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On Type 1 Semi-algebras of Continuous Functions

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§ 1. Introduction

We denote by $C(E)$ the Banach algebra of all real valued continuous functions on a compact Hausdorff space E normed by the uniform norm, and by $C^+(E)$ the class of all $f \in C(E)$ with $f(x) \geqq 0$ $(x \in E)$. A non-empty subset F of $C(E)$ is called a cone if $f+g$ and $\alpha f\,(\alpha \geqq 0)$ belong to F whenever f and g belong to F. A non-empty subset F of $C(E)$ is called a semi-algebra if F is a cone and fg belongs to F whenever f and g belong to F, and is said to be closed if it is uniformly closed. Given a non-negative integer N, a semi-algebra F is said to be of type N if $f^N/(1+f)$ belongs to F whenever f belongs to F. Given a subset F of $C(E)$, let us define by F a partial order $\leqq_F$ in E by taking $x \leqq_F y$ if and only if $f(x) \leqq f(y)$ for every $f \in F$. Given a partial order $\leqq$ in E, a real valued function defined on E is said to be monotonically increasing if $f(x) \leqq f(y)$ whenever $x \leqq y$. Let us denote by 1 the unit function: $1(x)=1$.

It has been shown by F. F. Bonsall [1, p. 134] that a closed semi-algebra F with the unit is the class of all $f \in C^+(E)$ monotonically increasing with respect to $\leqq_F$.

A closed semi-algebra of type N generated by $u_1, u_2, \cdots, u_n \in C^+(E)$ is the smallest closed semi-algebra of type N containing $u_1, u_2, \cdots, u_n$.

It has been shown also by F. F. Bonsall [2, p. 138] that the closed semi-algebra of type 2 in $C([0, 1])$ generated by $\chi(x)=x$ is the class of all $f \in C^+([0, 1])$ non-decreasing, convex, and satisfying $f(0)=0$. We may define a partial order in R as follows. For each two pairs (x, y), $(x', y') \in R \times R$, let us put $(x, y) \leqq (x', y') \Leftrightarrow x \leqq x',\ y \leqq y'$. In this note we shall prove that a closed semi-algebra of type 1 in $C([0, 1] \times [0, 1])$ generated by $x(x, y)=x$ and $y(x, y)=y$ is the class of all $f \in C^+([0, 1] \times [0, 1])$ monotonically increasing and also satisfying $f(0, 0)=0$. By using this fact we shall be able to give a generalization of F. F. Bonsall's theorem.

§ 2. A type 1 semi-algebra in $C([0, 1])$

We denote by χ the identity mapping: $\chi(x)=x$. Let f be the real valued continuous function defined on a subset of R which contains the range of g where $g\in C(E)$. Then their composition $f\circ g\in C(E)$ may be defined. Let us denote by $B([0, \alpha]; \chi)$ the smallest semi-algebra of type 1 containing χ and contained in $C^+([0, \alpha])$ for $\alpha>0$. It is the class of all $f\in C^+([0, \alpha])$ that can be built up from the function χ in a finite number of steps by addition, multiplication, multiplication by a non-negative number, and the operation $r\rightarrow r/(1+r)$. In particular for $\alpha=1$, we denote the class by $B(\chi)$. We denote by $\overline{B([0, \alpha]; \chi)}$ the closure, by the uniform norm, of $B([0, \alpha]; \chi)$ in $C([0, \alpha])$, and we call this the closed semi-algebra of type 1 generated by χ. In particular for $\alpha=1$, we denote it by $B(\chi)$. Let $f, g\in C^+(E)$. Let us write $f\leqq g$ if and only if $f(t)\leqq g(t)$ for every $t\in E$. Then $\leqq$ defines a partial order in $C^+(E)$. Let B be the class of all $f\in C^+([0, 1])$ vanishing at 0 and non-decreasing.

Our present purpose is to prove $B=B(\chi)$. The function P, defined for $x\geqq 0$ by $P(x)=x/(1+x)$, is a non-negative, non-decreasing, continuous function on $[0, \infty)$ vanishing at 0. Let $f, g\in B$ and $\alpha\geqq 0$, then $f+g$, fg, αf, and $P\circ f$ also belong to B. Thus B is a semi-algebra of type 1. Since $\chi\in B$ and B is closed, we have $\overline{B(\chi)}\subset B$. Thus our problem is to show $B\subset\overline{B(\chi)}$, for which we need the following lemmas and theorem by F. F. Bonsall [2, p. 138]:

LEMMA 1. $A(\chi)\subset B(\chi)$.

PROOF. If $f\in B(\chi)$, then $f^2/(1+f)=f/(1+f)f\in B(\chi)$ since $B(\chi)$ is a semi-algebra of type 1. Therefore $B(\chi)$ is a semi-algebra of type 2 containing χ. Hence $A(\chi)\subset B(\chi)$ by the definition of $A(\chi)$.

LEMMA 2. *For* $a\in[0, 1]$ *the function* χ_a *defined on* $[0, 1]$ *by* $\chi_a(x)=\max\{0, x-a\}$ *belongs to* $\overline{B(\chi)}$.

PROOF. By Bonsall [2, p. 138], χ_a belongs to $\overline{A(\chi)}$. Therefore χ_a belongs to $\overline{B(\chi)}$ since $\overline{A(\chi)}\subset\overline{B(\chi)}$ by Lemma 1.

Theorem $\overline{A(\chi)}$ is the class of all $f\in C^+([0, 1])$, non-decreasing, convex, and vanishing at 0.

THEOREM 1. $B=\overline{B(\chi)}$.

PROOF. As has been shown, we have only to prove $B\subset\overline{B(\chi)}$. Suppose that $f\in B$ and $f\neq 0$. Let ε be an arbitrary positive number. Then there exists a non-negative integer N such that

$$|f(x)-f(y)|<\varepsilon \quad x, y \in [(k-1)/N, k/N] \quad (k=1, 2, \cdots, N) . \tag{2-1}$$

Put $x_k=k/N$ $(k=0, 1, \cdots, N)$ and

$$\mu_k=f(x_k)-f(x_{k-1}) \qquad (k=1, 2, \cdots, N) . \tag{2-2}$$

Then there exists a positive number λ_k satisfying the inequality $|N\lambda_k\mu_k|\leqq\varepsilon/N$ $(k=1, 2, \cdots, N)$. Let us define three functions from $[0, 1]$ into R as:

$$f_k(x)=\mu_k x/(\lambda_k+x) ,$$
$$g_k(x)=f_k\circ\chi_{x_{k-1}}(x)=\mu_k\chi_{x_{k-1}}/(\lambda_k+\chi_{x_{k-1}}) \qquad (k=1, 2, \cdots, N) ,$$

and

$$g=\sum_{i=1}^{N} g_i .$$

Then follows $g_j(x_k)=0$ $(k=1, 2, \cdots, N)$ $(j=k+1, k+2, \cdots, N)$, and

$$g(x_k)=\sum_{i=1}^{k} g_i(x_k) \qquad (k=1, 2, \cdots, N). \tag{2-3}$$

On the other hand from (2-2) it follows that

$$f(x_k)=\mu_1+\mu_2+ \cdots +\mu_k \qquad (k=1, 2, \cdots, N) . \tag{2-4}$$

Since $|f_k(x)-\mu_k|=|-\lambda_k\mu_k/(\lambda_k+x)|\leqq|N\lambda_k\mu_k|\leqq\varepsilon/N$ for every $x\in[x_i, 1]$, holds

$$|g_k(x)-\mu_k|\leqq\varepsilon/N \quad x\in[x_k, 1] \qquad (k=1, 2, \cdots, N) . \tag{2-5}$$

From (2-3)-(2-5) it follows that

$$|g(x_k)-f(x_k)|=\left|\sum_{i=1}^{k} g_i(x_i)-\sum_{i=1}^{k}\mu_i\right|\leqq\sum_{i=1}^{k}|g_i(x_i)-\mu_i|\leqq\varepsilon \qquad (k=1, 2, \cdots, N) . \tag{2-6}$$

For $t\in[0, 1]$ there exists a non-negative integer i such that $x_i\leqq t\leqq x_{i+1}$. From (2-1) and (2-6) it follows that

$$|g(x_i)-f(t)|\leqq|g(x_i)-f(x_i)|+|f(x_i)-f(t)|<2\varepsilon , \tag{2-7}$$

and

$$|g(x_{i+1})-f(t)|\leqq|g(x_{i+1})-f(x_{i+1})|+|f(x_{i+1})-f(t)|<2\varepsilon . \tag{2-8}$$

From (2-7), (2-8), and the non-decreasingness of g, it follows that

$$|g(t)-f(t)|<2\varepsilon .$$

Since g_k belongs to $\overline{B(\chi)}$ as $\chi_{x_{k-1}}$ belongs to $\overline{B(\chi)}$ by Lemma 2 ($k=1, 2, \cdots, N$), g also belongs to $\overline{B(\chi)}$. Hence f belongs to $\overline{B(\chi)}$, and our theorem is proved.

§ 3. A closed semi-algebra of type 1 in $C([0, 1]\times[0, 1])$

For each two pairs (x, y), $(x', y')\in R$, we have defined in § 1 the order $(x, y)\leqq(x', y')$ as $x\leqq x'$ and $y\leqq y'$. Let us put $I=\{(x, y): 0\leqq x\leqq 1, 0\leqq y\leqq 1\}$. Let us denote by $B(x, y)$ the smallest semi-algebra of type 1 in $C(I)$ that contains the two functions $x(x, y)=x$ and $y(x, y)=y$. It is the class of all $f\in C^+(E)$ that can be built up from the variables x and y in a finite number of steps by the operations of addition, multiplication, multiplication by a non-negative number, and $r\rightarrow r/(1+r)$. We call $\overline{B(x, y)}$, the closure of $B(x, y)$, the closed semi-algebra of type 1 generated by x and y. We also denote by $B(x)$ (*resp.* $B(y)$) the smallest semi-algebra of type 1 in $C(I)$ that contains x (*resp.* y). Let C be the class of all $f\in C^+(I)$ vanishing at (0, 0), and monotonically increasing in the sense that $f(x, y)\leqq f(x', y')$ whenever $(x, y)\leqq(x', y')$.

Our present purpose is to prove that $\overline{B(x, y)}=C$. We begin by proving the following lemma.

LEMMA 3. *Let F be a closed semi-algebra of type* 1 *in $C(E)$. If $g\in B([0, \alpha]; \chi)$, $f\in F$, and the range of f is contained in* $[0, \alpha]$, *then $g\circ f\in F$.*

PROOF. Given an arbitrary positive number ε, there exists $g'\in B([0, \alpha]; \chi)$ such that $|g(t)-g'(t)|<\varepsilon$ $(0\leqq t\leqq\alpha)$. Consequently, $|g\circ f(x)-g'\circ f(x)|<\varepsilon$ for every $x\in E$, and $g\circ f\in F$ since $g'\circ f\in F$.

THEOREM 2. $\overline{B(x, y)}=C$.

PROOF. Since it is obvious that $\overline{B(x, y)}\subset C$ as in the proof of theorem 1, we have only to show $\overline{B(x, y)}\supset C$.

Suppose $f\in C$ and $\sup\{f(x, y); 0\leqq x\leqq 1, 0\leqq y\leqq 1\}=1$. Given an arbitrary positive number ε, there exists a positive integer n such that $1/n<\varepsilon$. Put $z_k=k/n$, $(k=0, 1, \cdots, n)$.

In R^3 we make use of the notations as:

$$K=\{(x, y, z); 0\leqq x\leqq 1, 0\leqq y\leqq 1, 0\leqq z\leqq f(x, y)\}$$
$$K_i=\{(x, y, z_i); (x, y, z_i)\in K\}$$
$$I_i=\{(x, y, z_i); 0\leqq x\leqq 1, 0\leqq y\leqq 1\} \qquad (i=1, 2, \cdots, n).$$

Put $d((0, 0, z_1), K_1)=\delta_1$. Then $\delta_1>0$. Put also $L_i=\{(x, y, z_i);$

$(x, y, z_{i-1}) \in K_{i-1}\}$ for $2 \leqq i \leqq n$. Then $\overline{I_i - L_i} \cap K_i = \varnothing$. Since $d(\overline{I_i - L_i}, K_i) = \delta_i > 0$, there exists a positive integer m_i such that $1/m_i < \delta_i/4$ and $m_i > n$. Let us divide I_i $(1 \leqq i \leqq n)$ into m_i^2 equal intervals in R^2, and define $I^i_{s,t} = \{(x, y, z_i);\ (s-1)/m_i \leqq x \leqq s/m_i,\ (t-1)/m_i \leqq y \leqq t/m_i$ $(s = 1, 2, \cdots, n)$ $(t = 1, 2, \cdots, n)$ (Fig. 1, 2). Let J_i be the union of all the sets $I^i_{s,t}$ such that $K_i \cap I^i_{s,t} = \varnothing$, and let $x^i_p = p/m_i$ $(p = 0, 1, \cdots, m_i)$.

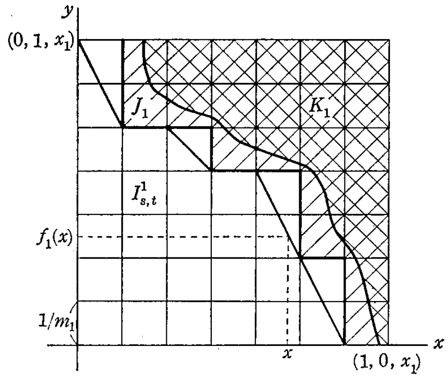

Fig. 1

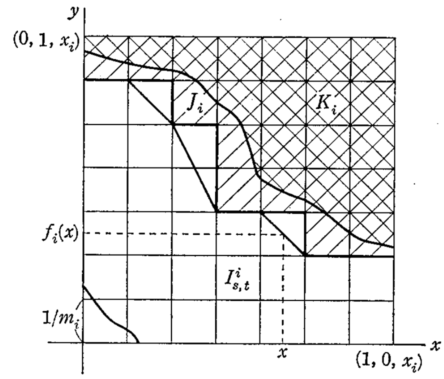

Fig. 2

We define a function f_i from $[0, 1]$ into $[0, 1]$ as follows:

For $x \in [0, 1]$, let j be the integer with $x^i_j \leqq x \leqq x^i_{j+1}$.

If $(x^i_{j+1}, y, z_i) \notin J_i$ for every $y \in [0, 1]$, then the following two cases may be distinguished: (i) $(x^i_l, y, z_i) \notin J_i$ for every $y \in [0, 1]$ $(l = 1, 2, \cdots, j)$. (ii) $(x^i_l, y, z_i) \notin J_i$ for every $y \in [0, 1]$ $(l = j+2, \cdots, n)$. In the first case put $f_i(x) = 1$. In the second case put $f_i(x) = 0$.

If there exists $y' \in [0, 1]$ such that $(x^i_{j+1}, y, z_i) \in J_i$, then denote the minimal value of such y' by y^i_{j+1}. (iii) In addition if there exists $y'' \in [0, 1]$ such that $(x^i_j, y'', z_i) \in J_i$, then denote the minimal value of such y'' by y^i_j, and put

$$f_i(x) = (y^i_j - y^i_{j+1})/(x^i_j - x^i_{j+1})x + (x^i_j y^i_{j+1} - x^i_{j+1} y^i_j)/(x^i_j - x^i_{j+1})$$

where $y^i_j \geqq y^i_{j+1}$. (iv) On the other hand if $(x^i_l, y, z_i) \notin J_i$ for every $y \in [0, 1]$ $(l = 1, 2, \cdots, j)$, then put $f_i(x) = y^i_{j+1}/(x^i_{j+1} - x^i_j)x - x^i_j y^i_{j+1}/(x^i_{j+1} - x^i_j)$.

Thus defined f_i is a non-negative non-increasing continuous function defined on $[0, 1]$ whose range is $[0, f_i(0)]$. Put $f'_i(x) = f_i(0) - f_i(x)$. Then f'_i is a non-negative non-decreasing continuous function defined on $[0, 1]$ and satisfies $f'_i(0) = 0$.

Then define a real valued continuous function g_i on $[-f_i(0), 1]$ as:

$$g_i(t) = 0 \quad (-f_i(0) \leqq t \leqq -1/m_i) \quad (f_i(0) > 1/m_i)\,.$$
$$g_i(t) = (m_i/n)t + 1/n\,, \quad (-1/m_i \leqq t \leqq 0)\,.$$
$$g_i(t) = 1/n\,, \quad (0 \leqq t \leqq 1)\,.$$

Thus the function $t \to g_i(t - f_i(0))$ is non-negative, non-decreasing, continuous on $[0, 1 + f_i(0)]$, and takes 0 at 0, satisfying $g_i(y - f_i(x)) =$

$g_i(y+f_i'(x)-f_i(0))$. By theorem 2, it is obvious that $y \in \overline{B(y)}$ and $f_i'(x) \in \overline{B(x)}$. Since $\overline{B(x)} \subset \overline{B(x, y)}$ and $\overline{B(y)} \subset \overline{B(x, y)}$, holds $y+f_i'(x) \in \overline{B(x, y)}$. From these facts and Lemma 3, it follows that the function $(x, y) \rightarrow g_i(y-f_i(x))$ is an element of $\overline{B(x, y)}$. Let $g(x, y)=\sum_{i=1}^{n} g_i(y-f_i(x))$. Then g is also an element of $\overline{B(x, y)}$.

From (3-1), it follows that $0 \leqq g_i(y-f_i(x)) \leqq 1/n$ for every $(x, y) \in I$. If $(x, y, z_i) \in K_i$, then $g_i(y-f_i(x))=1/n$ since $f_i(x)<y$. If $(x, y, z_i) \in \overline{I_i-L_i}$ for $(i=2, 3, \cdots, n)$, then $g_i(y-f_i(x))=0$ since $y-f_i(x)<-1/m_i$. Since the ratation $(x, y, z_i) \in \overline{I_i-L_i}$ is equivalent to the relation $(x, y, z_{i-1}) \notin K_{i-1}$, $(x, y, z_{i-1}) \notin K_{i-1}$ implies $g_i(y-f_i(x))=0$.

Let $(x, y) \in I$.

If $(x, y, z_i) \notin K_i$ $(i=1, 2, \cdots, n)$, then $0 \leqq f(x, y) \leqq 1/n$, and $0 \leqq g(x, y) \leqq 1/n$ since $g_i(y-f_i(x))=0$ $(i=2, 3, \cdots, n)$. Hence $|f(x, y)-g(x, y)|<\varepsilon$.

If there exists i such that $(x, y, z_j) \in K_j$ $(j=1, 2, \cdots, i)$ and $(x, y, z_j) \notin K_j$ $(j=i+1, i+2, \cdots, n)$, then $i/n \leqq f(x, y) \leqq (i+1)/n$, and $i/n \leqq g(x, y) \leqq (i+1)/n$ since $g_j(y-f_j(x))=1/n$ $(j=1, 2, \cdots, i)$ and $g_j(y-f_j(x))=0$ $(j=i+2, \cdots, n)$. Hence $|f(x, y)-g(x, y)|<\varepsilon$.

If $(x, y, z_i) \in K_i$ $(i=1, 2, \cdots, n)$, then $f(x, y)=1$, and $g(x, y)=1$ since $g_k(x, y)=1/n$ $(k=1, 2, \cdots, n)$. Hence it is obvious that $|f(x, y)-g(x, y)|<\varepsilon$.

From all the considerations made above, it follows that $f \in \overline{B(x, y)}$ since $B \in \overline{B(x, y)}$ and $|f(x, y)-g(x, y)|<\varepsilon$ for every $(x, y) \in I$, and our theorem is proved.

§ 4. Closed semi-algebras of type 1 in $C(E)$

Given f and g belonging to $C(E)$, let $f \vee g$ and $f \wedge g$ denote, as usual, the function defined by $(f \vee g)(t)=\max(f(t), g(t))$, $(f \wedge g)(t)=\min(f(t), g(t))$. A subset F of $C(E)$ is said to be a lattice if $f \vee g$ and $f \wedge g$ belong to F whenever f and g belong to F.

Let F be a closed semi-algebra of type 1 in $C(E)$, and let $N(F)=\{x : f(x)=0 \ (f \in F)\}$.

In the first place we shall prove that F is a lattice, and by using the fact we shall also prove that F is the class of all $f \in C^+(E)$ vanishing on $N(F)$ and monotonically increasing with respect to $\leqq_F$. Further, we shall prove that F has the unit if $N(F)$ is empty. To prove this we need the following notations and also the theorems by F. F. Bonsall [1, p. 128].

Given a cone A which is the subset of $C(E)$, let $\Gamma_0(A)$ denote the set of all quadruples $(s, s'; \alpha, \alpha')$ $(s, s' \in E, \alpha, \alpha' \geqq 0)$ satisfying $\alpha f(s) \leqq \alpha' f(s')$ for every $f \in A$. Also let $K(\Gamma_0(A))$ be the class of all

$f \in C(E)$ satisfying the inequality $\alpha f(s) \leqq \alpha' f(s)$ for every $(s, s'; \alpha, \alpha') \in \Gamma_0(A)$. Then the theorem of F. F. Bonsall is as follows:

Theorem of F. F. Bonsall. A closed convex cone A is a lattice if and only if $A = K(\Gamma_0(A))$.

Now we shall prove the following theorem.

THEOREM 3. *F is a lattice.*

PROOF. Let us remark that the functions α and β, defined on I by $\alpha(x, y) = \max(x, y)$ and $\beta(x, y) = \min(x, y)$, belong to $\overline{B(x, y)}$ by Theorem 3. If ε is an arbitrary positive number, then there exist two functions α' and β' belonging to $B(x, y)$ such that

$$|\alpha(x, y) - \alpha'(x, y)| < \varepsilon$$

and

$$|\beta(x, y) - \beta'(x, y)| < \varepsilon \qquad ((x, y) \in I).$$

Since $\alpha(f, g) = f \vee g$ and $\beta(f, g) = f \wedge g$ for $f, g \in F$ with $f \leqq 1$, $g \leqq 1$, and $\alpha'(f, g)$, $\beta'(f, g) \in F$, we have $f \vee g \in F$ and $f \wedge g \in F$, which completes our proof.

THEOREM 4. *F is the class of all $f \in C^+(E)$ vanishing on $N(F)$ and monotonically increasing with respect to $\leqq_F$.*

PROOF. Remark that $F = K(\Gamma_0(F))$ by [1, p. 128].

Take $(s, s'; \alpha, \alpha') \in \Gamma_0(F)$ with $\alpha > 0$ and $s \notin N(F)$. Then $\alpha \leqq \alpha'$. For evidently holds $\alpha\mu g(s)/(\lambda + g(s)) \leqq \alpha' \mu g(s')/(\lambda + g(s'))$ for arbitrary positive numbers λ and μ since $\mu g/(\lambda + g) \in F$, and consequently $\alpha/\alpha' \leqq \mu g(s')(\lambda + g(s))/\mu g(s)(\lambda + g(s'))$, from which follows $\alpha/\alpha' \leqq 1$ since the left hand side converges to 0 as λ converges to 0. In addition holds $f(s) \leqq f(s')$ for every $f \in F$. Assume that there exists $f \in F$ such that $f(s) > f(s')$. If $f(s') = 0$, then $\alpha = 0$, which contradicts the assuption. If $f(s') > 0$, then for every positive integer m, $\alpha f^m(s) \leqq \alpha f^m(s')$ or equivalently $f^m(s)/f^m(s') \leqq \alpha'/\alpha$. From $f(s)/f(s') > 1$, it follows that $f^m(s)/f^m(s') = (f(s)/f(s'))^m$ diverges to $+\infty$ as $m \to +\infty$, which is a contradiction. Thus holds $f(s) \leqq f(s')$ for every $f \in F$. From the previous consideration, for every $(s, s'; \alpha, \alpha') \in \Gamma_0(F)$ the following three cases may be distinguished: (i) $\alpha = 0$, (ii) $\alpha > 0$, and $s \notin N(F)$, (iii) $s \in N(F)$.

Suppose that $f \in C^+(E)$ vanishes on $N(F)$ and $f(x) \leqq f(y)$ $(x, y \in E, x \leqq_F y)$. Let $(s, s'; \alpha, \alpha') \in \Gamma_0(F)$. (i) If $\alpha = 0$, then $0 = \alpha f(s) \leqq \alpha' f'(s')$. (ii) If $\alpha > 0$, $s \notin N(F)$, then $\alpha \leqq \alpha'$ and $s \leqq_F s'$ from the above fact. Therefore $\alpha f(s) \leqq \alpha f(s') \leqq \alpha' f(s')$. (iii) If $s \in N(F)$, then $0 = \alpha f(s) \leqq \alpha' f(s')$.

Therefore in any case $f \in K(\Gamma_0(F))=F$, which completes our proof.

THEOREM 5. *If $N(F)$ is empty, then F has the unit.*

PROOF. For every $x \in E$ there exists $f_x \in F$ with $f_x(x)>0$, and a neighborhood U_x of x. Since E is compact, there exist a finite number of neighborhoods $U_{x_1}, U_{x_2}, \cdots, U_{x_n}$ such that $E \subset \bigcup_{i=1}^{n} U_{x_i}$. Put $g'=\sum_{i=1}^{n} f_{x_i}$. Then $g' \in F$ and $g'(x)>0$ for every $x \in E$. Further there exists a positive number λ such that $\sup\{\lambda g'(x) ; x \in E\}=1$. Put $\lambda g'=g$, and $\inf\{g(x) ; x \in E\}=a$. Then $0<a \leqq 1$. By Theorem 1 there exists $h \in \overline{B(\chi)}$ such that $h(x)=1$ on $[a, 1]$. Now from Lemma 3 follows $h \circ g \in F$, which proves our theorem since $h \circ g=1$.

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* Added in proof:

The author, to her greatest regect, did not know that Prof. Bonsall had already got Theorems 4 and 5. Though his proofs are different from hers.