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On a pole of the resolvent of positive operators

Fukiko Takeo

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo

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§1. Introduction.

For a positive operator T in an ordered Banach space, the following theorem has been shown by S. Karlin [2].

THEOREM (S. Karlin).
$\frac{1}{n}\sum_{i=0}^{n-1} T^i$ *converges uniformly to a non-zero operator if and only if* 1 *is a simple pole of the resolvent* $R(\lambda, T)$ *of* T.

S. Karlin has shown also that if 1 is a pole of $R(\lambda, T)$ of order k, then $\frac{\Gamma(k+1)}{n^k}\sum_{i=0}^{n-1} T^i \to (T-I)^{k-1}P$, where P is a projection operator. The converse, however, doesn't hold in case of $k \geqq 2$ as shown in §2.

In this paper, we shall give a necessary and sufficient condition for 1 to be a pole of order k of the resolvent, which is the extension of the above theorem of Karlin.

§2. Results.

EXAMPLE.
Consider $T = \begin{pmatrix} 1 & 1 & & & & 0 \\ & 1 & & & & \\ & & 1-1/2 & & & \\ & & & 1-1/3 & & \\ & & & & \ddots & \\ & 0 & & & 1-1/n & \\ & & & & & \ddots \end{pmatrix}$

belonging to $L(l^\infty)$.

Then the sequence $\left\{\frac{2}{n^2}\sum_{i=0}^{n-1} T^i\right\}$ converges uniformly to

$(T-I)P = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 0 \\ & 0 & & \ddots \end{pmatrix}$, where $P = \begin{pmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & 0 & & \\ & & & 0 & \\ & 0 & & & \ddots \end{pmatrix}$

But 1 is not a pole of $R(\lambda, T)$.

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The above example shows that the convergence of a sequence $\left\{\frac{\Gamma(k+1)}{n^k}\sum_{i=0}^{n-1} T^i\right\}$ is not sufficient for 1 to be a pole of $R(\lambda, T)$. Therefore, by using another sequence of polynomials, we obtain the following theorem.

THEOREM 1.

Let T be a positive operator with $r(T)=1$ in an ordered Banach space E[1]*. Then the following are equivalent.*

i) *1 is a pole of the resolvent $R(\lambda, T)$ of order at most k.*

ii) *$f_{k,n}(T)$ converges*[2] *to an operator P_k as $n\to\infty$, where*

$$f_{k,n}(\lambda)=\sum_{s=1}^{k} a_{k,s}\frac{g_{k,s,n-1}(\lambda)}{g_{k,s,n-1}(1)}$$

$$g_{k,s,q}(\lambda)=\sum_{i_k=0}^{q}\cdots\cdots\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}\frac{(i_1+s-1)!}{i_1!}\lambda^{i_1}$$

and $a_{k,s}=(-1)^{s+1}\binom{k}{s}\binom{2k+s-2}{k-1}\Big/\binom{2k-2}{k-1}$.

The coefficients $\{a_{k,s}\}$ in Theorem 1 are solutions of the following simple equations :

$$\sum_{s=1}^{k} a_{k,s}=1$$

$$\sum_{s=1}^{k}\frac{1}{k+j+s-1}a_{k,s}=\frac{1}{k+j-1},\quad 1\leqq j\leqq k-1. \qquad (*)$$

By calculation, we obtain

$$f_{k,n}^{(j)}(\lambda)=\sum_{s=1}^{k} a_{k,s}\frac{g_{k,s+j,n-j-1}(\lambda)}{g_{k,s,n-1}(1)}$$

$$g_{k,s,q}(1)=\binom{q+s+k-1}{q}(s-1)\,!$$

and

$$f_{k,n}^{(j)}(1)=\frac{(n-1)!}{(n-j-1)!}\sum_{s=1}^{k} a_{k,n}\frac{(s+j-1)!\,(s+k-1)!}{(s-1)!\,(s+k+j-1)!}$$

Therefore, by using the relation (*), we get

$f_{k,n}(1)=1$ and $f_{k,n}^{(j)}(1)=0$ for $1\leqq j\leqq k-1$.

PROOF OF THE THEOREM 1.

Suppose i) is satisfied. Then by the result of S. Karlin stated in §1, the sequence $\left\{\frac{k!}{n^k}\sum_{i=0}^{n-1} T^i\right\}$ converges to $(T-I)^{k-1}P$, where P is a projection. Therefore we have

1) An ordered Banach space means that it is generated by a proper, closed and normal cone. For these definitions, see [3].

2) In this paper, convergence is understood to be in the uniform operator topology.

$$\frac{T^n}{n^k} \longrightarrow 0 \quad (n \to \infty) \tag{**}$$

by using the relation

$$\frac{T^n}{n^k} = \left(\frac{n+1}{n}\right)^k \frac{1}{(n+1)^k} \sum_{i=0}^{n} T^i - \frac{1}{n^k} \sum_{i=0}^{n-1} T^i.$$

By calculation, we have, for $1 \leqq s \leqq k$

$$g_{k,s,n-1}(T)\,(T-I)^k = \frac{(n+s-2)!}{(n-1)!}\,T^{n+k-1}$$
$$+ \sum_{m=1}^{s-1} (-1)^m \binom{k}{m} \frac{(s-1)!}{(s-m-1)!}\, T^{k-m} g_{m,s-m,n-1}(T)$$
$$+ (-1)^s (s-1)! \sum_{m=0}^{k-s} \binom{n+m+s-2}{n-1} \binom{k-m-1}{s-1} T^{k-s-m} (T-I)^m.$$

So by using the relation (**) and

$$g_{k,s,n-1}(1) = \frac{(n-s+1)\,(n-s+2)\cdots\cdots(n+k-1)}{s(s+1)\cdots\cdots(s+k-1)} \simeq n^{k+s-1} \tag{***}$$

we get

$$f_{k,n}(T)\,(T-I)^k = \sum_{s=1}^{k} a_{k,s} \frac{g_{k,s,n-1}(T)}{g_{k,s,n-1}(1)}\,(T-I)^k \to 0$$

as $n \to \infty$.

By virtue of a result of N. Dunford [1, Theorem 3. 6], we deduce that $f_{k,n}(T)$ converges to a projection as $n \to \infty$, by considering $(\lambda-1)^k$ for $P(\lambda)$ in [1. Th. 3.6].

Next we suppose that ii) is satisfied. We consider the functions

$$h_{k,1,n}(T) = \frac{n+2k-1}{n} f_{k,n+1}(T) - f_{k,n}(T)$$

and

$$h_{k,m,n}(T) = \frac{n+2k-1}{n}\, h_{k,m-1,n+1}(T) - T \cdot h_{k,m-1,n}(T)$$

for $2 \leqq m \leqq k$.

Then by using the condition that $f_{k,n}(T)$ converges to a projection, we see, for $1 \leqq m \leqq k$

$$h_{k,m,n}(T) \longrightarrow 0 \qquad (n \to \infty).$$

By calculation, we have, for $1 \leqq m \leqq k$

$$h_{k,m,n}(T) = \sum_{s=1}^{k-m} \sum_{r=0}^{m-1} \frac{(m-1)!\,(s+r+k-1)!}{r!\,(s-1)!} \binom{k-s-r}{m-r-1} a_{k,s+r}$$
$$\times \left\{ \frac{(n-1)!}{(n+k+s+m-3)!} g_{k-1,s,n+m-1}(T) + \frac{(k-s-m+1)(n-1)!}{(n+k+s+m-2)!} g_{k,s,n+m-1}(T) \right\}$$
$$+ \frac{(n-1)!}{(n+2k-2)!} \sum_{r=0}^{m-1} \frac{(m-1)!\,(2k-m+r)!}{r!\,(k-m)!}\, a_{k,k-m+r+1}\, g_{k-1,k-m+1,n+m-1}(T)$$
$$+ 0\left(\frac{1}{n}\right)$$

and

$$h_{k,k,n}(T)\cdot(T-I)^{k-2}=\frac{(n-1)!\,(k-1)!}{(n+2k-2)!}\sum_{r=1}^{k}\frac{(r+k-1)!}{(r-1)!}a_{k,r}$$
$$\times\sum_{i=0}^{n-1}T^{i+2k-2}+0\left(\frac{1}{n}\right).$$

Since $\sum_{r=1}^{k}\frac{(r+k-1)!}{(r-1)!}a_{k,r}$ is nonzero, we have

$$\frac{1}{n^{2k-1}}\sum_{i=0}^{n-1}T^{i}\longrightarrow 0\quad(n\longrightarrow\infty)\tag{*1}$$

by using the relation $h_{k,k,n}(T)\cdot(T-I)^{k-2}\to 0$. In the same way, from (*1) and $h_{k,k-1,n}(T)\cdot(T-I)^{k-1}\to 0$, we have

$$\frac{T^n}{n^{2k-2}}\longrightarrow 0\quad(n\longrightarrow\infty)\tag{*2}$$

and then, from (*1), (*2) and $h_{k,k-2,n}(T)\cdot(T-I)^{k}\to 0$,

$$\frac{T^n}{n^{2k-3}}(T-I)\longrightarrow 0\quad(n\longrightarrow\infty).$$

A repetition of this argument clearly yields

$$\frac{T^n}{n^{2k-2-j}}(T-I)^{j}\longrightarrow 0\quad(n\longrightarrow\infty)\tag{*j+2}$$

for $1\leqq j\leqq k-2$.
By using the relation (***), (*1)$\sim$(*k) and

$$f_{k,n}(T)\cdot(T-I)^{2k-2}=\sum_{s=1}^{k}\frac{a_{k,s}}{g_{k,s,n-1}(1)}$$
$$\times\sum_{i=0}^{(s-1)\wedge(k-2)}(-1)^{i}\binom{k}{i}\binom{n+s-i-2}{s-i-1}(s-1)!\;T^{n+k-1}(T-I)^{k-2-i}$$
$$+\frac{a_{k,k}}{g_{k,k,n-1}(1)}(-1)^{k-1}k!\sum_{i=0}^{n-1}T^{i+k-1}+0(n^{k-s}),$$

we get

$$f_{k,n}(T)\cdot(T-I)^{2k-2}\longrightarrow 0\quad(n\longrightarrow\infty).$$

Put $P_k(T-I)^j=Q_{k,j}$, $1\leqq j\leqq 2k-3$.
Then by using the relation $P_k(T-I)^{2k-2}=0$, we have, for $i\geqq 1$

$$P_k\cdot T^i=P_k+\sum_{s=1}^{r}\binom{i}{s}Q_{k,s},\ \text{where } r=i\wedge(2k-3).$$

Therefore we have

$$f_{k,n}(T)\cdot P_k=f_{k,n}(1)\cdot P_k+\sum_{j=1}^{2k-3}\frac{f_{k,n}^{(j)}(1)}{j!}Q_{k,j}.$$

Since the coefficients $\{a_{k,s}\}$ satisfy the equations $f_{k,n}(1)=1$ and $f_{k,n}^{(j)}(1)=0$ for $1 \leqq j \leqq k-1$, we see that for $j \geqq k$, $f_{k,n}^{(j)}(1)$ is nonzero and tends to infinity as $n \to \infty$. If $Q_{k,j} \neq 0$ for some $j \geqq k$, $\|f_{k,n}(T) \cdot P_k\|$ tends to infinity, which contradicts that $f_{k,n}(T) \cdot P_k$ converges to P_k^2. Therefore $Q_{k,k}=0$, which means that $P_k^2=P_k$ and 1 is a pole of the resolvent of order at most k, since $r(T)=1$ belongs to the spectra of T by positivity of T. This completes the proof of the Theorem.

The positivity of Theorem 1 plays the role to get $\dfrac{T^n}{n^k} \to 0$ as $n \to \infty$ from the condition i) in the Theorem and also for 1 to belong to the spectra of T. So we we can replace positivity by the condition $\dfrac{T^n}{n^k} \to 0$ as follows.

THEOREM 2. *Let T be an operator with $r(T)=1$ in a Banach space, satisfing $\dfrac{T^n}{n^k} \to 0$ as $n \to \infty$. Then the following are equivalent.*

i′) *1 is either in $\rho(T)$ or else a pole of $R(\lambda, T)$ of order at most k.*
ii) *$f_{k,n}(T)$ defined in Theorem 1 converges to an operator P_k as $n \to \infty$.*

References

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