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A Characterization of Some Type of *PSL* (2, *q*)

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§1. Introduction.

Let G be a finite group. A subgroup A is called a strongly self-centralizing subgroup if the centralizer of any nonidentity element of A is A. In this paper under some conditions we shall determine a finite simple group which has two types of strongly self-centralizing subgroups and one type of subgroup with a property similar to a strongly self-centralizing subgroup.

Main Theorem. *If G is a finite simple group with three special types of subgroups A, B and C such that*
1. A is a strongly self-centralizing subgroup of order n_1, and $N_G(A)/A$ is a group of order 2
2. B is a subgroup of order n_2 with a property that the centralizer of any nonidentity element of B is $B\times\langle j\rangle$, where j is an involution and $N_G(B)/B$ is an abelian group of type (2, 2)
3. C is a strongly self-centralizing subgroup of order n_3 and $N_G(C)/C$ is a cyclic group of order $\frac{n_3-1}{2}$
and satisfies the following conditions
(0) $|G|\leqq n_1^2 n_3^2 \frac{n_2-1}{2}$
(i) $n_3<5n_2$
(ii) $n_3<2n_2+n_1+6$
(iii) $5\leqq n_1,\ n_2$
then G is isomorphic to one of PSL (2, q) with 2-Sylow groups of order 4.

Of course any *PSL* (2, *q*) with 2-Sylow groups of order 4 has these subgroups and satisfies these conditions. We remark that A and B are abelian groups of odd orders since they have fixed-point-free automorphisms of order 2. Moreover C is an elementary abelian p_3-group, since nonidentity elements of C form 2 conjugacy classes of the same numbers. A, B and C are T.I. sets in G and Hall subgroups in G. ([1]). The involutions of a finite simple group with a strongly self-centralizing subgroup of type A, form a single conjugacy class. (Cor. 1. [1]). See [3] on the known simple groups with a certain type of strongly self-centralizing subgroup.

The main process of the proof of this theorem is as follows.

I. We plan to use the theory of exceptional characters. (see [1], [3]). If H is a subgroup of G, a collection of conjugacy classes D in H is a collection of special classes if whenever $y \in D$ and $x^{-1}yx \in H$ for some $x \in G$ then in fact $x \in H$. Suppose $\theta_1, \cdots\cdots, \theta_n$ is a family of irreducible characters of H, $n>1$ maximal with the property that $\theta_1(x)=\cdots\cdots=\theta_n(x)$ for all x in H not members of special classes. Then:

a. There exist n irreducible characters $\chi_1, \cdots\cdots, \chi_n$ of G (the exceptional characters) such that $\chi_{i|H}=\varepsilon\theta_i+\theta$, where $\varepsilon=\pm 1$ (independent of i) and θ is a character of H containing the θ_i with equal multiplicity. If χ is an irreducible character of G not one of χ_i, $\chi_{|H}$ contains the θ_i with equal multiplicity.

b. If $x \in G$ is not conjugate to any element of D, then $\chi_1, \cdots\cdots, \chi_n$ all have the same value on x. (In particular, if $G \neq H$ the class of 1 cannot be special and all χ_i have the same degree.)

Moreover, two different maximal families lead to disjoint families of exceptional characters.

From A, B and C we shall pick up some collections of special classes in their normalizers and make some families of exceptional characters of G.

II. We shall divide the families of exceptional characters and some special irreducible characters into blocks with respect to some prime numbers. Here we shall determine a part of the character table by using a theorem of modular representation:

THEOREM 1. (IV 6.4 p. 245 [2]) *Let p and q be distanct primes. Suppose that y is a p-element in G and x is a q-element in G such that no conjugate of x commutes with any conjugate of y.*

(*i*) *Let $B(p)$ be a fixed p-block of G and let $B(q)$ be a fixed q-block of G. Then $\Sigma\, \chi_s(x)\chi_s(y)=0$ where χ_s ranges over all the irreducible characters which lie in $B(p)$* and *$B(q)$.*

(*ii*) *There exists a nonprincipal irreducible character which is in the principal p-block and also in the principal q-block.*

III. The conditions (0)$\sim$(ii) and the orthogonality relation for $c \in C^{\#}$ $(=C-\{1\})$, will lead to the fact that there are only a few irreducible characters except the characters constructed in II. For the calculation of the orthogonality relation for $c \in C^{\#}$ here, we must determine the principal p_3-block beforehand in II. (In fact for determination of the principal p_3-block, division into p_1-blocks for p_1 dividing n_1 and p_2-blocks for p_2 dividing n_2 are useful by the existence of theorem 1.)

IV. By III there are only a few conjugacy classes except special classes in I and 1 and involution. We shall examine orders of the remaining classes and verify nonexistence of the remaining classes. Then we can verify the result by theorem 1. (Harada [1]).

§ 2. Exceptional characters with respect to A (see [1]) and division into blocks (I·II)

$A^{\#}$ in $N_G(A)$ is a collection of special classes. Associated with the family of irreducible characters of $N_G(A)$ of degree 2 which vanish on $N_G(A)-A$, we get $\frac{n_1-1}{2}$ exceptional characters of G: $X_1, \cdots\cdots, X_{\frac{n_1-1}{2}}$. Since $|N_G(A): A|=2$, there is a special nonprincipal nonexceptional irreducible character Y. Moreover $Y(A^{\#})=\varepsilon_1(\varepsilon_1=\pm 1)$, $Y(1)=kn_1+\varepsilon_1$ and $X_i(1)=kn_1+2\varepsilon_1$ for all i, where k is a positive integer. All nonprincipal irreducible characters of G except $\{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$ and Y, have degrees divisible by n_1. Let p_1 be a prime number dividing n_1. Dividing irreducible characters into, p_1-blocks, each of these characters makes one block of defect 0 since their degrees are divisible by the highest power of p_1. As the p_1'-classes have defect 0 or maximal defect, 1_G, Y, $\{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$ are divided into some p_1-blocks of maximal defect.

On the other hand, the order of a finite simple group with a strongly self-centralizing subgroup of type A can be written as follows:

Lemma 1. *$|G|=n_1(kn_1+\varepsilon_1)(kn_1+2\varepsilon_1)m^2$. $\left(m=\frac{|C_G(\tau)|}{kn_1+\varepsilon_1-Y(\tau)}\right)$, where τ is an involution in $N_G(A)$. Moreover m is an integer and $m^2\equiv 1 (mod\ n_1)$.*

Proof. See [1].

§ 3. Exceptional characters with respect to B and division into blocks (I·II)

$B^{\#}$, $jB^{\#}$ is a collection of special classes in $N_G(B)$. Since $N_G(B)$ is a direct product of a cyclic group $\langle j\rangle$ and a generalized dihedral group $B\langle i\rangle$, where i is an involution, the character table can be determined as follows:

	1	$B^{\#}$	i	$jB^{\#}$	j	ji
λ_1	1	1	1	1	1	1
λ_2	1	1	-1	1	1	-1
λ_1'	1	1	1	-1	-1	-1
λ_2'	1	1	-1	-1	-1	1
$\{\theta_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)$	2		0		2	0
$\{\theta_r'\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)$	2		0		-2	0

the orthogonality relations give

$$\sum_r \theta_r(b) = \sum_r \theta_r(jb) = \sum_r \theta'_r(b) = -\sum_r \theta'_r(jb) = -1, \text{ and}$$
$$\sum_r |\theta_r(b)|^2 = \sum_r |\theta'_r(b)|^2 = n_2 - 2 \quad \cdots\cdots(1)$$

for $b \in B^{\#}$.

The families of the θ_r and the θ'_r provide two families of exceptional characters of G, the η_r and the η'_r:

$$\eta_{r\,|\,N_G(B)} = \varepsilon\,\theta_r + \eta, \quad \eta'_{r\,|\,N_G(B)} = \varepsilon'\,\theta'_r + \eta' \qquad (\varepsilon = \pm 1,\ \varepsilon' = \pm 1),$$

and the multiplicity of θ_r in η and θ'_r in η' is independent of r. Each family has $\frac{n_2-1}{2}$ members. From the constant multiplicity (§1. I **a**.) and the fact that the other characters of $N_G(B)$ have rational integral values on $b \in B^{\#}$ independent of b, $\chi(b)$ are rational integers independent of b for any nonexceptional irreducible character χ of G. The same is true for η and η'. And these statements are also true for $jb \in jB^{\#}$.

The orthogonality relation for b becomes

$$|C_G(b)| = \sum_r |\eta_r(b)|^2 + \sum_r |\eta'_r(b)|^2 + \sum |\chi(b)|^2,$$

where the last sum ranges over all nonexceptional characters. Let $\eta(b) = x$, $\eta'(b) = x'$. Then

$$2n_2 = \sum_r |\theta_r(b)|^2 + \varepsilon x \sum_r (\theta_r(b) + \overline{\theta_r(b)}) + \frac{n_2-1}{2}x^2 + \sum_r |\theta'_r(b)|^2$$
$$+ \varepsilon' x' \sum_r (\theta'_r(b) + \overline{\theta'_r(b)}) + \frac{n_2-1}{2}x'^2 + \sum |\chi(b)|^2.$$

Then by (1)

$$2n_2 = 2n_2 - 4 + \frac{n_2-1}{2}x^2 + \frac{n_2-1}{2}x'^2 - 2\varepsilon x - 2\varepsilon' x' + \sum |\chi(b)|^2 \cdots\cdots(2)$$

Since $n_2 \geqq 5$ and x and x' are rational integers, $\frac{n_2-1}{2}x^2 - 2\varepsilon x \geqq 2x^2 - 2\varepsilon x \geqq 0$ and $\frac{n_2-1}{2}x'^2 - 2\varepsilon' x' \geqq 2x'^2 - 2\varepsilon' x' \geqq 0$.

The orthogonality relation for jb becomes

$$|C_G(jb)| = \sum_r |\eta_r(jb)|^2 + \sum_r |\eta'_r(jb)|^2 + \sum |\chi(jb)|^2.$$

Let $\eta(jb) = y$ and $\eta'(jb) = y'$. Then

$$2n_2 = \sum_r |\theta_r(jb)|^2 + \varepsilon y \sum_r (\theta_r(jb) + \overline{\theta_r(jb)}) + \frac{n_2-1}{2}y^2 + \sum_r |\theta'_r(jb)|^2$$
$$+ \varepsilon' y' \sum_r (\theta'_r(jb) + \overline{\theta'_r(jb)}) + \frac{n_2-1}{2}y'^2 + \sum |\chi(jb)|^2.$$

Then by (1)

$$2n_2=2n_2-4+\frac{n_2-1}{2}y^2+\frac{n_2-1}{2}y'^2-2\varepsilon y+2\varepsilon' y'+\sum|\chi(jb)|^2 \quad \cdots\cdots(3)$$

Since $n_2\geqq 5$ and y and y' are rational integers, $\frac{n_2-1}{2}y^2-2\varepsilon y\geqq 2y^2-2\varepsilon y\geqq 0$ and $\frac{n_2-1}{2}y'^2+2\varepsilon' y'\geqq 2y'^2+2\varepsilon' y'\geqq 0$.

Because the terms in the last sum of (2) and (3) are rational integers and $\chi(b)\equiv\chi(jb)$, $\eta(b)\equiv\eta(jb)$, $\eta'(b)\equiv\eta'(jb)$ (mod 2), there are five possibilities as follows.

Case 1. Only four characters $\xi_1(=1_G)$, ξ_2 ξ_3 and ξ_4 have nonzero values on $B^\#$ and $jB^\#$ (in fact ± 1). $x=x'=y=y'=0$.

Case 2. Only four characters $\xi_1(=1_G)$, ξ_2, ξ_3 and ξ_4 have nonzero values on $B^\#$ and $jB^\#$ (in fact ± 1). $n_2=5$, $x=\varepsilon$ and $x'=\varepsilon'$, or $x=\varepsilon$ and $x'=0$, or $x=0$ and $x'=\varepsilon'$.

Case 3. Only three characters have nonzero values on $B^\#$ and $jB^\#$. $n_2=7$, $x=\varepsilon$ and $x'=0$, or $x=0$ and $x'=\varepsilon'$.

Case 4. Only two characters have nonzero values on $B^\#$ and $jB^\#$. $n_2=7$ and $x=\varepsilon$ and $x'=\varepsilon'$, or $n_2=9$ and $x=\varepsilon$ and $x'=0$, or $n_2=9$ and $x=0$ and $x'=\varepsilon'$.

Case 5. Only the principal character has nonzero values on $B^\#$ and $jB^\#$.

Let p_2 be a prime number dividing n_2 and let B_{p_2} be a p_2-Sylow group of order p_2^d. Let $B_{p_2'}$ be a Hall p_2'-subgroup of B. Now we wish to determine the principal p_2-block.

LEMMA 2. *In case* 1, $\frac{p_2^d-1}{2}$ *irreducible characters* η_r *associated with the* θ_r *of the form* $(\phi\cdot\varphi_1+\bar{\phi}\cdot\varphi_1)\cdot\rho_1$, *where* ϕ *is a nonprincipal irreducible character of* B_{p_2}, *and* φ_1 *and* ρ_1 *are the principal characters of* $B_{p_2'}$ *and* $\langle j\rangle$, *are in the principal* p_2*-block* $B_0(p_2)$. *And in* $B_0(p_2)$ *there is only one nonprincipal irreducible character other than these* η_r.

PROOF. In case 1 for a fixed p_2-element $b_0\in B$, and any p_2'-element $u\in C_G(b_0)=B\times\langle j\rangle$,

$$\eta_r(b_0u)=\varepsilon(\phi(b_0)+\overline{\phi(b_0)})(\varphi_1\cdot\rho)(u) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

Since there exists one to one correspondence (in fact inclusion) from the set of all p_2-blocks of $C_G(b_0)$ onto the set of all p_2-blocks of p_2'-group $C_G(b_0)/B_{p_2}$ where all p_2-blocks have p_2-defect 0, there is only one irreducible Brauer character in any p_2-block of $C_G(b_0)$ and Cartan matrix of $C_G(b_0)$ is a scalar matrix with diagonal entries p_2^d. (Th. 4. 5, p. 277 [2], Lemma 66.1, p. 401 [4]) Here $\varphi_1\cdot\rho_1$ in (4) can be regarded unique irreducible Brauer character in the principal block of $C_G(b_0)$. So, by the second main theorem (Th. 6. 1, p. 242 [2]) and theorem 6.2 (p. 286 [2]) these η_r are contained in $B_0(p_2)$. Hence there is only one column of higher decomposition numbers for B_0 (p_2)

for the section of b_0. The inner product of this column and itself is p_2^d, since the Cartan matrix of the principal p_2-block of $C_G(b_0)$ is p_2^d. Now we number these η_r: $\eta_1, \eta_2, \cdots\cdots \eta_{\frac{p_2^d-1}{2}}$. Then

$$\sum_{r=1}^{\frac{p_2^d-1}{2}} |\eta_r(b_0u)|^2 = \sum|\psi(b_0\}+\overline{\psi(b_0)}|^2 = p_2^d-2. \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

On the other hand $1_G(=\xi_1)$, $\{X_i\}$ $\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$ and Y form a collection of p_1-blocks S which includes the principal p_1-block $B_0(p_1)$. If we apply theorem 1 for $S\cap B_0(p_2)$, then we conclude that Y must be contained in $B_0(p_2)$ and $Y(b_0)=-\varepsilon_1$. From (5) 1_G, Y and $\{\eta_r\}\left(1\leqq r\leqq\frac{p_2^d-1}{2}\right)$ are all members of $B_0(p_2)$. (q.e.d.)

Since all p_2'-classes have defect 0 or maximal, all p_2-blocks have defect 0 or maximal. Then only exceptional characters and ξ_1, ξ_2, ξ_3 and ξ_4 which don't vanish on all $B^{\#}$, can belong to p_2-blocks of maximal defect. Hence we can take this Y out of ξ_2, ξ_3, ξ_4: say ξ_4.

Since this column of higher decomposition numbers is orthogonal to the columns of degrees, we conclude as follows :

LEMMA 3. *In case* 1

	$\varphi_1\cdot\rho_1$	*degree*
$\xi_1=1_G$	1	1
$\{\eta_r\}\left(1\leqq r\leqq\frac{p_2^d-1}{2}\right)$		kn_1
$\xi_4=Y$	$-\varepsilon_1$	$kn_1+\varepsilon_1$

Furthermore the sum of higher decomposition numbers for $\{\eta_r\}\left(1\leqq r\leqq\frac{p_2^d-1}{2}\right)$ *is* ε_1.

Next we wish to determine other p_2-blocks of maximal defect in case 1.

LEMMA 4. *In case* 1, p_2^d *irreducible characters* η_r *associated with* θ_r *of the form* $(\psi\cdot\varphi_i+\bar{\psi}\cdot\bar{\varphi}_i)\cdot\rho_1$, *where* ψ *is an irreducible character of* B_{p_2} *and* φ_i *is a fixed nonprincipal irreducible character of* B_{p_2}' *and* ρ_1 *is the principal character of* $\langle j\rangle$, *make one* p_2-*block of maximal defect. These types of blocks amount to* $\frac{|B_{p_2}'|-1}{2}$.

PROOF. With the same notations of lemma 2,

$$\eta_r(b_0u)=\varepsilon\psi(b_0)\,(\varphi_i\cdot\rho_1)(u)+\varepsilon\bar{\psi}(b_0)(\bar{\varphi}_i\cdot\rho_1)(u).$$

Here each of $\varphi_i\cdot\rho_1$ and $\bar{\varphi}_i\cdot\rho_1$ can be regarded unique irreducible Brauer character in some blocks of $C_G(b)$. By the second main theorem (Th. 6.1, p. 242 [2]) these η_r are contained in a same block. Calculating the inner product of the column of higher decomposition numbers for $\varphi_i\cdot\rho_1$ we get

$$\sum_{\mathrm{all}\,\phi}|\varepsilon\phi(b_0)|^2=p_2^d.$$

So there exists no irreducible character other than these η_r in this block. As the number of the choice of φ_i and $\bar{\varphi}_i$ is $\frac{|B_{p_2'}|-1}{2}$, the last assertion holds. (q.e.d.)

LEMMA 5. *In case* 1, p_2^d *irreducible characters* η_r' *associated with* θ_r *of the form* $(\phi\cdot\varphi_i+\bar{\phi}\cdot\bar{\varphi}_i)\cdot\rho_1'$, *where* ϕ *is an irreducible character of* B_{p_2}, *and* φ_i *is a fixed nonprincipal irreducible character of* $B_{p_2'}$ and ρ_1' *is unique nonprincipal irreducible character of* $\langle j\rangle$, *make one* p_2-*block of maximal defect.*
These types of blocks amount to $\frac{|B_{p_2'}|-1}{2}$.

PROOF. In the same way as lemma 4. (q.e.d.)

LEMMA 6. *In case* 1 *there exists only one nonprincipal maximal defect* p_2-*block other than lemma* 3 *and* 4. *This block consists of* $\frac{p_2^d-1}{2}$ *irreducible characters* η_r' *associated with* θ_r' *of the form* $(\phi\cdot\varphi_1+\bar{\phi}\cdot\varphi_1)\,\rho_1'$ *and* ξ_2 *and* ξ_3.

PROOF. As the number of the blocks of maximal defect coincides with the number of p_2'-classes of maximal defect, there are $|B_{p_2'}|+1$ blocks of maximal defect. Then the first assertion holds by lemma 2, 4 and 5. The fact that $\frac{p_2^d-1}{2}$ irreducible characters η_r' of the above form are contained in a same block can be verified similarly to lemma 2. Furthermore in the same way this remaining block has only two more characters other than these η_r'. On the other hand ξ_2 and ξ_3 have nonzero rational integral values on $B^\#$. Then ξ_2 and ξ_3 must be these (in fact the values are ± 1). We remark $\sum_{r=1}^{\frac{p_2^d-1}{2}}\eta_r'(b_0)=-\varepsilon'$. (q.e.d.)

Hereafter we call this block $B_1(p_2)$, and number these η_r' : $\eta_1'\cdots\cdots$, $\eta'_{\frac{p_2^d-1}{2}}$.

Next we shall deal with case 2, 3, 4 and 5.

LEMMA 7. *In case* 2, $B_0(p_2)$ *consists of* η_1, η_2, $\xi_1(=1_G)$ *and* $\xi_4(=Y)$ *with the same values as case* 1 *in lemma* 3. *Furthermore* $B_1(p_2)$ *consists of* η_1', η_2', ξ_2 *and* ξ_3 *as in case* 1 *and these two blocks are all* p_2-*blocks of maximal defect.*

PROOF. In case $x=\varepsilon$ and $x'=\varepsilon'$, we have for a fixed 5-element $b_0\in B^\#$,

$$\eta_r(b_0\cdot 1)=\varepsilon\theta_r(b_0\cdot 1)+\eta(b_0\cdot 1)=\varepsilon(\phi(b_0)+\bar{\phi}(b_0))\rho_1(1)+\varepsilon \quad (r=1,\ 2)\cdots(6)$$

where ϕ is a nonprincipal irreducible character of B and ρ_1 is the principal character of $\langle j\rangle$, since B is a cyclic group of order 5.

$$\eta_r(b_0 j)=\varepsilon\theta_r(b_0 j)+\eta(jb_0)=\varepsilon(\phi(b_0)+\bar{\phi}(b_0))\cdot\rho_1(j)+y \quad (r=1,\ 2)\cdots\cdots(7)$$

(6) and (7) mean η_1 and η_2 have nonzero higher decomposition numbers for the section of b_0 for ρ_1, which can be regarded unique irreducible Brauer character in the principal block of $C_G(b_0)$. By the second main theorem (Th. 6.1 p. 242 [2]) and theorem 6.2 (p. 286 [2]) η_1 and η_2 are in $B_0(p_2)$ and y must be ε. Then this higher decomposition number for ρ_1 is $\varepsilon\theta_r(b_0)+\varepsilon$. Since

$$\sum_{r=1}^{2}|\varepsilon\theta_r(b_0)+\varepsilon|^2=n_2-2-2+\frac{n_2-1}{2}=n_2-2,$$

we can see that the remaining members of $B_0(p_2)$ are 1_G and Y with the same values as in case 1 in lemma 3 in the same way.

Next

$$\begin{aligned}\eta'_r(b_0\cdot 1)&=\varepsilon'\theta'_r(b_0\cdot 1)+\eta'(b_0\cdot 1)=\varepsilon'(\phi(b_0)+\bar{\phi}(b_0))\rho'_1(1)+\varepsilon'\\ \eta'_r(b_0\cdot j)&=\varepsilon'\theta'_r(b_0\cdot j)+\eta'(jb_0)=\varepsilon'(\phi(b_0)+\bar{\phi}(b_0))\rho'_1(j)+y'\\ &=-\varepsilon'(\phi(b_0)+\bar{\phi}(b_0))+y' \qquad (r=1,\ 2)\end{aligned}$$

Suppose $y'=\varepsilon'$, then η'_1 and η'_2 have nonzero higher decomposition numbers for ρ_1 which is unique irreducible Brauer character in the principal block of $C_G(b_0)$. This contradicts that η'_1 and η'_2 can not belong to $B_0(p_2)$ by theorem 6.2 (p. 286 [2]). Then $y'=-\varepsilon'$ and η'_1 and η'_2 are contained in a same block of maximal defect. Since their higher decomposition numbers are $\varepsilon'\theta'_r(b_0)+\varepsilon'(r=1,\ 2)$ and $\sum_{r=1}^{2}|\varepsilon'\theta'_r(b_0)+\varepsilon'|^2=n_2-2$, the remaining members in this block are ξ_2 and ξ_3. Furthermore we remark

$$\sum_{r=1}^{2}\eta'_r(b_0)=\sum_{r=1}^{2}(\varepsilon'\theta'_r(b_0)+\varepsilon')=-\varepsilon'+2\varepsilon'=\varepsilon'.$$

As p'_2-classes of maximal defect are only 1 and j, all blocks of maximal defect are $B_0(p_2)$ and this block $B_1(p_2)$.

In case $x=\varepsilon$ and $x'=0$ and in case $x=0$ and $x'=\varepsilon'$ these results can be verified in the same way. We remark $\sum_{r=1}^{2}\eta'_r(b_0)=-\varepsilon'$ in the second case and $+\varepsilon'$ in the third case. (q.e.d.)

LEMMA 8. *In case* 3 *all* p_2-*blocks of maximal defect are* $B_0(p_2)=\{1_G, \{\eta_r\}$ $(1\leqq r\leqq 3)$, $Y=\xi_4\}$ *with the same values as in case 1 in lemma 3 and* $B_1(p_2)=\{\{\eta'_r\}$ $(1\leqq r\leqq 3)$, $\xi_2\}$. ξ_3 *cannot appear.*

PROOF. Suppose $x=\varepsilon$ and $x'=0$. Then $\eta_r(1\leqq r\leqq 3)$ have higher decomposition numbers $\varepsilon\theta_r(b_0)+\varepsilon$ for ρ_1 and belong to $B_0(p_2)$ in the same way as in case 2. But since $\sum_r |\varepsilon\theta_r(b_0)+\varepsilon|^2=n_2-1$, $B_0(p_2)$ must be $\{1_G, \{\eta_r\}$ $(1\leqq r\leqq 3)\}$, which contradicts theorem 1 (ii) for $B_0(p_1)\cap B_0(p_2)$. Then $x=0$ and $x'=\varepsilon'$. In this case $\eta_r(1\leqq r\leqq 3)$ have higher decomposition numbers $\varepsilon\theta_r(b_0)$ for ρ_1 and $B_0(p_2)=\{1_G, \{\eta_r\}(1\leqq r\leqq 3), Y\ (=\xi_4)\}$ with the same degrees in case 1 in the same way. Since each $\eta'_r(1\leqq r\leqq 3)$ has higher decomposition number $\varepsilon'\theta'_r(b_0)+\varepsilon'$ for ρ'_1 and $\sum_r|\varepsilon'\theta'_r(b_0)+\varepsilon'|^2=n_2-1$, $B_1(p_2)$ must be $\{\{\eta'_r\}(1\leqq r\leqq 3), \xi_2\}$. We remark $\sum_{r=1}^{3}\eta'_r(b_0)=\sum_{r=1}^{3}(\varepsilon'\theta'_r(b_0)+\varepsilon')=-\varepsilon'+3\varepsilon'=2\varepsilon'$.(q.e.d.)

Lemma 9. *Cases* 4 *and* 5 *cannot occur.*

Proof. In case 5 $B_0(p_2)=\{1_G, \{\eta_r\}\}$, which contradicts theorem 1 (ii) for $B_0(p_1)\cap B_0(p_2)$.

In case 4 suppose $x=\varepsilon$. Then this contradicts theorem 1 (ii) in the same way as the first part of the proof of lemma 8. In case 4 suppose $n_2=9$ and $x=0$ and $x'=\varepsilon'$. Then $B_0(p_2)=\{1_G, \{\eta_r\}\ (1\leqq r\leqq 4), Y\}$ and $B_1(p_2)=\{\{\eta'_r\}(1\leqq r\leqq 4)\}$. In this case each η'_r has higher decomposition number $\varepsilon'\theta'_r(b_0)+\varepsilon'$ for ρ'_1. Since the column of higher decomposition numbers is orthogonal to the column of degrees,

$$\eta'_r(1)\sum_{r=1}^{4}(\varepsilon'\theta'_r(b_0)+\varepsilon')=0.$$

Hence

$$\eta'_r(1)(-\varepsilon'+4\varepsilon')=0.$$

Then $\eta'_r(1)=0$, which is a contradiction. (q.e.d.)

§4. Exceptional characters with respect to C and division into p_3-blocks (I·II)

$C^{\#}$ is a collection of special classes in $N_G(C)$. The families of two irreducible characters φ_1, φ_2 of degree $\frac{n_3-1}{2}$ of $N_G(C)$ which vanish on $N_G(C)-C$, provide two exceptional characters ζ_1 and ζ_2 of G.

$$\zeta_{1|N_G(C)}=\varepsilon_3\varphi_1+\zeta,\quad \zeta_{2|N_G(C)}=\varepsilon_3\varphi_2+\zeta.$$

Let $\zeta(C^{\#})=z$, where z is a rational integer. And $\chi(C^{\#})$ is a rational integer for any nonexceptional character χ. Since C is a p_3-group and 1 is the only p'_3-class of maximal defect and other classes have defect 0, p'_3-blocks of G have defect 0 except the principal block $B_0(p_3)$.

The orthogonality relation for $c\in C^{\#}$ becomes

$$\begin{aligned}|C_G(c)|&=|\varepsilon_3\varphi_1(c)+\zeta(c)|^2+|\varepsilon_3\varphi_2(c)+\zeta(c)|^2+|\xi_1(1)|^2\\&\quad+\frac{n_2-1}{2}\,|\eta_r(c)|^2+\sum|\chi(c)|^2\end{aligned}$$

where the last sum ranges over all the other characters. Since $\varphi_1(c)+\varphi_2(c) = -1$ and $|\varphi_1(c)|^2+|\varphi_2(c)|^2=\frac{n_3+1}{2}$ from the orthogonality relations in $N_G(C)$, this equation becomes

$$\begin{aligned} n_3 &= |\varphi_1(c)|^2+|\varphi_2(c)|^2+\varepsilon_3 z(\varphi_1(c)+\overline{\varphi_1(c)}+\varphi_2(c)+\overline{\varphi_2(c)}) \\ &\quad +2z^2+1+\frac{n_2-1}{2}\ |\eta_r(c)|^2+\sum|\chi(c)|^2 \\ &= \frac{n_3+1}{2}+2z^2-2\varepsilon_3 z+1+\frac{n_2-1}{2}\ |\eta_r(c)|^2+\sum|\chi(c)|^2 \quad \cdots\cdots\cdots(8) \end{aligned}$$

Next we wish to determine the principal p_3-block $B_0(p_3)$.

LEMMA 10. *(i)* $\{\zeta_1, \zeta_2\}$ *equal* $\{\xi_2, \xi_3\}$ *or equal none of* $\left\{\xi_1(=1_G), \xi_2, \xi_3, Y, \{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right), \{\eta_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right), \{\eta'_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)\right\}$.

(ii) $\left\{\eta'_r\left(1\leqq r\leqq\frac{n_2-1}{2}\right)\right\}$ *belong to* $B_0(p_3)$.

(iii) *In any case of (i) at least one of* $\{\xi_2, \xi_3\}$ *belongs to* $B_0(p_3)$.

(iv) *There exists no character which is in a* p_2*-block of defect 0 for any* p_2 *diviping* n_2 *and in a* p_3*-block of defect 0 except* $\{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$.

PROOF. (i) holds because ζ_1 and ζ_2 have the same value on elements which are not conjugate to $C^\#$.

(ii) Suppose one of $\{\eta'_r\}$ is in a p_3-block of defect 0. Then $\eta'_r(1)$ is divisible by n_1n_3 and then each η'_r is in a p_3-block of defect 0. From the equation $\sum_{\text{all}\chi}|\chi(1)|^2=|G|$ we get $\sum_r|\eta'_r(1)|^2<|G|$ and then $\frac{n_2-1}{2}n_1^2n_3^2<|G|$, which contradicts the condition (0).

(iii) If not, $B_0(p_3)\cap B_1(p_2)=\left\{\{\eta'_r\}\left(1\leqq r\leqq\frac{p_2^d-1}{2}\right)\right\}$. Applying theorem 1 (i) for $B_0(p_3)\cap B_1(p_2)$, $\sum_{r=1}^{\frac{p_2^d-1}{2}}\eta'_r(b_0)\eta'_r(c)=0$ for a fixed $b_0\in B^\#_{p_2}$ and any $c\in C^\#$. Since $\eta'_r(c)$ is independent of r and $\sum_{p=1}^{\frac{p_2^d-1}{2}}\eta'_r(b_0)\neq 0$ by remarks in the proof of lemma 6, 7 and 8, we have $\eta'_r(c)=0$ for any $c\in C^\#$. This contradicts (ii).

(iv) If there exists such a character, its degree must be divisible by n_2n_3. Since it can not be Y nor 1_G, its degree must be also divisible by n_1. Then in the orthogonality relation $\sum_{\text{all}\chi}|\chi(1)|^2=|G|$, this character contributes at least $n_1^2n_2^2n_3^2$, which contradicts the condition (0). (q.e.d.)

From the fact that $|G|=n_1(kn_1+\varepsilon_1)(kn_1+2\varepsilon_1)\dfrac{|C_G(\tau)|^2}{(kn_1+\varepsilon_1-Y(\tau))^2}$

(lemma 1) and $(n_3, |C_G(\tau)|)=1$, two cases occur as follows:
Case (1) n_3 divides $kn_1+\varepsilon_1$
Case (2) n_3 divides $kn_1+2\varepsilon_1$

LEMMA 11. *In case* (1) $X_i(C^{\#})=\varepsilon_1\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$, $\eta_r(C^{\#}) = -\varepsilon_1\left(1\leqq r\leqq\frac{n_2-1}{2}\right)$, $Y(C^{\#})=0$.

PROOF. Since $X_i(1)\ (=kn_1+2\varepsilon_1)$, $\eta_r(1)\ (=kn_1)$ are not divisible by n_3, $\{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$, and $\{\eta_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)\in B_0(p_3)$. Applying theorem 1 for $S\cap B_0(p_3)=\left\{1_G,\ \{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right)\right\}$

$$\sum_{S\cap B_0(p_3)}\chi(a)\,\overline{\chi(C^{\#})}=1+\sum_{i=1}^{\frac{n_1-1}{2}}X_i(a)\,\overline{X_i(C^{\#})}=1-\varepsilon_1\overline{X_i(C^{\#})}=0$$

for p_1-element $a\in A^{\#}$. Hence $X_i(C^{\#})=\varepsilon_1$. Applying theorem 1 for $B_0(p_2)\cap B_0(p_3)=\left\{1_G,\ \{\eta_r\}\left(1\leqq r\leqq\frac{p_2^d-1}{2}\right)\right\}$

$$\sum_{B_0(p_2)\cap B_0(p_3)}\chi(b_0)\,\overline{\chi(C^{\#})}=1+\sum_{r=1}^{\frac{p_2^d-1}{2}}\eta_r(b_0)\,\overline{\eta_r(C^{\#})}=1+\varepsilon_1\overline{\eta_r(C^{\#})}=0$$

for a p_2-element $b_0\in B^{\#}$, because $\sum_{r=1}^{\frac{p_2^d-1}{2}}\eta_r(b_0)=\varepsilon_1$ by lemma 3, 7 and 8. Hence $\eta_r(C^{\#})=-\varepsilon_1$. Since this value is independent of r, $\eta_r(C^{\#})=-\varepsilon_1$ for $1\leqq r\leqq\frac{n_2-1}{2}$. (q.e.d.)

LEMMA 12. *In case* (2) $X_i(C^{\#})=0$ $\left(1\leqq i\leqq\frac{n_1-1}{2}\right)$, $Y(C^{\#})=-\varepsilon_1$ and $\eta_r(C^{\#})=-2\varepsilon_1\left(1\leqq r\leqq\frac{n_2-1}{2}\right)$.

PROOF. Similarly to the proof of lemma 11. (q.e.d.)

LEMMA 13. *Case* (2) *cannot occur.*

PROOF. Since $\{\eta'_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)\in B_0(p_3)$ by lemma 10 (ii), η'_r is not zero on at least one element $c\in C^{\#}$ and it is independent of r. We can put this absolute value ($\geqq 1$) into the equation (8) as well as the values of lemma 12. Then (8) becomes

$$n_3\geqq\frac{n_3+1}{2}+2z^2-2\varepsilon_3 z+1+\frac{n_2-1}{2}\,|\eta_r(c)|^2+\frac{n_2-1}{2}|\eta'_r(c)|^2+|Y(c)|^2$$

$$\geqq \frac{n_3+1}{2}+2z^2-2\varepsilon_3 z+1+\frac{n_2-1}{2}\cdot 4+\frac{n_2-1}{2}\cdot 1+1.$$

Hence $n_3 \geqq 5n_2+4z^2-4\varepsilon_3 z \geqq 5n_2$, which contradicts the condition (i). (q.e.d.)

LEMMA 14. *Let $\mathfrak{A}$ be the set of irreducible characters* $\{\xi_1, \xi_2, \xi_3, Y, \{X_i\} \left(1 \leqq i \leqq \frac{n_1-1}{2}\right), \{\eta_r\} \left(1 \leqq r \leqq \frac{n_2-1}{2}\right) \{\eta'_r\} \left(1 \leqq r \leqq \frac{n_2-1}{2}\right)\}$. *Let $\mathfrak{B}$ be the set of representatives of the conjugacy classes $\{1, j, A^{\#}, B^{\#}, jB^{\#}, C^{\#}\}$. There are five cases in case* 1 *and* 2.

(a) $n_3=2n_2+n_1$. *$\mathfrak{A}$ is the set of all irreducible characters and $\mathfrak{B}$ represents all conjugacy classes.*

(b)-(1) $n_3=2n_2+n_1+2$. *There exists only one irreducible character except $\mathfrak{A}$ and only one class except $\mathfrak{B}$.*

(b)-(2) $n_3=2n_2+n_1+2$. *There exist only two irreducible characters except $\mathfrak{A}$ and only two classes except $\mathfrak{B}$.*

(c)-(1) $n_3=2n_2+n_1+4$. *There exist only two irreducible characters except $\mathfrak{A}$ and only two classes except $\mathfrak{B}$.*

(c)-(2) $n_3=2n_2+n_1+4$. *There exist only three irreducible characters except $\mathfrak{A}$ and only three classes except $\mathfrak{B}$.*

In case 3 the equation and the number of the classes are the same as (b)-(1) *or* (c)-(1).

PROOF. Since η'_r is not zero on at least one element $c \in C^{\#}$, in the same way as the proof of lemma 13, we can put this absolute value ($\geqq 1$) into the equation (8) as well as the values of lemma 11. Then (8) becomes

$$\begin{aligned} n_3 &\geqq \frac{n_3+1}{2}+2z^2-2\varepsilon_3 z+1+\frac{n_2-1}{2}\,|\eta_r(c)|^2+\frac{n_2-1}{2}\,|\eta'_r(c)|^2 \\ &\quad +\frac{n_1-1}{2}\,|X_i(c)|^2 \\ &\geqq \frac{n_3+1}{2}+2z^2-2\varepsilon_3 z+1+\frac{n_2-1}{2}+\frac{n_2-1}{2}+\frac{n_1-1}{2} \\ &= \frac{n_3}{2}+2z^2-2\varepsilon_3 z+n_2+\frac{n_1}{2}. \end{aligned}$$

Hence

$$n_3 \geqq 2n_2+n_1+4z^2-4\varepsilon_3 z.$$

Since z is a rational integer, the minimal value of $4z^2-4\varepsilon_3 z$ is 0 and minimal nonzero value is 8. From the condition (ii) $n_3<2n_2+n_1+6$, we get $n_3=2n_2+n_1$ or $2n_2+n_1+2$ or $2n_2+n_1+4$, because n_3 and n_1 are odd. Furthermore $4z^2-4\varepsilon_3 z=0$ and $\eta'_r(c)=\pm 1$ and hence $\eta'_r(C^{\#})=\pm 1$ by equal multiplicity in § 1, I, **a**.

In case (a) by the equation (8) $B_0(p_3)=\left\{1_G, \zeta_1, \zeta_2, \{X_i\}\left(1 \leqq i \leqq \frac{n_1-1}{2}\right), \{\eta_r\}\left(1 \leqq r \leqq \frac{n_2-1}{2}\right), \{\eta'_r\}\left(1 \leqq r \leqq \frac{n_2-1}{2}\right)\right\}$. Suppose $\{\xi_2, \xi_3\} \neq \{\zeta_1, \zeta_2\}$, then neither ξ_2 nor ξ_3 can belong to $B_0(p_3)$, which contradicts lemma 10 (iii). Hence $\{\xi_2, \xi_3\}=\{\zeta_1, \zeta_2\}$. In this case any character except $\mathfrak{A}$ is in a p_2-block of defect 0 for any p_2 dividing n_2 and in a p_3-block of defect 0, which contradicts lemma 10 (iv). Then $\mathfrak{A}$ is the set of all irreducible characters.

In case (b) by the equation (8) $B_0(p_3)=\left\{1_G, \zeta_1, \zeta_2, \{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right),\right.$ $\{\eta_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)$, $\left.\{\eta'_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right),\ Z\right\}$ wher Z is a new character. We remark that a nonexceptional character has common value on $C^{\#}$ by equal multiplicity in §1, I, **a**. If $\{\zeta_1, \zeta_2\}=\{\xi_2, \xi_3\}$, Z is not in the set $\mathfrak{A}$. If $\{\zeta_1, \zeta_2\}\neq\{\xi_2, \xi_3\}$, Z is one of $\{\xi_2, \xi_3\}$ by lemma 10 (iii). In the first case $\mathfrak{A}\cup Z$ is the set of all irreducible characters and in the second case $\mathfrak{A}\cup\{\zeta_1, \zeta_2\}$ is the set of all irreducible characters by lemma 10 (iv).

In case (c) from the equation (8), $B_0(p_3)=\left\{1_G, \zeta_1, \zeta_2, \{X_i\}\left(1\leqq i\leqq\frac{n_1-1}{2}\right),\right.$ $\{\eta_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right)$, $\left.\{\eta'_r\}\left(1\leqq r\leqq\frac{n_2-1}{2}\right),\ Z_1,\ Z_2\right\}$, where Z_1 and Z_2 are new characters. If $\{\zeta_1, \zeta_2\}=\{\xi_2, \xi_3\}$, then neither Z_1 nor Z_2 is in $\mathfrak{A}$, and $\mathfrak{A}\cup Z_1\cup Z_2$ is the set of all irreducible characters by lemma 10 (iv). If $\{\zeta_1, \zeta_2\}\neq\{\xi_2, \xi_3\}$, then at least one of Z_1, Z_2 is equal to ξ_2 or ξ_3 by lemma 10 (iii). Anyway $\mathfrak{A}\cup\zeta_1\cup\zeta_2$ or $\mathfrak{A}\cup\zeta_1\cup\zeta_2$ with only one more character is the set of all irreducible characters by lemma 10 (iv).

Since the number of $\mathfrak{A}$ and $\mathfrak{B}$ are same, the number of the remaining classes can be determined in each case. (q.e.d.)

§ 5. Nonexistence of the remaining conjugacy classes (IV)

LEMMA 15. *If* $m^2=1$, *then* $k=2$.

PROOF. If $m^2=1$, then $|G|=n_1(kn_1+\varepsilon_1)(kn_1+2\varepsilon_1)$. On the other hand kn_1 which is the degrees of η_r, divides $|G|$. Then $k=1$ or 2. Suppose $k=1$, then $n_3\geqq 2n_2+n_1>n_1+1$. But now $n_3|Y(1)=n_1+\varepsilon_1$, which is a contradiction. (q.e.d.)

LEMMA 16. *In case* (a) *G is isomorphic to one of PSL* (2, *q*) *with 2-Sylow groups of order 4.*

PROOF. Since $\frac{n_3-1}{2}=\left(\frac{2n_2+n_1-1}{2}\right)$ is an order of a cyclic group and is greater than n_2 and $\frac{n_1}{2}$, must be $2n_2$ or n_1. In the first case, $n_1=2n_2+1$. Since n_2 divides $kn_1+2\varepsilon_1$ which is now $2n_2k+k+2\varepsilon_1$, $k=2$ and $\varepsilon_1=-1$, or n_2 divides $k+2\varepsilon_1$. Suppose n_2 divides $k+2\varepsilon_1$, then $n_2\leqq k+2\varepsilon_1$ and then $n_2-2\varepsilon_1\leqq k$ and it contradicts the condition (0). Because if $m^2\neq 1$, then

$$\begin{aligned}|G|&=n_1(kn_1+\varepsilon_1)(kn_1+2\varepsilon_1)m^2\geqq n_1\{n_1(n_2-2\varepsilon_1)+\varepsilon_1\}\{n_1(n_2-2\varepsilon_1)+2\varepsilon_1\}(n_1+1)\\&>n_1^4(n_2-2)^2\end{aligned}$$

and by the condition (0)

$$|G| \leqq n_1^2 n_3^2 \frac{n_2-1}{2} = n_1^2 (2n_2+n_1)^2 \frac{n_2-1}{2} < 2n_1^4 (n_2-1),$$

then since $n_2 \geqq 5$, $(n_2-2)^2 > 2(n_2-1)$ which is a contradiction. And if $m^2=1$, then $k=2$ by lemma 15, which contradicts $n_2-2\varepsilon_1 \leqq k$.

Then $k=2$ and $\varepsilon_1=-1$. Then $|G|=n_1(2n_1-1)(2n_1-2)m^2=4n_1n_2n_3m^2$. Let 2^α be the order of a 2-Sylow group of G. Since $\mathfrak{B}$ is the set of representatives of all conjugacy classes in case (a), m^2 can be divisible only by a power of 2. Hence

$$2^{\alpha-2}=m^2=\frac{|C_G(\tau)|^2}{(2n_1-1-Y(\tau))^2}=\frac{n_2^2\, 2^{2\alpha}}{(4n_2+1-Y(\tau))^2}, \quad \cdots\cdots\cdots\cdots\cdots(9)$$

since τ is in a center of a 2-Sylow group. $-(4n_2-1) \leqq Y(\tau) \leqq 4n_2-1$. Because G is simple and $Y(\tau)$ cannot be a trace of a scalar matrix. Then $2 \leqq |4n_2+1-Y(\tau)| \leqq 8n_2$. Hence $(4n_2+1-Y(\tau))=2n_2$, $4n_2$ or $8n_2$. If it is $2n_2$, (9) implies $2^{\alpha-2}=2^{2\alpha-2}$, which is a contradiction. If it is $4n_2$, (9) implies $\alpha=2$ and $m^2=1$. Then by theorem 1 [1] G is isomorphic to one of *PSL* (2, q), Futhermore the order of a 2-Sylow group is 4 now. If it is $8n_2$, then (9) implies $\alpha=4$ and $m^2=4\equiv 1 \pmod{n_1}$. Then $n_1=3$. This is a contradiction.

In the second case that $\frac{n_3-1}{2}=n_1$, $n_1=2n_2-1$. We can verify that G is isomorphic to one of *PSL* (2, q) with 2-Sylow groups of order 4 similarly. (q.e,d.)

REMARK. The order of *PSL* (2, q) (q: odd) is $\frac{(q-1)q(q+1)}{2}$. Now q is n_3. If $q\equiv 1 \pmod 4$, then the first case occurs: $\frac{q-1}{4}=n_2$ and $\frac{q+1}{2}=n_1$. If $q\equiv -1 \pmod 4$ then the second case occurs: $\frac{q+1}{4}=n_2$ and $\frac{q-1}{2}=n_1$.

Next we shall deal with cases (b) and (c) and wish to conclude that neither case can occur by considering orders of the remaining classes except $\mathfrak{B}$. We can deal with these four cases by almost similar method. Here we shall describe only the most complicated case (c)-(2).

LEMMA 17. *In case* (c)-(2) *possible orders of the remaining classes except* $\mathfrak{B}$ *are as follows:*

(r, q, s), (r, r, q), (r, r, r), $(r, 2r, q)$, $(r, r, 2r)$, $(r, 2r, 2r)$, $(r, 2r, 4)$, $(r, r, 4)$, $(r, 4, 4)$, $(r, q, 4)$, $(r, 4, 8)$, $(4, 4, 4)$, $(4, 4, 8)$, $(4, 8, 8)$, $(4, 8, 16)$, $(3, 9, 6)$, $(3, 9, 9)$,

where r, q *and* s *are distinct odd primes.*

PROOF. First of all we must verify that if there exists order r^2, $r=3$. Suppose there is only one class of order r^2. Since the automorphism of an element v of order r^2 is an abelian group of order $r(r-1)$, $N_G(v)/C_G(v)$ must be so and there exists an element of order rt, where t is any prime dividing $r-1$. Then $r=3$. Furthermore the orders are (3, 9, 6). Suppose there are 2 classes of order r^2. In this case the orders are (r, r^2, r^2). $N_G(v)/C_G(v)$ for an element v of order r^2 acts fixed-point-free on the elements of order r^2 in $\langle v\rangle^{\#}$ and any two elements of order r^2 in $\langle v\rangle^{\#}$ conjugate in G are conjugate in $N_G(v)$, Hence

$$r(r-1)=2|N_G(v)/C_G(v)| \text{ or } |N_G(v)/C_G(v)|.$$

Then $|N_G(v)/C_G(v)|=\dfrac{r\,(r-1)}{2}$ or $r\,(r-1)$. But since there is not an element of order $2r$, $|N_G(v)/C_G(v)| =\dfrac{r\,(r-1)}{2}$ and $r=3$ and the orders are (3, 9, 9).

Next we must verify the orders (r, q, rq) cannot occur, Since the conjugacy class of order rq is unique, $N_G(v)/C_G(v)$ is an abelian group of order $(r-1)(q-1)$ for an element v of order rq. Here $r-1$ and $q-1$ are even. Since an elementary abelian 2-Sylow group of $N_G(v)$ cannot act fixed-point-free on $\{r, q\}$-group $C_G(v)$, this is a contradiction. (q.e.d.)

LEMMA 18. *In case* (c)-(2)

(i) $n_1 < 3n_2-4$.

(ii) $\dfrac{n_3-1}{2} > n_2,\ \dfrac{n_1}{2}$. $\quad \dfrac{n_3-1}{2} \geqq 10.$

(iii) $\dfrac{n_3-1}{2} =n_1,\ 2n_2,\ r,\ q,\ s$ or $2r$.

PROOF. (i) By the condition (i). (ii) Because $\dfrac{n_3-1}{2} = \dfrac{2n_2+n_1+3}{2}$ and $2n_2+n_1\leqq 2\times 5+7$ by the condition (ii). (iii) The assertion holds by lemma 17 and (ii). Because $\dfrac{n_3-1}{2}$ is an order of a cyclic group. We remark $\dfrac{n_3-1}{2} \neq 16$ since n_3 is a power of p_3. (q.e.d.)

Next we shall examine the cases in this lemma (iii) case by case.

LEMMA 19. *In case* (c)-(2) *the case* $\dfrac{n_3-1}{2}=n_1$ *cannot occur.*

PROOF. In this case $n_1=2n_2+3$ and $n_3=4n_2+7$. Since n_2 divides $kn_1+2\varepsilon_1$, which equals $2kn_2+3k+2\varepsilon_1$, $n_2\leqq 3k+2\varepsilon_1$. Hence $\dfrac{n_2-2\varepsilon_1}{3} \leqq k$. Suppose $m^2=1$. Then by lemma 15, $k=2$ and n_2 divides $4n_2+6+2\varepsilon_1$. This is a contradiction. Then $m^2\neq 1$. Then

$$|G| \geqq n_1^2 \left(\frac{n_2 - 2\varepsilon_1}{3} n_1 + 2\varepsilon_1 \right) \left(\frac{n_2 - 2\varepsilon_1}{3} n_1 + \varepsilon_1 \right) (n_1 + 1)$$
$$> n_1^2 (2n_2 + 3)^2 \left(\frac{n_2 - 2}{3} \right)^2 \quad \cdots\cdots (10)$$

On the other hand by the condition (0),

$$|G| \leqq n_1^2 (4n_2 + 7)^2 \frac{n_2 - 1}{2} \quad \cdots\cdots (11)$$

By easy caluculation we can see the right side of (10) > the right side of (11) for $n_2 \geqq 21$ and this is a contradiction. If $n_2 < 21$ there are only finite possibilities for n_1, n_2 and n_3. By the condition (0) and the fact that $m^2 \equiv 1$ (mod n_1) we can verify no case can occur. (q.e.d.)

We remark that if we get an equation of n_1 and n_2, we can deal with the case similarly as above and gain a contradiction.

LEMMA 20. *In case* (c)-(*2*) *the case* $\frac{n_3 - 1}{2} = 2n_2$ *cannot occur.*

PROOF. In this case $n_1 = 2n_2 - 3$. We can deal with this case similarly. (q.e.d.)

LEMMA 21. *In case* (c)-(*2*) *the case* $\frac{n_3 - 1}{2} = r$ *or* s *cannot occur.*

PROOF. In case that it is r and there is only 1 class of order r, or in case that it is s, $r-1$ (or $s-1$) is $2 \times$(a divisor of n_2) or 4. Now since $10 \leqq 2n_2 < 2n_2 + n_1 + 1 = 2(r-1)$, (or $2(s-1)$), $2n_2 + n_1 + 1 = 4n_2$. In this case $n_1 = 2n_2 - 1$. We can deal with this case similarly.

In case that $\frac{n_3 - 1}{2} = r$ and there are only 2 classes of order r, the possibilities of orders of the remaining classes are (r, r, q), $(r, r, 2r)$ and $(r, r, 4)$. Since an abelian group $N_G(v)/C_G(v)$ acts fixed-point-free on $\langle v \rangle^{\#}$ for an element v of order r and any two elements of $\langle v \rangle^{\#}$ conjugate in G are conjugate in $N_G(v)$, $|N_G(v)/C_G(v)| = r-1$ or $\frac{r-1}{2}$. We can deal with the case that it is $r-1$ as well as the previous paragraph. Suppose $|N_G(v)/C_G(v)| = \frac{r-1}{2}$. Since $\frac{n_2}{2}$, $\frac{n_1}{4} < \frac{2n_2 + n_1 + 1}{4} = \frac{n_3 - 3}{4} = \frac{r-1}{2}$ and $4 < \frac{2n_2 + n_1 + 1}{4} = \frac{r-1}{2}$, $\frac{r-1}{2} = 2 \times$(a divisor of n_2), $\frac{n_1}{3}$, n_1 n_2 or q. If $\frac{r-1}{2} = 2 \times$(a divisor of n_2), $2n_2 + n_1 + 1 = 8n_2$ or $\frac{8}{3} n_2$, since $2n_2 + n_1 + 1 > 2n_2$. If it is $8n_2$, then $n_1 = 6n_2 - 1$, which contradicts lemma 18 (i). If it is $\frac{8}{3} n_2$, then $n_1 = \frac{2}{3} n_2 - 1$. We can deal with this case similarly. If $\frac{r-1}{2}$

$=\frac{n_1}{3}$, then $n_1=6n_2+3$, which contradicts lemma 18 (i). If $\frac{r-1}{2}=n_1$, then $n_1=\frac{2n_2+1}{3}$. We can deal with this case similarly. If $\frac{r-1}{2}=n_2$, then $n_1=2n_2-1$ and we can deal with this case similarly. If $\frac{r-1}{2}=q$, then $q-1=2\times$(a divisor of n_2) since there is only 1 class of order q. Then $\frac{r-1}{2}=2n_2+1$ or $\frac{2}{3}n_2+1$ since $\frac{r-1}{2}>\frac{n_2}{2}$. In the first case $n_1=6n_2+3$, which contradicts lemma 18 (i). We can deal with the second case similarly.

In case that $\frac{n_3-1}{2}=r$ and there are 3 classes of order r, the possibility of orders of the remaining classes is (r, r, r). $|N_G(v)/C_G(v)|=r-1$, $\frac{r-1}{2}$ or $\frac{r-1}{3}$ for an element v of order r by the same reason as the previous paragraph. We can deal with the first and second cases in the same way as the previous paragraph. Suppose it is $\frac{r-1}{3}$. Then it is $2\times$(a divisor of n_2). Then $2n_2<2n_2+n_1+1=12n_2$, $4n_2$ or $\frac{12}{5}n_2$. Hence $n_1=10n_2-1$, $2n_2-1$ or $\frac{2}{5}n_2-1$. The first case contradicts lemma 18 (i). We can deal with the second and third cases similarly to the proof of lemma 19. (q.e.d.)

LEMMA 22. *In case* (c)-(2) *the case* $\frac{n_3-1}{2}=q$ *cannot occur.*

PROOF. Suppose $\frac{n_3-1}{2}=q$. Then $q-1$ is an order of an abelian group since there is only 1 class of order q. Hence $2n_2<2n_2+n_1+1=2(q-1)=4n_2$ or $4r$ since $q-1=2\times$(a divisor of n_2) or $2r$. If it is $4n_2$, then $n_1=2n_2-1$ and we can deal with this case similarly. If it is $4r$, then $r-1=2\times$(a divisor of n_2), since the orders of the remaining classes are $(r, 2r, q)$. Since $2n_2<2n_2+n_1-3=4(r-1)=8\times$(a divisor of n_2), it must be $8n_2$ or $\frac{8}{3}n_2$. The first case contradicts lemma 18 (i) since $n_1=6n_2+3$. In the second case $n_1=\frac{2}{3}n_2+3$ and we can deal with this case similarly. (q.e.d.)

LEMMA 23. *In case* (c)-(2) *the case* $\frac{n_3-1}{2}=2r$ *cannot occur.*

PROOF. Suppose $\frac{n_3-1}{2}=2r$. If there is only 1 class of order r, $r-1=2\times$(a divisor of n_2) or 4. Otherwise the orders of the remaining classes are $(r, r, 2r)$. In this case $r-1=2\times$(a divisor of n_2) since $N_G(v)/C_G(v)$ for an element v of order $2r$ is an abelian group of order $r-1$. Anyway $2n_2<$

$2n_2+n_1-1=4(r-1)=8n_2, \dfrac{8}{3}n_2$ or 16. The first case contradicts lemma 18 (i) since $n_1=6n_2+1$. In the second case $n_1=\dfrac{2}{3}n_2+1$ and $r=2\left(\dfrac{2}{3}n_2+1\right)=2n_1$, which is a contradiction. In the third case $n_2=5$, $n_1=7$ and $n_3=21=3n_1$, which is a contradiction. (q.e.d.)

Then the main theorem is proved.

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