Tables of the King and Domino Polynomials for Polyominoes

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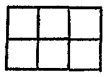
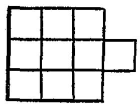
The rook polynomial has been defined for the enumeration of the number of ways for choosing non-taking rooks on a given chessboard, polyomino, or square animal[1,2]. The present authors have proposed the king polynomial and found that it is closely related not only to the domino paving problem but also to several important problems of physics and chemistry[3]. For the problem of paving dominoes the domino polynomial has also been proposed. The recursive relations for the newly defined polynomials are less simpler than that of the rook polynomial. Thus it is worth tabulating these two polynomials for the fundamental polyomino graphs.

A polyomino or a square animal is composed of square cells of the same size as graphs I and II. The non-taking number $r(G, k)$ for given



I



II

graph G is the number of ways for choosing non-taking kings which can take on any of the eight neighboring cells. The king polynomial $K_G(X)$ is defined as

$$K_G(X)=\sum_{k=0}^{m} r(G, k)X^k\,. \tag{1}$$

The value

$$K_G(1)=\sum_{k=0}^{m} r(G, k) \tag{2}$$

gives the total number of " king patterns " as III for I, where a circle stands for a king. In Table I $K_G(X)$ and $K_G(1)$ values are given for smaller polyominoes up to the heptominoes. For several series of rectangular polyominoes these numbers are given in Table II. For the sake of the limited space those heptominoes with odd number of vertices are omitted*.

* The numbers of smaller polyominoes have been enumerated and tabulated elsewhere[4,5].

As each of the $r(G, k)$ patterns for a given k has k circles, the total number of circles in the set of $K_G(1)$ king patterns for given graph G is equal to

$$K'_G(1)=\sum_{k=1}^{m} k\cdot r(G, k), \tag{3}$$

where $K'_G(X)$ is the first derivative of $K_G(X)$,

$$K'_G(X)=\sum_{k=1}^{m} k\cdot r(G, k)X^{k-1}. \tag{4}$$

The value $K'_G(1)$ is the sum of the contribution of each cell l with respect to the number c_l of circles assigned in the set of the $K_G(1)$ king patterns,

$$K'_G(1)=\sum_{l}^{\text{cells}} c_l. \tag{5}$$

Below the tables these numbers are given in the cells of the graphs.

It was found that for smaller polyominoes with even number ($N=2m$) of points there is a one-to-one relationship between the king pattern and the maximum matching pattern in which m edges span whole the graph as in IV, which in turn is equivalent to the " domino paving pattern" as in V. Thus in this case we have

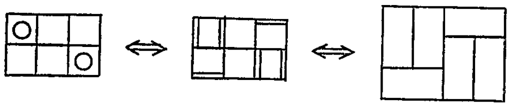
$$K_G(1)=K(G), \tag{6}$$

where $K(G)$ is the number of maximum matching, or in the chemistry terminology, the Kekulé number.

In order to keep the consistency of the one-to-one relationship (6) small correction terms are necessary. The corrected polynomial is called as the domino polynomial $D_G(X)$ which, by definition, always satisfies the relation

$$D_G(1)=K(G). \tag{7}$$

Detailed discussion on this polynomial is given elsewhere[3]. In Tables I and II only those $D_G(X)$'s are given which differ from the corresponding $K_G(X)$.



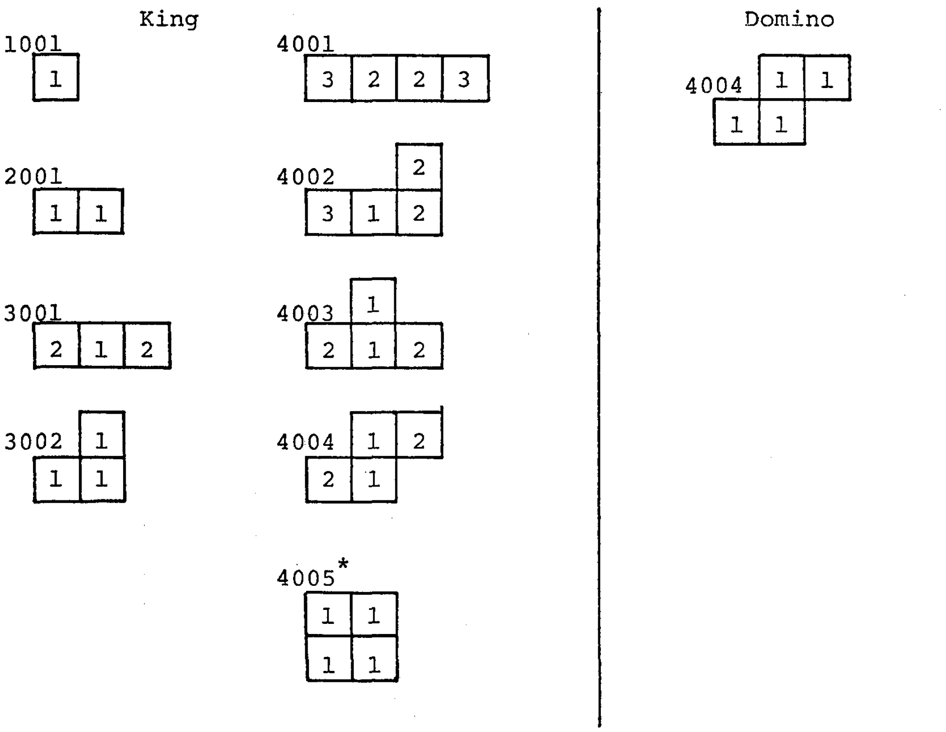
III IV V

References

1) J. Riordan: "An Introduction to Combinatorial Analysis", John Wiley and Sons, New York (1958).
2) C.L. Liu: "Introduction to Combinatorial Mathematics", McGraw-Hill, New York (1968).
3) A. Motoyama and H. Hosoya: Submitted to J. Math. Phys.
4) F. Harary, ed.: "Graph Theory and Theoretical Physics", Academic Press, London (1967), p. 36.
5) S. Hitotsumatsu: Suugaku Seminar (in Japanese), 9(8), 42 (1970).

Table I-1

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)
1001	2	1	1	1		
2001	3	2	1	2		
3001	5	5	1	3	1	
3002	4	3	1	3		
4001	8	10	1	4	3	
4002	7	8	1	4	2	
4003	6	6	1	4	1	
4004	{6	6	1	4	1	
	{5	4	1	4		
4005*	5	4	1	4		



* Since these polyominoes have odd number of vertices, no domino pattern is possible.

Table I-2

No.	K(1) D(1)	K′(1) D′(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)
5001	13	20	1	5	6	1
5002	11	15	1	5	5	
5003	10	13	1	5	4	
5004	{10 {9	13 11	1 1	5 5	4 3	
5005	10	13	1	5	4	
5006	12	18	1	5	5	1
5007	10	13	1	5	4	
5008	{9 {8	11 9	1 1	5 5	3 2	
5009	11	16	1	5	4	1
5010	8	9	1	5	2	
5011	{9 {6	11 5	1 1	5 5	3	
5012*	8	9	1	5	2	

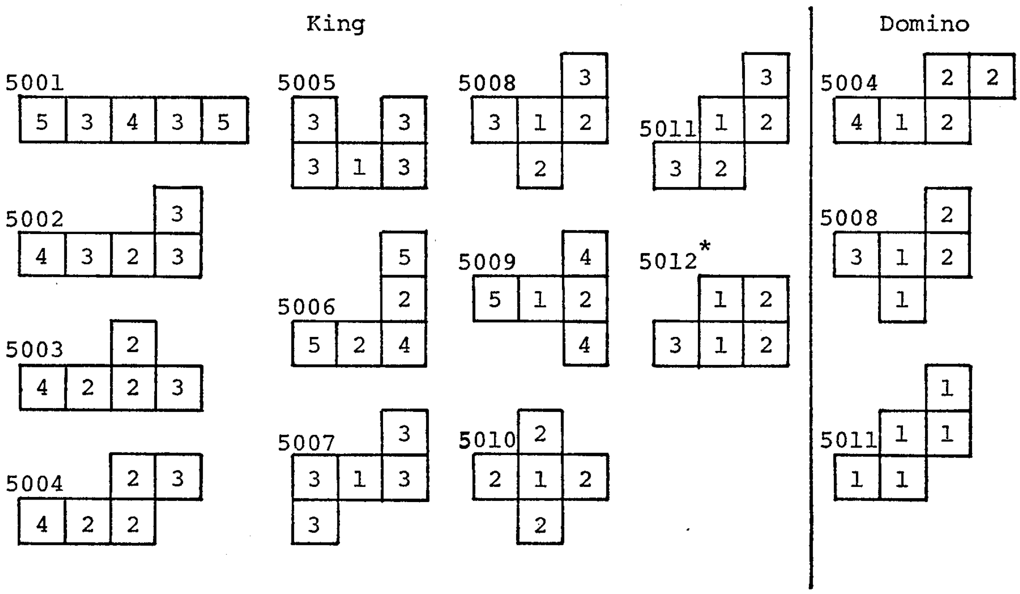
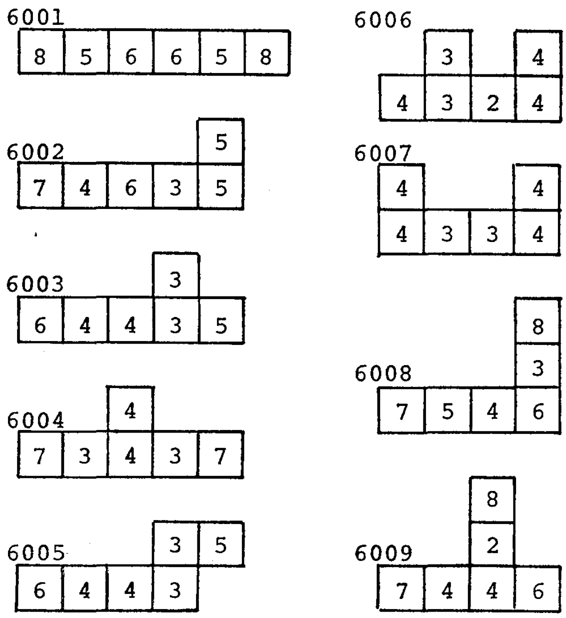


Table I-3

No.	K(1) D(1)	K′(1) D′(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)
6001	21	38	1	6	10	4
6002	18	30	1	6	9	2
6003	16	25	1	6	8	1
6004	17	28	1	6	8	2
6005	{16	25	1	6	8	1
	{14	20	1	6	7	
6006	14	20	1	6	7	
6007	15	22	1	6	8	
6008	19	33	1	6	9	3
6009	18	31	1	6	8	3

King



Domino

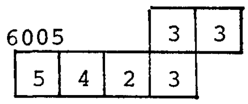


Table I-4

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)
6010	15	22	1	6	8	
6011	14	20	1	6	7	
6012	{15	23	1	6	7	1
	13	18	1	6	6	
6013	17	28	1	6	8	2
6014	{13	18	1	6	6	
	12	16	1	6	5	
6015	14	21	1	6	6	1
6016	11	14	1	6	4	
6017	{17	28	1	6	8	2
	16	26	1	6	7	2
6018	{14	20	1	6	7	
	13	18	1	6	6	

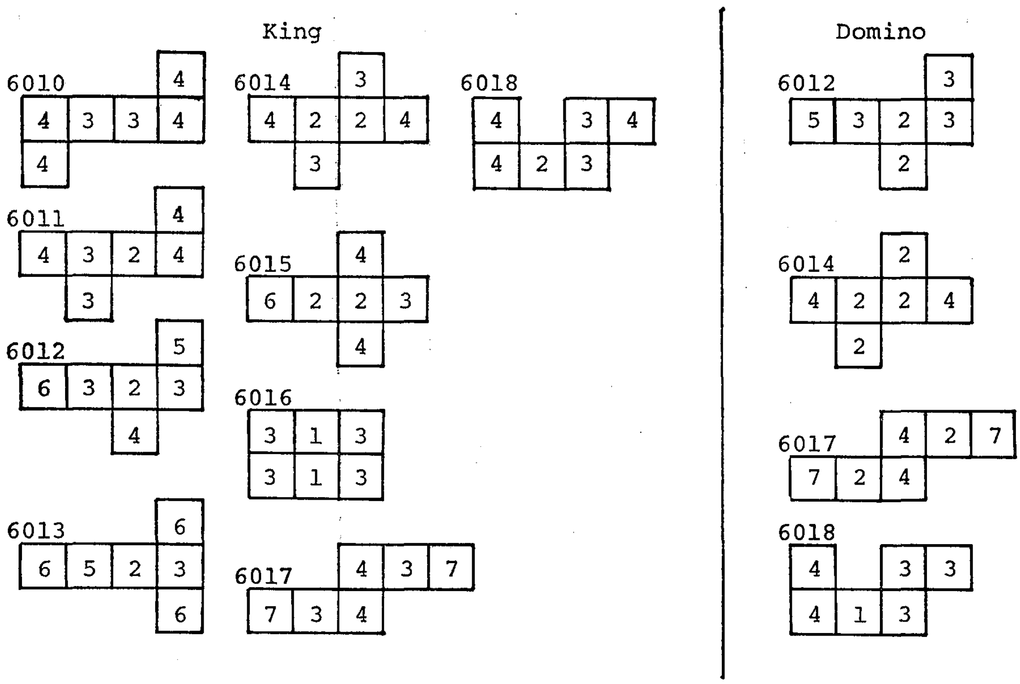


Table I-5

No.	K(1) D(1)	K′(1) D′(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)
6019	{15 11	23 14	1 1	6 6	7 4	1
6020	{15 14	23 21	1 1	6 6	7 6	1 1
6021	17	28	1	6	8	2
6022	17	28	1	6	8	2
6023	{14 13	20 18	1 1	6 6	7 6	
6024	{13 10	18 12	1 1	6 6	6 3	
6025	{16 15	26 24	1 1	6 6	7 6	2 2
6026	16	26	1	6	7	2

King

Domino

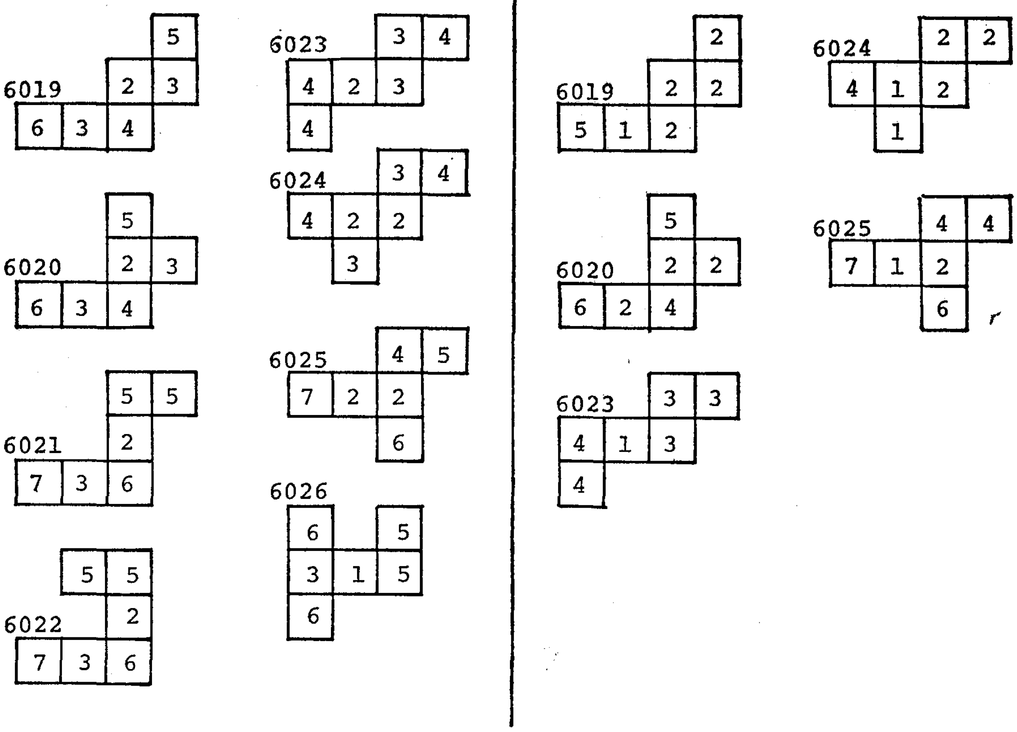


Table I-6

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)
6027	{14 11	21 14	1 1	6 6	6 4	1
6028	{13 7	18 6	1 1	6 6	6	
6029*	13	18	1	6	6	
6030*	12	16	1	6	5	
6031*	12	16	1	6	5	
6032*	12	16	1	6	5	
6033*	11	14	1	6	4	
6034*	13	19	1	6	5	1
6035*	13	19	1	6	5	1

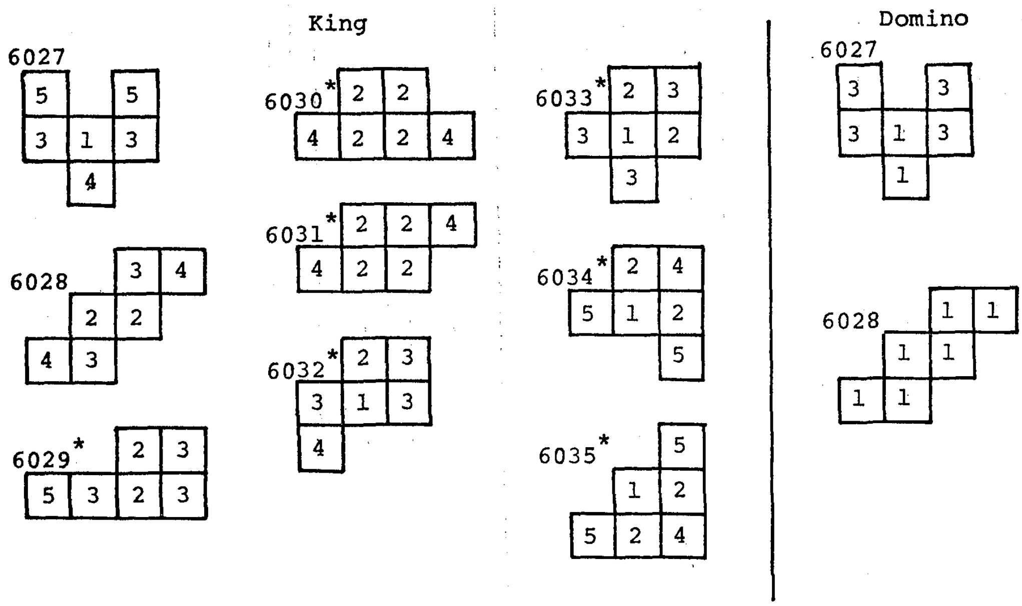


Table I-7

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7001	34	71	1	7	15	10	1
7002	29	56	1	7	14	7	
7003	26	48	1	7	13	5	
7004	27	51	1	7	13	6	
7005	{26 23	48 40	1 1	7 7	13 12	5 3	
7006	24	43	1	7	12	4	
7007	22	37	1	7	12	2	
7008	25	45	1	7	13	4	
7009	20	32	1	7	11	1	
7010	31	63	1	7	14	8	1
7011	29	58	1	7	13	7	1
7012	30	61	1	7	13	8	1
7013	25	45	1	7	13	4	

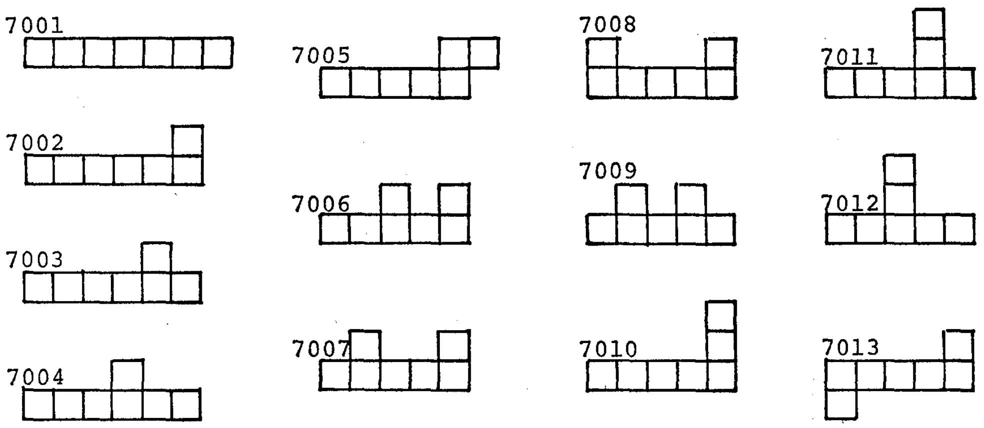


Table I-8

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7014	22	37	1	7	12	2	
7015	24	43	1	7	12	4	
7016	{24	43	1	7	12	4	
	21	35	1	7	11	2	
7017	28	55	1	7	13	6	1
7018	20	32	1	7	11	1	
7019	{22	38	1	7	11	3	
	20	33	1	7	10	2	
7020	22	38	1	7	11	3	
7021	25	48	1	7	11	5	1
7022	{16	23	1	7	8		
	15	21	1	7	7		
7023	{27	51	1	7	13	6	
	25	46	1	7	12	5	
7024	{20	32	1	7	11	1	
	18	27	1	7	10		
7025	{22	37	1	7	12	2	
	19	29	1	7	11		
7026	{24	43	1	7	12	4	
	17	25	1	7	9		

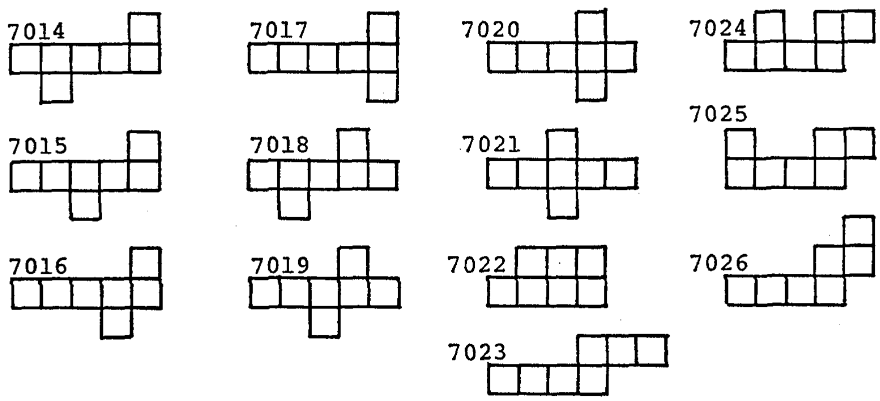


Table I-9

No.	K(1) D(1)	K′(1) D′(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7027	{24	43	1	7	12	4	
	22	38	1	7	11	3	
7028	27	51	1	7	13	6	
7029	26	49	1	7	12	6	
7030	26	49	1	7	12	6	
7031	27	51	1	7	13	6	
7032	24	43	1	7	12	4	
7033	26	48	1	7	13	5	
7034	25	46	1	7	12	5	
7035	{22	37	1	7	12	2	
	19	29	1	7	11		
7036	{20	32	1	7	11	1	
	18	27	1	7	10		
7037	{22	38	1	7	11	3	
	16	23	1	7	8		
7038	{25	46	1	7	12	5	
	23	41	1	7	11	4	

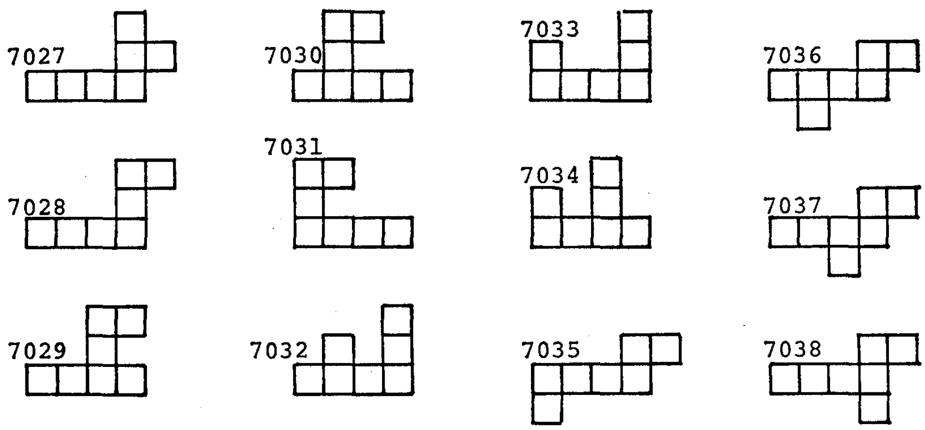


Table I-10

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7039	23	40	1	7	12	3	
7040	{21	35	1	7	11	2	
	{18	27	1	7	10		
7041	{21	35	1	7	11	2	
	{18	27	1	7	10		
7042	20	33	1	7	10	2	
7043	{20	33	1	7	10	2	
	{16	23	1	7	8		
7044	22	38	1	7	11	3	
7045	30	59	1	7	14	8	
7046	28	54	1	7	13	7	
7047	26	48	1	7	13	5	
7048	24	43	1	7	12	4	
7049	{25	46	1	7	12	5	
	{23	41	1	7	11	4	
7050	25	46	1	7	12	5	

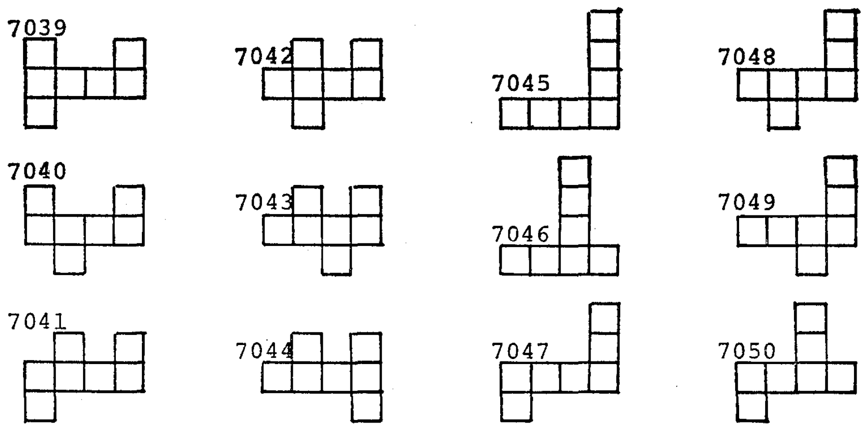


Table I-11

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7051	{23	41	1	7	11	4	
	22	39	1	7	10	4	
7052	24	44	1	7	11	5	
7053	{26	49	1	7	12	6	
	24	44	1	7	11	5	
7054	{24	43	1	7	12	4	
	23	41	1	7	11	4	
7055	{20	32	1	7	11	1	
	17	25	1	7	9		
7056	18	29	1	7	8	2	
7057	{16	24	1	7	7	1	
	12	15	1	7	4		
7058	{24	43	1	7	12	4	
	23	41	1	7	11	4	
7059	{22	38	1	7	11	3	
	18	29	1	7	8	2	
7060	{27	53	1	7	12	6	1
	26	51	1	7	11	6	1
7061	29	58	1	7	13	7	1
7062	27	53	1	7	12	6	1

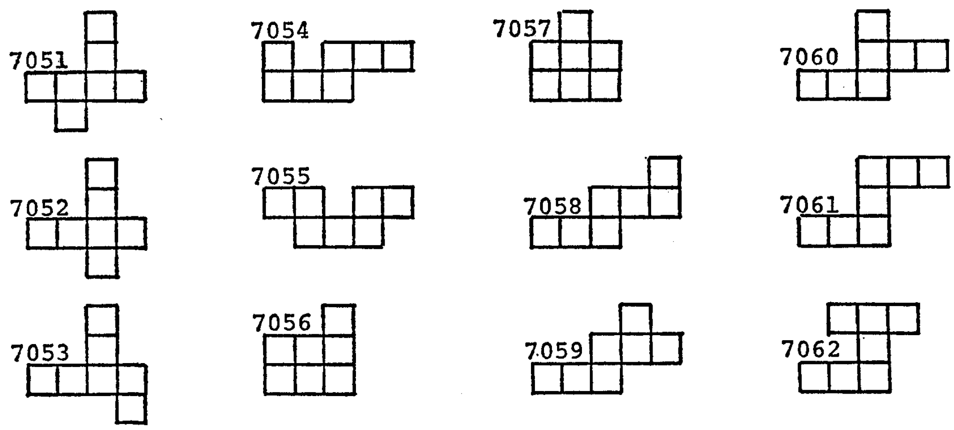


Table I-12

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7063	29	58	1	7	13	7	1
7064	26	51	1	7	11	6	1
7065	{22 {13	38 17	1 1	7 7	11 5	3	
7066	{23 {19	41 31	1 1	7 7	11 9	4 2	
7067	{20 {17	32 25	1 1	7 7	11 9	1	
7068	{24 {21	44 37	1 1	7 7	11 9	5 4	
7069	{24 {22	43 38	1 1	7 7	12 11	4 3	
7070	{21 {16	35 23	1 1	7 7	11 8	2	
7071	{21 {20	35 33	1 1	7 7	11 10	2 2	
7072	24	43	1	7	12	4	
7073	{22 {21	38 36	1 1	7 7	11 10	3 3	
7074	{20 {14	33 19	1 1	7 7	10 6	2	

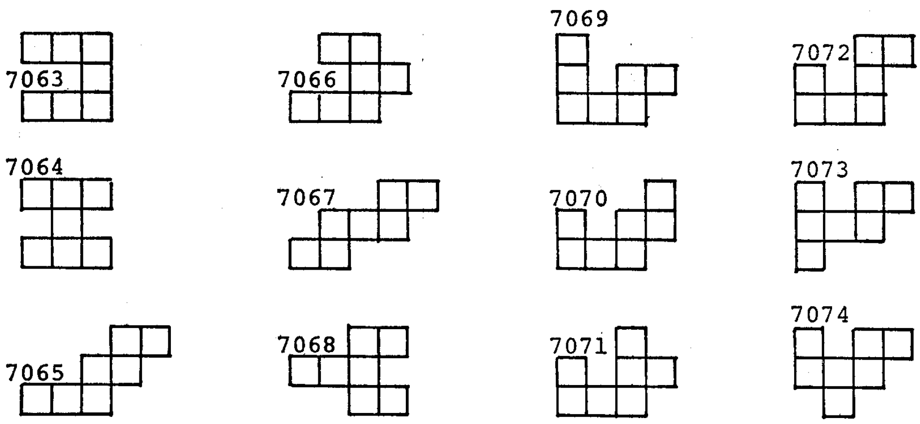
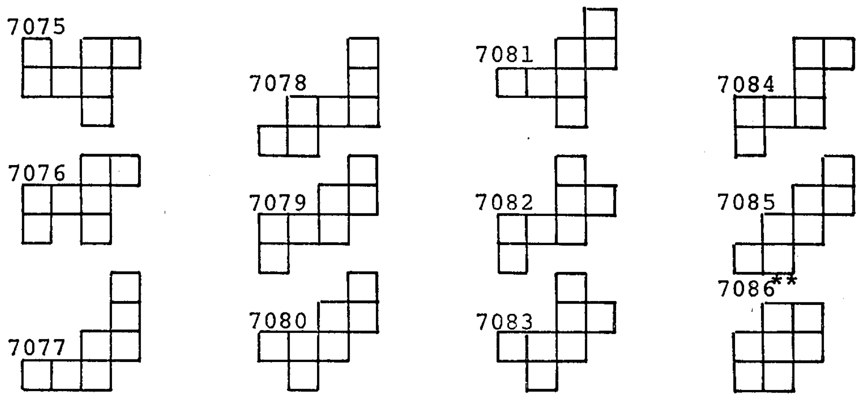


Table I-13

No.	K(1) D(1)	K´(1) D´(1)	r(G,0)	r(G,1)	r(G,2)	r(G,3)	r(G,4)
7075	23	41	1	7	11	4	
	22	39	1	7	10	4	
7076	23	41	1	7	11	4	
	22	39	1	7	10	4	
7077	25	46	1	7	12	5	
	20	34	1	7	9	3	
7078	24	43	1	7	12	4	
	22	38	1	7	11	3	
7079	21	35	1	7	11	2	
	16	23	1	7	8		
7080	19	30	1	7	10	1	
	12	15	1	7	4		
7081	23	41	1	7	11	4	
	19	32	1	7	8	3	
7082	21	35	1	7	11	2	
	20	33	1	7	10	2	
7083	19	30	1	7	10	1	
	16	24	1	7	7	1	
7084	24	43	1	7	12	4	
7085	19	30	1	7	10	1	
	8	7	1	7			
7086**	15	21	1	7	7		
	-	-					



** Although this heptomino has even number of vertices, no domino pattern is possible.

Table II-1

Lattice	K(1) D(1)	K′(1) D′(1)	r(G,k) k=0	1	2	3	4	5	6
2 × 4*	21	32	1	8	12				
2 × 5	43	82	1	10	24	8			
	41	78	1	10	22	8			
2 × 6*	85	188	1	12	40	32			
2 × 7	171	438	1	14	60	80	16		
	153	388	1	14	56	66	16		
3 × 3	35	69	1	9	16	8	1		
	36	70	1	10	16	8	1		
3 × 4	93	224	1	12	37	34	9		
	95	226	1	14	37	34	9		
3 × 5	269	805	1	15	67	105	65	15	1
	281	828	1	18	74	107	65	15	1

King

2 × 4*

5	3		

2 × 5

11	5	9		

2 × 6*

21	11	15			

2 × 7

43	21	33	25			

3 × 3

12	5	
	1	

3 × 4

29	19		
11	5		

3 × 5

88	47	74		
35	11	25		

Domino

2 × 5

11	4	9		

2 × 7

41	15	33	16			

3 × 3

12	5	
	2	

3 × 4

29	19		
11	6		

3 × 5

90	47	74		
36	17	26		

Table II-2

Lattice	K(1) D(1)	K´(1) D´(1)	r(G,k) k=0	1	2	3	4	5	6	7	8	9
3 × 6	747	2610	1	18	106	248	250	108	16			
	781	2684	1	22	126	258	250	108	16			
3 × 7	2115	8545	1	21	154	490	726	522	176	24	1	
	2245	8915	1	26	194	548	750	525	176	24	1	
4 × 4*	314	908	1	16	78	140	79					
4 × 5	1213	4384	1	20	135	382	454	194	27			
	1183	4202	1	26	155	378	410	186	27			
5 × 5	6427	28978	1	25	228	964	1987	1974	978	242	27	1
	6728	29780	1	35	300	1132	2082	1946	962	242	27	1

King

3 × 6

239	141	181			
93	35	55			

3 × 7

684	385	546	445			
269	93	175	121			

5 × 5

2002	1032	1680		
	359	779		
		1442		

4 × 4*

87	55		
	30		

4 × 5

358	177	297		
231	89	185		

Domino

3 × 6

244	144	181			
95	48	61			

3 × 7

711	393	558	445			
281	137	192	157			

5 × 5

2088	1034	1718		
	440	770		
		1444		

4 × 5

351	153	285		
231	80	186		