On Order Completions of Banach Lattices

Chiê Nara

Department of Mathematics, Faculty of Science,
Ochanomizu University

(Received September 10, 1976)

Introduction. In 1937, H. M. Macneille [4] obtained an order completion of a partially ordered space, containing the initial set as an order-dence subset, by generalizing the notion of Dedekind cuts. This result led naturally to the following problem: When a partially ordered space E has not only the order structure but also other structures (for example, group, vector, topological ones, etc.), does there exist an order completion of E with the structures which coincides with the given ones in E? A. H. Clifford [1] and P. Lorenzen [3] gave independently an affirmative answer to the problem in the case of Archimedean abelian groups and H. Nakano [5] and A. I. Yudin [7] did it in the case of Archimedean vector lattices.

The purpose of this paper is to solve the problem in the case of Banach lattices. In Section 2, the author extends the norm in a Banach lattice E to the vector lattice order completion F of E in order to make F a Banach lattice. Moreover, such F is determined uniquely up to Banach lattice isomorphisms which means linear, lattice and topological isomorphisms (Theorem). We call such F a *Banach lattice order completion.* If we strengthen the meaning of Banach lattice isomorphisms to that of isometric ones, then the Banach lattice order completion is not necessarily determined uniquely up to Banach lattice isomorphisms in this strong sense (Example 2). An arbitrary AM-space with the order unit, however, has a Banach lattice order completion determined uniquely up to isometric Banach lattice isomorphisms.

The author wishes to express her hearty thanks to Professor I. Sawashima and Professor F. Niiro for their kind encouragement and valuable advice given her throughout these researches.

§ 1. Preliminaries.

Let E be an ordered ($<$) set. For non-void majorized set A of E, we denote by A^* (resp. A_*) the set of all upper (resp. lower) buonds to A and $A^*{}_*$ the set of all lower bounds to A^*. If $A^*{}_*=A$, A is a *cut* of E. It is easy to see that

$$(A^*{}_*)^*=A^*$$

therefore $A^*{}_*$ is a cut. Especially, if A consists of only one element x, we write simply $x^*{}_*$ instead of $\{x\}^*{}_*$ and if A is a cut of E, A is directed above. A subset A of E is *order-dense* in E, if for $x \in E$ there exist a monotone increasing net $\{x_\alpha\}$ in A and a monotone decreasing net $\{y_\beta\}$ in A such that x is the least upper bound of $\{x_\alpha\}$ and the greatest lower bound of $\{y_\beta\}$. If for a pair of $x, y \in E$, the least upper bound denoted by $x \vee y$ and the greatest lower bound denoted by $x \wedge y$ exist in E, E is a *lattice*. A subset A of E is a *sublattice* if $x \vee y$ and $x \wedge y$ belong to A for each pair of $x, y \in A$. If for a majorized or minorized set A of E, the least upper bound denoted by $\vee A$ or the greatest lower bound denoted by $\wedge A$ exists respectively, E is *order complete*. We denote by $\boldsymbol{R}$ the field of all real numbers and use the symbols $\geqq$, sup and inf only for the usual order, the least upper bound and the greatest lower bound in $\boldsymbol{R}$, respectively. An ordered space E is an ordered vector space, if E is a vector space over $\boldsymbol{R}$ and if the following axiom is satisfied:

$$x \succ y \text{ implies } x+z \succ y+z \quad \text{and} \quad \lambda x \succ \lambda y$$

for $x, y, z \in E$ and $\lambda \geqq 0$. Moreover, if E is a lattice, it is a *vector lattice*. For an element x of E, $|x|$ means $x \vee 0 + (-x) \vee 0$. We understand by a *vector lattice isomorphism* of a vector lattice E_1 to a vector lattice E_2, an order preserving linear bijection. A vector lattice E is said to be *Archimedean* if E satisfies the following condition: $x, y \in E$ and $nx \prec y$ for all natural number n implies $x \prec 0$.

It is well known that for an Archimedean vector lattice E, there exist an order complete vector lattice F and a vector lattice isomorphism ρ of E to an order-dense vector sublattice of F. Moreover, such F is determined uniquely up to vector lattice isomorphisms ([5], [6]). We call such F and ρ a *vector lattice order completion* of E and the *embedding map* of E to F, respectively. We sketch here the way of an order completion by cuts which we will use in § 2.

Let $\widetilde{E}$ be the set of all cuts of E. Define an order $\succ$, an addition $(+)$, a scalar multiplication $(\cdot)$ in $\widetilde{E}$ and the mapping ρ of E into $\widetilde{E}$ as followos: For $\tilde{x}, \tilde{y} \in \widetilde{E}$ and $\lambda > 0$

$$\tilde{x} \succ \tilde{y} \quad \text{if} \quad \tilde{x} \supset \tilde{y}, \tag{1.1}$$

$$\tilde{x}(+)\tilde{y} = (\tilde{x}+\tilde{y})^*{}_* \quad \text{where} \quad \tilde{x}+\tilde{y} = \{x+y : x \in \tilde{x},\ y \in \tilde{y}\}, \tag{1.2}$$

$$\lambda(\cdot)\tilde{x} = \lambda\tilde{x} \quad \text{where} \quad \lambda\tilde{x} = \{\lambda x : x \in \tilde{x}\} \tag{1.3}$$

and for $x \in E$

$$\rho(x) = x^*{}_* . \tag{1.4}$$

Then we can prove that $\widetilde{E}$ and ρ are desired ones. If F is another vector lattice order completion, a vector lattice isomorphism σ of $\widetilde{E}$ to F as follows:

$$\sigma(\tilde{x}) = \wedge\{\rho'(a) : a \in E,\ \tilde{x} \prec \rho(a)\} \tag{1.5}$$

where ρ' is the embedding map of E to F.

Hereafter we use the symbols $\tilde{E}$ and $\tilde{0}$ always as the *vector lattice order completion by cuts* of E and the *zero element* in $\tilde{E}$, that is,

$$\tilde{0}=\rho(0)=0^*{}_*=\{x\in E:\ x\prec 0\}\,.$$

A vector lattice E is a Banach lattice if E is a Banach space whose norm satisfies the following condition:

$$|x|\succ|y| \text{ implies } \|x\|\geqq\|y\| \qquad \text{for} \quad x, y\in E\,.$$

It is clear that a Banach lattice is an Archimedean vector lattice. An AM-*space* is a Banach lattice whose norm satisfies the condition $\|x\vee y\|=\max\{\|x\|, \|y\|\}$. An element e of an AM-space is an order unit if and only if

$$\{x:\ |x|\prec e\}=\{x:\ \|x\|\leqq 1\}\,.$$

It is well known that an AM-space with the order unit is respresented by the Banach lattice $C(\Omega)$ of all continuous real-valued functions on a compact Hausdorff space Ω with the usual norm and order [2].

§ 2. Banach lattice order completions.

THEOREM. *Let E be a Banach lattice. Then there exists a Banach lattice F satisfying the following conditions:*

(1) *F is the vector lattice order completion of E.*

(2) *The embedding map of E to F is isometric.*

Moreover, such F is determined uniquely up to Banach lattice isomorphisms.

PROOF. Let $\tilde{E}$ be the vector lattice order completion by cuts and ρ the embedding map of E to $\tilde{E}$ defined by (1.4). Then we can define a norm in $\tilde{E}$ as follows:

$$\|\tilde{x}\|=\inf\{\|a\|:\ a\in\tilde{x}^*\} \qquad \text{for} \quad \tilde{x}\succ\tilde{0} \tag{2.1}$$

and

$$\|\tilde{x}\|=\|\,|\tilde{x}|\,\| \qquad\qquad \text{for} \quad \tilde{x}\in\tilde{E}.$$

For, if $\|\tilde{x}\|=0$, then for every $\varepsilon>0$ there exists $a\in|\tilde{x}|^*$ such that $\|a\|<\varepsilon$, which implies by the monotonity of the norm in E that $y=0$ for $0\prec y\in|\tilde{x}|$. Since $|\tilde{x}|\supset 0^*{}_*$ is clear, it follows $|\tilde{x}|=\tilde{0}$ from that every cut is directed above. Therefore $\tilde{x}=\tilde{0}$. For a pair of positive elements $\tilde{x}, \tilde{y}\in\tilde{E}$ and $\varepsilon>0$, there exist $a\in\tilde{x}$ and $b\in\tilde{y}$ such that $\|a\|<\|\tilde{x}\|+\varepsilon$ and $\|b\|<\|\tilde{y}\|+\varepsilon$. Hence we obtain

$$\|\tilde{x}(+)\tilde{y}\|\leqq\|\tilde{x}\|+\|\tilde{y}\|$$

since $a+b\in(\tilde{x}+\tilde{y})^*=(\tilde{x}(+)\tilde{y})^*$ and

$$\|\tilde{x}(+)\tilde{y}\|\leqq\|a+b\|<\|\tilde{x}\|+\|\tilde{y}\|+2\varepsilon\,.$$

It is clear that the norm satisfies the conditions of the monotonity and $\|\tilde{x}\|=\||\tilde{x}|\|$ for $\tilde{x}\in\tilde{E}$ and ρ is isometric.

Next we shall prove that $\tilde{E}$ is norm complete under the norm defined by (2.1). Let $\{\tilde{x}_n\}$ be a Cauchy sequence in $\tilde{E}$. We can assume without loss of generality that $\{\tilde{x}_n\}$ satisfies

$$\|\tilde{x}_n(-)\tilde{x}_{n+1}\|<\frac{1}{2^{n+1}}. \tag{2.2}$$

By (2.1) and (2.2), for each number n there exists $a_n\in E$ such that $a_n\in|\tilde{x}_n(-)\tilde{x}_{n+1}|^*$ and that $\|a_n\|<\frac{1}{2^n}$. Put $\tilde{y}_n=\sum_{j=1}^{n}|\tilde{x}_j(-)\tilde{x}_{j+1}|$. Then $\tilde{y}_n\prec\sum_{j=1}^{n}\rho(a_j)$. Since the sequence $\{\sum_{j=1}^{n}a_j\}$ is a Cauchy sequence in E, it converges to an element $a=\sum_{j=1}^{\infty}a_j$ in E. Therefore, we obtain $|\tilde{x}_n|(-)|\tilde{x}_1|\prec\tilde{y}_{n-1}\prec\rho(a)$, so that $\{\tilde{x}_n\}$ is majorized. Since $\tilde{E}$ is order complete, there exists an element $\tilde{x}_0$ of $\tilde{E}$ such that

$$\tilde{x}_0=\bigwedge_{n=1}^{\infty}\bigvee_{j=n}^{\infty}\tilde{x} \qquad \text{in } \tilde{E}.$$

Moreover, $\{\tilde{x}_n\}$ converges to $\tilde{x}_0$, because

$$|\tilde{x}_0(-)\tilde{x}_k|\prec\bigwedge_{n=1}^{\infty}\bigvee_{j=n}^{\infty}|\tilde{x}_j(-)\tilde{x}_k|\prec\sum_{j=k}^{\infty}|\tilde{x}_j(-)\tilde{x}_{j+1}|\prec\sum_{j=k}^{\infty}\rho(a_j)$$

implies $\|\tilde{x}_0(-)\tilde{x}_k\|<\frac{1}{2^{k-1}}$ by the monotonity of the norm in $\tilde{E}$. We conclude that $\tilde{E}$ is norm complete.

Let F be another Banach lattice satisfying the conditions (1), (2), and ρ' the embedding map of E to F. Define a vector lattice isomorphism σ of $\tilde{E}$ to F by (1.5). Then it is clear that

$$\|\sigma(\tilde{x})\|=\||\sigma(\tilde{x})|\|=\|\sigma(|\tilde{x}|)\|.$$

If $|\tilde{x}|\prec\rho(a)$ for $a\in E$, then $\sigma(|\tilde{x}|)\prec\rho'(a)$ and hence $\|\sigma(|\tilde{x}|)\|\leqq\|\rho'(a)\|=\|a\|$, which implies

$$\|\sigma(\tilde{x})\|\leqq\inf\{\|a\|:\ |\tilde{x}|\prec\rho(a),\ a\in E\}=\|\tilde{x}\|.$$

Thus σ is continuous and the inverse mapping σ^{-1} is also continuous by the open mapping theorem, which completes the proof.

DEFINITION. Let E be a Banach lattice. Then a Banach lattice F satisfying the conditions (1), (2) in Theorem is called a *Banach lattice order completion* of E.

We immediately obtain the following corollary.

COROLLARY. *The norm in $\tilde{E}$ defined by* (2.1) *is the strongest one among all norms which make $\tilde{E}$ a Banach lattice order completion of E.*

In this section, when we use the symbol $\|\cdot\|$ in $\tilde{E}$, it means the norm defined by (2.1).

Let E be an AM-space. We shall consider a functional $|||\cdot|||$ on E defined by

$$(2.3) \qquad |||\tilde{x}|||=\sup\{\|a\|: a\in\tilde{x},\ a>0\} \qquad \text{for} \quad 0<\tilde{x}\in\tilde{E}$$

and

$$|||\tilde{x}|||=|||\,|\tilde{x}|\,||| \qquad \text{for} \quad \tilde{x}\in\tilde{E}.$$

LEMMA 1. *Suppose E be an* AM-*space with the order unit. Then $|||\cdot|||$ is a norm in $\tilde{E}$ and $(\tilde{E}, |||\cdot|||)$ is a Banach lattice order completion of E.*

PROOF. Let e be an order unit of E and $\tilde{x}\in E$. Then $|||\tilde{x}|||\cdot e$ always belong to $\tilde{x}^*$, which implies $|||\tilde{x}|||=\|\tilde{x}\|$. By Theorem we have the conclusion.

Here we give an example of an AM-space without the order unit such that $|||\cdot|||$ defines a norm but $(\tilde{E}, |||\cdot|||)$ is not norm complete.

EXAMPLE 1. Suppose E be the closed subspace

$$\left\{f\in C[0,1]: f\left(\frac{1}{2^n}\right)=\frac{1}{2^n}f(1) \text{ for every natural number } n\right\}$$

of the AM-space $C[0,1]$. Then $|||\cdot|||$ is a norm in $\tilde{E}$. In order to prove this, it is sufficient to show that $|||\tilde{x}(+)\tilde{y}|||\leqq|||\tilde{x}|||+|||\tilde{y}|||$ for positive elements $\tilde{x}, \tilde{y}$ in $\tilde{E}$. We denote by B the set

$$\left\{\omega\in(0,1): \omega\neq\frac{1}{2^n} \text{ for every natural number } n\right\}$$

of $[0,1]$. Let $\tilde{x}, \tilde{y}\in\tilde{E}$ and $a\in\tilde{x}^*$. Then for $\omega_0\in B$ and a neighbourhood $U\subset B$ of ω_0, there exists an element $h\in C[0,1]$ such that $h(\omega_0)=|||\tilde{x}|||$, $h(\omega)=\|a\|$ for $\omega\in U$ and $|||\tilde{x}|||\leqq h(\omega)\leqq\|a\|$ otherwise by Urysohn's lemma. Hence $h\wedge a\in E$, therefore, $h\wedge a\in\tilde{x}^*$ and $h\wedge a(\omega_0)=|||\tilde{x}|||$. Put $f=h\wedge a$. Then $f\in\tilde{x}^*$ and $f(\omega_0)=|||\tilde{x}|||$. Since, similarly, there exists an element $g\in\tilde{y}^*$ such that $g(\omega_0)=|||\tilde{y}|||$, it follows from $f+g\in(\tilde{x}+\tilde{y})^*=(\tilde{x}(+)\tilde{y})^*$ that for $z\in\tilde{x}(+)\tilde{y}$,

$$z(\omega_0)\leqq(f+g)(\omega_0)=|||\tilde{x}|||+|||\tilde{y}||| \qquad \text{for} \quad \omega_0\in B,$$

and moreover by the density of B in $[0,1]$ and the continuity of z,

$$z(\omega)\leqq|||\tilde{x}|||+|||\tilde{y}||| \qquad \text{for} \quad \omega\in[0,1],$$

which implies $|||\tilde{x}(+)\tilde{y}|||\leqq|||\tilde{x}|||+|||\tilde{y}|||$.

Let χ_n be the characteristic functions of $\left[\frac{1}{2^n}, 1\right]$ and A_n the set $\{f\in E: f<\chi_n\}$ of E. Then, A_n is a cut of E, that is, $A_n\in\tilde{E}$, and satisfies $|||A_n|||=1$ and $\|A_n\|=2^n$ for every n, which contradicts the open mapping theorem if $(\tilde{E}, |||\cdot|||)$ is norm complete. Therefore $(\tilde{E}, |||\cdot|||)$ is not norm complete.

LEMMA 2. *If* $|||\cdot|||$ *is a norm and moreover makes* $\tilde{E}$ *a Banach lattice, the norm* $|||\cdot|||$ *is the weakest one among all norms in* $\tilde{E}$ *which make* $\tilde{E}$ *a Banach lattice.*

The proof is clear, so we omit it.

Now we get immediately the following proposition by Corollary and Lemma 2.

PROPOSITION. *If* E *is the Banach lattice* $C(\Omega)$, *the Banach lattice order completion of* E *can be determined uniquely up to isometric Banach lattice isomorphisms.*

Finally we show an example of an AM-space without the order unit such that $|||\cdot|||$ makes $\tilde{E}$ a Banach lattice but two Banach lattice order completions $(\tilde{E}, ||\cdot||)$ and $(\tilde{E}, |||\cdot|||)$ are not isometric each other.

EXAMPLE 2. Suppose E be the closed subspace

$$\left\{f \in C[0, 1] : f\left(\frac{1}{2}\right) = \frac{1}{2} f(1)\right\}$$

of the AM-space $C[0, 1]$. We can prove by the similar way to Example 1 that $|||\cdot|||$ is a norm in $\tilde{E}$ and $|||\tilde{x}||| \leqq ||\tilde{x}|| \leqq 2|||\tilde{x}|||$. Then by Theorem and Lemma 2 $(\tilde{E}, |||\cdot|||)$ is a Banach lattice. Let χ be the characteristic function of $\left[\frac{1}{2}, 1\right]$ and $A = \{f \in E : f \prec \chi\}$. Then A is a cut of E, that is, $A \in \tilde{E}$, and moreover $|||A||| = 1$ and $||A|| = 2$ hold, which shows that two Banach lattice order completions $(\tilde{E}, ||\cdot||)$ and $(\tilde{E}, |||\cdot|||)$ are not isometric each other.

References

[1] A.H. Clifford: Partially ordered Abelian groups, Ann. of Math., **25** (1940), 465-473.

[2] S. Kakutani: Concrete representation of abstract (M)-spaces, Ann. of Math., **42** (1941), 994-1024.

[3] P. Lorenzen: Abstrakte Begründung der multiplikativen Idealtheorie, Math. Zeit., **45** (1939), 533-553.

[4] H.M. Macneille: Partially ordered sets, Trans. A.M.S., **42** (1937), 416-460.

[5] H. Nakano: Eine spectral theorie, Proc. Phy.-Math. Soc. Japan, **23** (1941), 485-511.

[6] H.H. Schaefer: Banach lattices and positive operators, Springer-Verlag Berlin Heidelberg, (1974), 56.

[7] A.I. Yudin: On the extention of partially ordered linear spaces, Uch. zap. LGU, **12** (1941), 57-61.

Added in Proof: Professor T. Ando kindly gave me an information that the similar problem had been investigated in Russian papers by G. Ja Lozanovskii, V.A. Solov'ev and B.Z. Vulikh.