infty}(\check{G}\sigma_p-\check{G}\tau_p)d\nu_{2m+1} \\ &=\lim_{p\to\infty}\sum_{m=0}^{q}\int(G\nu_{2m}-G\nu_{2m+1})d\sigma_p \\ &\quad-\lim_{p\to\infty}\sum_{m=0}^{q}\int(G\nu_{2m}-G\nu_{2m+1})d\tau_p \\ &\leqq\liminf_{p\to\infty}\int g\,d\sigma_p\leqq\liminf_{p\to\infty}\int N\lambda\,d\sigma_p<+\infty\,. \end{aligned}$$

Consequently $\sum_{m=0}^{\infty}\int d\nu_{2m}\leqq\liminf_{p\to\infty}\int N\lambda\,d\sigma_p<+\infty$ and hence $\mu_0:=\sum_{m=0}^{\infty}\nu_{2m}$ is a positive measure supported by K. Let f be a non-negative continuous function on X with compact support and η be a measure in $\mathcal{F}(\check{G})$ with $\check{G}\eta\geqq f$ on X. Then

$$\int f d\nu_{2m+1} \leqq \int \check{G}\eta d\nu_{2m+1} = \int G\nu_{2m+1} d\eta \leqq \int G\nu_{2m} d\eta = \int \check{G}\eta d\nu_{2m} .$$

Hence

$$\sum_{m=0}^{\infty} \int f d\nu_{2m+1} \leqq \sum_{m=0}^{\infty} \int \check{G}\eta d\nu_{2m} < +\infty .$$

This implies that the positive linear functional: $f \mapsto \sum_{m=0}^{\infty} \int f d\nu_{2m+1}$ on $\mathcal{K}(X)$ (the vector space of continuous real-valued functions on X with compact support) is a positive measure μ_1 supported by F. Obviously $\sum_{m=0}^{\infty} G\nu_{2m} = G\mu_0$ and $\sum_{m=0}^{\infty} G\nu_{2m+1} = G\mu_1$. For any $\beta \in \mathcal{L}(\check{G})$ we obtain

$$\int G\mu_1 d\beta = \sum_{m=0}^{\infty} \int G\nu_{2m+1} d\beta \leqq \sum_{m=0}^{\infty} \int G\nu_{2m} d\beta \leqq \sum_{m=0}^{\infty} \int \check{G}\beta d\nu_{2m} < +\infty .$$

Thus both $G\mu_1$ and $G\mu_0$ are finite on G-n.e. on X and it follows that $g = G\mu_0 - G\mu_1$ G-n.e. on X. We remark that $g = G\mu_0 - G\mu_1$ on X if F is compact. Therefore $\mu_0 - \mu_1$ is a (G, N)-condensertype measure of λ onto $\langle K, F \rangle$.

We define

$$\mathcal{S}(\check{G}) := \left\{ u = \lim_{n\to\infty} \check{G}\sigma_n ;\ \sigma_n \in \mathcal{L}(\check{G}),\ \check{G}\sigma_n \leqq \check{G}\sigma_{n+1},\ \int u d\beta < +\infty \quad \text{for all} \right.$$
$$\left. \beta \in \mathcal{F}(G) \right\} .$$

Putting $G = N$, we have easily the following corollary.

COROLLARY 4. *Assume that* $G \prec G$. *Then* (G, G) *has the property* (b') *if and only if* G *has the following properties* (d), (e'):

(d) *Any* $\mu \in \mathcal{L}(G)$ *is balayable onto any closed set* F,

(e') *Let* K *be a non-G-negligible compact set and* F *be a closed set with* $A(K) \cap F = \emptyset$. *Then there exist* $u, v \in \mathcal{S}(\check{G})$ *such that*

$$u - v \geqq 1 \ G\text{-}n.e. \ on \ K, \qquad u - v = 0 \ G\text{-}n.e. \ on \ F,$$

$$0 \leqq u - v \leqq \check{G}\eta \ G\text{-}n.e. \ on \ X \ for \ some \ \eta \in L(\check{G}) .$$

PROPOFITION 3. *Assume that* G *is non-degenerate and* $G \prec G$. *Further assume that* G *has the following property* (f):

(f) *For any* $\lambda \in \mathcal{F}(\check{G})$, *for any* $\varepsilon > 0$ *and for any compact set* K *there exist* $u, v \in S(\check{G})$ *and a compact set* K' *such that*

$$v = \lim_{n\to\infty} \check{G}\sigma_n, \quad \operatorname{supp}(\sigma_n) \subset K'$$

$$u - v \leqq \varepsilon \ G\text{-}n.e. \ on \ K, \qquad u - v \geqq \check{G}\lambda \ G\text{-}n.e. \ on \ CK', \tag{4.2}$$

$$0 \leqq u - v \ G\text{-}n.e. \ on \ X .$$

Then (G, G) *has the property* (b').

PROOF. First we remark that we can choose $u, v \in \mathcal{S}(\check{G})$ in (f) satisfying (4.2) and

$$u-v=\check{G}\lambda \ G\text{-n.e. on } CK', \qquad 0 \leqq u-v \leqq \check{G}\lambda \ G\text{-n.e. on } X.$$

In fact, since $(u-v)\wedge\check{G}\lambda=u\wedge(\check{G}\lambda+v)-v$ and $u\wedge(\check{G}\lambda+v)\in\mathcal{S}(\check{G})$, we can take $(u-v)\wedge\check{G}\lambda$ instead of $u-v$. By Proposition 4 in [8] any $\mu\in\mathcal{L}(G)$ is balayable to any closed set. Further, let K be a non-G-negligible compact set and F be a closed set with $K\cap F=\emptyset$. Choose $\lambda_0\in\mathcal{F}(\check{G})$ satisfying $\check{G}\lambda_0\geqq 2$ on K. By the assumption (f) there exist $u, v\in\mathcal{S}(\check{G})$ and a compact set K' with $K\subset K'$ such that

$$u-v=\check{G}\lambda_0 \ G\text{-n.e. on } CK', \qquad u-v\leqq 1 \ G\text{-n.e. on } K$$

$$0\leqq u-v\leqq\check{G}\lambda_0 \ G\text{-n.e. on } X.$$

Then we obtain, putting $w_1:=\check{G}\lambda_0+v-u$,

$$w_1\geqq 1 \ G\text{-n.e. on } K, \qquad w_1=0 \ G\text{-n.e. on } CK'.$$

$$0\leqq w_1\leqq\check{G}\lambda_0 \ G\text{-n.e. on } X.$$

Since $(F\cap K')\cap K=\emptyset$ and $\check{G}\prec\check{G}$, it follows from Corollary 1 that there exists $w_2=\check{G}\alpha-\check{G}\beta(\alpha, \beta\in\mathcal{L}(\check{G}))$ such that

$$w_2=\frac{1}{2}\check{G}\lambda_0\geqq 1 \ G\text{-n.e. on } K, \qquad w_2=0 \ G\text{-n.e. on } F\cap K',$$

$$0\leqq w_2\leqq\frac{1}{2}\check{G}\lambda_0 \ G\text{-n.e. on } X.$$

We can write $w_1\wedge w_2=u_1-u_2(u_1, u_2\in\mathcal{S}(\check{G}))$ G-n.e. on X and obtain

$$u_1-u_2\geqq 1 \ G\text{-n.e. on } K, \qquad u_1-u_2=0 \ G\text{-n.e. on } F$$

$$0\leqq u_1-u_2\leqq\frac{1}{2}\check{G}\lambda_0 \ G\text{-n.e. on } X.$$

Thus (G, G) has the property (b') by Corollary 4.

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