Natural Science Report, Ochanomizu University, Vol. 25, No. 1, 1974 7
お茶の水女子大学 自然科学報告 第25巻 第1号

On Kählerian Manifolds of σ-Positive Curvature Operator

Shun-ichi Tachibana

Department of Mathematics, Faculty of Science
Ochanomizu University, Tokyo

(Received March 15, 1974)

Introduction. Let M^n be a Riemannian manifold. If there exists a positive constant k such that

$$(\sharp) \qquad -R_{ijkl}u^{ij}u^{kl} \geqq 2k\, u_{ij}u^{ij}$$

holds good for any 2-form $u=(u_{ij})$ at any point, then M^n is called to be of positive curvature operator. For a space of constant sectional curvature k the equality sign in ($\sharp$) is valid. M. Berger [1] has proved that the second Betti number $b_2(M)$ vanishes for a compact orientable M^n of positive curvature operator. Then D. Meyer [4] has obtained $b_i(M)=0$ for $i=1, \cdots, n-1$. We are interested in its Kählerian analogy. As it is impossible to impose the condition ($\sharp$) on Kählerian manifolds, the condition will be modified as (*) in § 2 below. (*) would be suitable. For, the equality in (*) is valid for a space of constant holomorphic sectional curvature k, and with (*) we can get a satisfactory result about the Betti numbers. In § 3～§ 5 we shall give Kählerian analogies to the results for M^n which have been given in [4], [5], [9] and [10].

§ 1. Preliminaries. Let M^n $(n>1)$ be a Riemannian manifold. Denote by g_{ji}, $R_{hji}{}^{l}$ and $R_{ji}=R_{hji}{}^{h}$ the metric, the curvature and the Ricci tensors respectively. We shall represent tensors by their components with respect to the natural base and the summation convention is assumed, unless otherwise stated. ∇ means the operator of the covariant differential, and $\Delta=d\delta+\delta d$ denotes the Laplacian operator, viz. $\Delta f=-\nabla^r\nabla_r f$ for a 0-form f and

$$(\Delta u)_i=-\nabla^r\nabla_r u_i+R_{ir}u^r, \qquad p=1,$$

$$(\Delta u)_{i_1\cdots i_p}=-\nabla^r\nabla_r u_{i_1\cdots i_p}+\sum_h R_{i_h}{}^{r}u_{i_1\cdots r\cdots i_p}$$
$$+\sum_{h<k} R_{i_h i_k}{}^{rs}u_{i_1\cdots r\cdots s\cdots i_p}, \qquad p\geqq 2,$$

for a p-form $u=(1/p!)u_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$.

If a non-zero p-form u satisfies $\Delta u=\lambda u$ with a constant λ, it is called a proper form of Δ corresponding to the proper value λ. Especially, a harmonic p-form is proper corresponding to $\lambda=0$.

The quadratic forms $F_p(u)$ of u are defined by

$$F_1(u)=R_{ij}u^iu^j,$$

$$F_p(u)=\frac{1}{(p-1)!}\Big(R_{rs}u^{ri_2\cdots i_p}u^s{}_{i_2\cdots i_p}+\frac{p-1}{2}R_{rsjh}u^{rsi_3\cdots i_p}u^{jh}{}_{i_3\cdots i_p}\Big).$$

If we denote

$$\langle u, v\rangle=\frac{1}{p!}u_{i_1\cdots i_p}v^{i_1\cdots i_p}, \qquad |u|^2=\langle u, u\rangle,$$

$$|\nabla u|^2=\frac{1}{p!}\nabla_j u_{i_1\cdots i_p}\nabla^j u^{i_1\cdots i_p},$$

the following Weizenböck formula holds good for any p-form u:

(1.1) $$\tfrac{1}{2}\Delta(|u|^2)=\langle u, \Delta u\rangle-|\nabla u|^2-F_p(u).$$

A Kählerian manifold K^{2m} of real dimension $n=2m$ is a Riemannian manifold admitting a parallel tensor field $F_i{}^h$ such that

$$F_i{}^rF_r{}^j=-\delta_i^j, \qquad F_{ji}=-F_{ij}.$$

In the sequel, we shall consider only Kählerian manifolds K^{2m} and assume that $m>1$.

The fundamental form Ω is the 2-form given by

$$\Omega=\tfrac{1}{2}F_{ji}dx^j\wedge dx^i.$$

The curvature tensor and the Ricci tensor satisfy [12]

$$F_l{}^rR_{rhji}=-F_h{}^rR_{lrji},$$

$$\tfrac{1}{2}F^{rs}R_{rsji}=F^{rs}R_{rjsi}=-S_{ji}=S_{ij},$$

where

$$S_{ji}=F_j{}^tR_{ti}.$$

Let $u=u_idx^i$ be a Killing 1-form, viz. a vector field u^i satisfying

$$\nabla_ju_i+\nabla_iu_j=0.$$

It is well known that such a u satisfies

$$\nabla_h\nabla_ju_i+R_{rhji}u^r=0.$$

A Killing 1-form u will be called *F-special with c*, if it satisfies

$$\nabla_h\nabla_ju_i+\tfrac{1}{4}c(g_{hj}u_i-g_{hi}u_j-F_{hj}\tilde{u}_i+F_{hi}\tilde{u}_j+2F_{ji}\tilde{u}_h)=0$$

where c is a constant and $\tilde{u}_i=F_i{}^ru_r$.

The curvature tensor of a space of constant holomorphic sectional curvature k satisfies

$$R_{rhji}=\tfrac{1}{4}k(g_{hj}g_{ri}-g_{rj}g_{hi}+F_{hj}F_{ri}-F_{rj}F_{hi}-2F_{rh}F_{ji}).$$

Any Killing 1-form in such a space is special with $c=k$.

The holomorphically projective curvature tensor $P_{rhj}{}^i$ of K^{2m} is defined as

$$P_{rhj}{}^i = R_{rhj}{}^i + \frac{1}{2(m+1)}(R_{rj}\delta_h{}^i - R_{hj}\delta_r{}^i + S_{rj}F_h{}^i - S_{hj}F_r{}^i + 2S_{rh}F_j{}^i).$$

A necessary and sufficient condition for $P_{rhj}{}^i=0$ is that K^{2m} is a space of constant holomorphic curvature, [11].

§ 2. **The Betti numbers.** Let us associate to a p-form u $(2m \geqq p \geqq 2)$ a scalar $\sigma_p(u)$ by

$$\sigma_p(u) = \frac{1}{p!}(u_{i_1\cdots i_p}u^{i_1\cdots i_p} + F^{hr}F^{js}u_{hji_3\cdots i_p}u_{rs}{}^{i_3\cdots i_p} + F^{hr}u_{hri_3\cdots i_p}F^{js}u_{js}{}^{i_3\cdots i_p}).$$

The following lemma corresponds to the one of D. Mayer [4] in Riemannian manifolds of positive curvature operator.

LEMMA 2.1. *We assume that there exists a constant $k>0$ such that*

$$(*) \qquad -R_{hjil}u^{hj}u^{il} \geqq \tfrac{1}{2}\, k\, \sigma_2(u)$$

holds good for any 2-form u at any point. Then the following inequalities are valid:

$$(2.1) \qquad F_1(u) \geqq \tfrac{1}{2}(m+1)k|u|^2, \qquad p=1,$$

$$(2.2) \qquad F_p(u) \geqq \frac{(m+1)p}{2}k\Big\{|u|^2 - \frac{p-1}{2(m+1)}\sigma_p(u)\Big\}, \qquad 2m \geqq p \geqq 2.$$

PROOF. For each fixed $i_1, \cdots, i_p$ we define a local 2-form $u^{(i_1\cdots i_p)} = \frac{1}{2}\, u_{jh}{}^{(i_1\cdots i_p)}dx^j \wedge dx^h$ as follows. Let P be any point and take an orthonormal base field around P. With respect to this base we put

$$u_{jh}{}^{(i_1\cdots i_p)} = \sum_r (u_{i_1\cdots i_{r-1}ji_{r+1}\cdots i_p}\delta_{hi_r} - u_{i_1\cdots i_{r-1}hi_{r+1}\cdots i_p}\delta_{ji_r}).$$

A long calculation shows that

$$-\sum_{\substack{i_1,\cdots,i_p \\ h,j,i,l}} R_{hjil}u_{hj}{}^{(i_1\cdots i_p)}u_{il}{}^{(i_1\cdots i_p)} \geqq \tfrac{1}{2}\, k \sum_{i_1,\cdots,i_p} \sigma_2(u^{(i_1\cdots i_p)})$$

is equivalent to (2.1) or (2.2). q. e. d.

It should be noticed that if the space under consideration is of constant holomorphic curvature k, the equality holds in each of (*), (2.1) and (2.2).

LEMMA 2.2. *For any p-form u $(2m \geqq p \geqq 2)$ we have*

$$u_{i_1\cdots i_p}u^{i_1\cdots i_p} \geqq |F^{hr}F^{js}u_{hji_3\cdots i_p}u_{rs}{}^{i_3\cdots i_p}|.$$

PROOF. We take an orthonormal base at a point with respect to which the components of F_{ji} are all zero except

$$F_{\mu\mu^*} = -F_{\mu^*\mu} = 1, \qquad \mu = 1, \cdots, m\,;\ \mu^* = m+\mu.$$

The metric tensor has components δ_{ij} and $u^{i_1\cdots i_p} = u_{i_1\cdots i_p}$ with respect to the base. If we put

$$A=\sum_{\substack{h,j,r,s\\ i_3,\cdots,i_p}} F_{hr}F_{js}u_{hji_3\cdots i_p}u_{rsi_3\cdots i_p}$$

and denote $u_{hj(i)}=u_{hji_3\cdots i_p}$ for simplicity, then it holds that

$$A=2\sum_{\mu,\nu,i}(u_{\mu\nu(i)}u_{\mu^*\nu^*(i)}-u_{\mu\nu^*(i)}u_{\mu^*\nu(i)})=2\vec{\mathbf{a}}\cdot\vec{\mathbf{b}},$$

where

$$\vec{\mathbf{a}}=(u_{\mu\nu(i)},\ u_{\mu\nu^*(i)}),\qquad \vec{\mathbf{b}}=(u_{\mu^*\nu^*(i)},\ -u_{\mu^*\nu(i)}).$$

Consequently we have

$$2|\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}|\leqq 2|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\leqq|\vec{\mathbf{a}}|^2+|\vec{\mathbf{b}}|^2=\sum_{i_1,\cdots,i_p}(u_{i_1\cdots i_p})^2$$

which completes the proof. q. e. d.

Now, consider the operator Λ [6, p. 170], which associate to a p-form u $(2m\geqq p\geqq 2)$ a $(p-2)$-form

$$\Lambda u=\frac{1}{(p-2)!}v_{i_1\cdots i_{p-2}}dx^{i_1}\wedge\cdots\wedge dx^{i_{p-2}}$$

with

$$v_{i_1\cdots i_{p-2}}=\tfrac{1}{2}F^{rs}u_{rsi_1\cdots i_{p-2}}.$$

When $\Lambda u=0$ holds, u is called effective. Any 0-form and 1-form are defined to be effective.

Let u be any effective p-form $(m+1\geqq p\geqq 2)$. Then we have from Lemma 2.2.

$$2|u|^2\geqq|\sigma_p(u)|.$$

Substituting this equation into (2.2), we get

$$F_p(u)\geqq\tfrac{1}{2}p(m+2-p)k|u|^2. \tag{2.3}$$

By this equation and (2.1), $F_p(u)$ is positive definite on the space of effective p-forms $(m+1\geqq p\geqq 1)$. Thus we know that there does not exist an effective harmonic p-form except the zero form. Consequently we have

THEOREM 2.3. *The p-th Betti number of a compact Kählerian manifold K^{2m} satisfying* (*) *is* 0 *or* 1 *according as p is odd or even,* $(2m\geqq p\geqq 0)$.

§ 3. A condition to be a space of constant holomorphic curvature. Consider a compact Riemannian manifold M^n and define a scalar function K by

$$K=R_{rs}R^{rjih}R^s{}_{jih}+\tfrac{1}{2}R^{lmpq}R_{lmjh}R_{pq}{}^{jh}+2R^{ljmh}R_{lpmq}R^p{}_j{}^q{}_h.$$

Then it is well known [2] that the following integral formula holds good:

$$\int_M \{K-|\nabla_r R_{jih}{}^r|^2\}d\sigma = -\tfrac{1}{2}\int_M |\nabla_r R_{jih}{}^k|^2\, d\sigma$$

where $d\sigma$ denotes the canonical measure of M^n and $|A_{jih}|^2 = A_{jih}A^{jih}$.

Hence if the curvature tensor satisfies

$$\nabla_r R_{jih}{}^r = 0, \tag{3.1}$$

then we have

$$\int_M K\, d\sigma = -\tfrac{1}{2}\int_M |\nabla_r R_{jih}{}^k|^2\, d\sigma \leqq 0. \tag{3.2}$$

By virtue of the identity

$$\nabla_r R_{jih}{}^r = \nabla_j R_{ih} - \nabla_i R_{jh} \tag{3.3}$$

the condition (3.1) is satisfied when M^n or K^{2m} has one of the following properties:

(i) the Ricci tensor is proportional to the metric tensor,
(ii) the Ricci tensor is parallel,
(iii) the conformal curvature tensor vanishes identically, and the scalar curvature is constant,
(iii)′ the Bochner curvature tensor of K^{2m} vanishes identically, and the scalar curvature is constant, [8].

In a recent paper [10], the present author got the following

THEOREM. *If a compact Riemannian manifold M^n $(n>2)$ of positive curvature operator satisfies* (3.1), *then M^n is a space of constant curvature.*

The purpose of this section is to get a similar theorem for Kählerian manifolds satisfying (*).

Let P be any point of a Kählerian manifold K^{2m} and consider all the quantities to be written with respect to an orthonormal base field around P. For each fixed k, j, i, h we define a local field of skew symmetric tensor $u_{lm}{}^{(kjih)}$ by

$$u_{lm}{}^{(kjih)} = R_{ljih}\delta_{mk} + R_{klih}\delta_{mj} + R_{kjlh}\delta_{mi} + R_{kjil}\delta_{mh} - R_{mjih}\delta_{lk} - R_{kmih}\delta_{lj} - R_{kjmh}\delta_{li} - R_{kjim}\delta_{lh}.$$

Then it follows after long calculations that

$$\sum_{l,m} F_{lm}u_{lm}{}^{(kjih)} = 0,$$

$$\sum_{\substack{k,j,i,h\\ l,m}} u_{lm}{}^{(kjih)}u_{lm}{}^{(kjih)} = 8\{(2m-1)|R_{kjih}|^2 - 2|R_{ji}|^2\},$$

$$\sum_{\substack{k,j,i,h\\ l,m,r,s}} F_{lr}F_{ms}u_{lm}{}^{(kjih)}u_{rs}{}^{(kjih)} = 24\{|R_{kjih}|^2 - 2|R_{ji}|^2\}.$$

Hence we have

$$\sum_{k,j,i,h} \sigma_2(u^{(kjih)}) = 8(m+1)\left\{|R_{kjih}|^2 - \frac{4}{m+1}|R_{ji}|^2\right\} \tag{3.4}$$
$$= 8(m+1)|P_{kjih}|^2.$$

On the other hand we can get

$$\sum_{\substack{k,j,i,h\\ l,m,p,q}} R_{lmpq} u_{lm}{}^{(kjih)} u_{pq}{}^{(kjih)} = -16K. \tag{3.5}$$

Now we assume that K^{2m} satisfies (*) and (3.1), then we have by (3.4) and (3.5)

$$K \geqq \frac{(m+1)k}{4}|P_{kjih}|^2$$

and by (3.3)

$$0 \leqq \frac{(m+1)k}{4}\int_K |P_{kjih}|^2 d\sigma \leqq -\tfrac{1}{2}\int_K |\nabla_k R_{jih}{}^l|^2 d\sigma \leqq 0.$$

Thus we have $P_{kjih}=0$ and hence the following theorem has been proved.

THEOREM 3.1. *If a compact Kählerian manifold K^{2m} satisfies* (*) *and*

$$\nabla_r R_{jih}{}^r = 0,$$

then K^{2m} is a space of constant holomorphic curvature.

§ 4. **The proper value.** Let L be the operator which associates to a p-form u a $(p+2)$-form $Lu=\Omega\wedge u$. As is well known [6, p. 180], every p-form u $(m+1\geqq p\geqq 0)$ has a unique representation as a sum

$$u = \sum_{h=0}^{r} L^h \phi_{p-2h} \tag{4.1}$$

where ϕ_{p-2h} are effective $(p-2h)$-forms and $r=[p/2]$. If $p-2h$ is the largest number such that ϕ_{p-2h} does not vanish identically in (4.1), we shall call it the height of u.

Now let u be a proper p-form $(m+1\geqq p\geqq 0)$: $\Delta u=\lambda u$. Then we have from (4.1)

$$\sum_h L^h \Delta\phi_{p-2h} = \sum_h L^h(\lambda\phi_{p-2h})$$

because Δ commutes with L [6, p. 176]. As Δ commutes with Λ, ϕ_{p-2h} are effective too, and hence

$$\Delta\phi_{p-2h} = \lambda\phi_{p-2h}$$

hold by the uniqueness of the representation. Hence the discussion about λ for p-forms of height q reduces to one for effective q-forms.

We shall find a lower bound of λ for effective p-forms in a compact Kählerian manifold K^{2m} satisfying (*).

First, the following lemma is known.

LEMMA 4.1. (*Gallot-Meyer*, [5]). *For any p-form u $(n-1\geqq p\geqq 1)$ in an n dimensional Riemannian manifold, the following inequality holds:*

$$|\nabla u|^2\geqq\frac{1}{p+1}|du|^2+\frac{1}{n-p+1}|\delta u|^2.$$

Let u be a proper effective p-form $(m+1\geqq p\geqq 1)$ corresponding to λ. By integration of the Weizenböck formula (1.1) we have

$$(u, \Delta u)\geqq||\nabla u||^2+\int_K F_p(u)d\sigma$$

where $(\ ,\ )$ and $||\ ||$ mean the global inner product and the global length respectively. If we take account of Lemma 4.1 and (2.1) or (2.3) it follows that

$$\begin{aligned}(u, \Delta u)&\geqq\frac{1}{p+1}||du||^2+\frac{1}{2m-p+1}||\delta u||^2+\tfrac{1}{2}\,p(m+2-p)k||u||^2\\&\geqq\frac{1}{2m-p+1}(||du||^2+||\delta u||^2)+\tfrac{1}{2}\,p(m+2-p)k||u||^2.\end{aligned}$$

Thus we can get the following

THEOREM 4.2. *Let K^{2m} be a compact Kählerian manifold satisfying* (*). *Then the proper value λ for effective p-forms $(m+1\geqq p\geqq 0)$ satisfies*

$$\lambda\geqq\frac{(2m-p+1)p(m+2-p)}{2(2m-p)}k.$$

§ **5. F-special Killing 1-form.** Consider a compact Kählerian manifold K^{2m} satisfying (*). Let u be a coclosed proper 1-form: $\delta u=0$, $\Delta u=\lambda u$. Then we have by (4.1)

$$\lambda||u||^2\geqq\tfrac{1}{2}\,||du||^2+\tfrac{1}{2}(m+1)k||u||^2,$$

from which we get

$$\lambda\geqq(m+1)k. \tag{5.1}$$

If the equality holds in (5.1), the u must be Killing, which follows from the proof of theorem 2.4 in [9]. Hence we have

LEMMA 5.1. *In a compact Kählerian manifold K^{2m} satisfying* (*) *if there exists a coclosed proper* 1*-form u corresponding to $\lambda=(m+1)k$, then u is Killing.*

We are interested in the case when K^{2m} actually admits the 1-form u stated in Lemma 5.1.

Let us consider a Killing 1-form $u=u_i dx^i$ and we assume that u satisfies

$$\Delta u=\lambda u,\qquad \lambda=\text{constant}. \tag{5.2}$$

If we put $v_{ji}=\nabla_j u_i$, then

$$v_{ji}=-v_{ij}, \qquad \nabla_h v_{ji}=-R_{rhji}u^r \tag{5.3}$$

hold good. From (5.2) and (5.3) we get

$$R_{ir}u^r=\tfrac{1}{2}\lambda u_i, \qquad \nabla^r\nabla_r u_i=-\tfrac{1}{2}\lambda u_i.$$

On the other hand if we put $\tilde{u}_j=F_j{}^i u_i$, we have

$$\begin{aligned}\nabla_h\nabla^j\tilde{u}_j=\nabla_h(F^{ji}\nabla_j u_i)&=-F^{ji}R_{rhji}u^r\\ &=-2\,F_h{}^iR_{ir}u^r=-\lambda\tilde{u}_h.\end{aligned}$$

Now let us define A_{jih} by

$$A_{jih}=\nabla_j v_{ih}+t_{jih},$$

where

$$t_{jih}=\tfrac{1}{4}k(g_{ji}u_h-g_{jh}u_i-F_{ji}\tilde{u}_h+F_{jh}\tilde{u}_i+2\,F_{ih}\tilde{u}_j)$$

and k is the constant appeared in (*). If A_{jih} vanish identically, u is F-special with k.

We shall integrate $|A_{jih}|^2=A_{jih}A^{jih}$ over K^{2m} with respect to the canonical measure $d\sigma$ and get an integral formula.

$|A_{jih}|^2$ is the sum of the following A_1, A_2 and A_3.

$$\begin{aligned}A_2&=2\nabla^j v^{ih}t_{jih}=-k\,R_{rjih}u^r(g^{ji}u^h-F^{ji}\tilde{u}^h+F^{ih}\tilde{u}^j)\\ &=-2\lambda\,k|u|^2,\\ A_3&=t_{jih}t^{jih}=(m+1)k^2|u|^2,\\ A_1&=\nabla^j v^{ih}\nabla_j v_{ih}=-\nabla^j v^{ih}R_{rjih}u^r\\ &=-\nabla^j(v^{ih}R_{rjih}u^r)+v^{ih}\nabla^jR_{rjih}u^r+v^{ih}R_{rjih}v^{jr}.\end{aligned}$$

If we neglect here and below the terms which vanish under the integral sign, A_1 can be written as

$$A_1=v^{ih}\nabla^jR_{rjih}u^r-R_{rjih}v^{rj}v^{ih}.$$

By virtue of the identity (3.3) the first term of A_1 becomes

$$\begin{aligned}v^{ih}\nabla^jR_{rjih}u^r&=2\nabla_iR_{hr}v^{ih}u^r\\ &=2\{\nabla_i(R_{hr}v^{ih}u^r)-R_{hr}\nabla_iv^{ih}u^r-R_{hr}v^{ih}v_i{}^r\}\\ &=\tfrac{1}{2}\lambda^2|u|^2-2\,R_{hr}v^{hi}v^r{}_i.\end{aligned}$$

Hence we have

$$A_1=\tfrac{1}{2}\lambda^2|u|^2-2\,F_2(v).$$

On the other hand, it holds from Lemma 2.1 that

$$F_2(v)\geqq(m+1)k\Big\{|v|^2-\frac{1}{2(m+1)}\sigma_2(v)\Big\}. \tag{5.4}$$

We shall write the right hand side of (5.4) in terms of $|u|^2$.

First we have

(5.5) $$2|v|^2=\nabla_i u_j \nabla^i u^j=-u_j \nabla^i \nabla_i u^j=\tfrac{1}{2}\lambda|u|^2.$$

Next, from the definition of $\sigma_2(v)$ it follows that

(5.6) $$2\sigma_2(v)=\nabla_i u_j \nabla^i u^j+F^{ir}F^{js}\nabla_i u_j \nabla_r u_s+(F^{ij}\nabla_i u_j)^2.$$

As is well known, any Killing 1-form being contravariant analytic [12, p. 86], u satisfies

$$F_r{}^i \nabla_i u_j=-F_j{}^i \nabla_r u_i.$$

Hence

$$F^{ir}F^{js}\nabla_i u_j \nabla_r u_s=-F_r{}^i \nabla_i u_j F^{js}\nabla^r u_s$$
$$=F_j{}^i \nabla_r u_i F^{js}\nabla^r u_s=\nabla_r u_i \nabla^r u^i.$$

The last term of $\sigma_2(v)$ becomes

$$(F^{ij}\nabla_i u_j)^2=(\nabla_i \tilde{u}^i)^2=\nabla_i \tilde{u}^i \nabla_j \tilde{u}^j=-\tilde{u}^i \nabla_i \nabla_j \tilde{u}^j=\lambda|u|^2.$$

Substituting these values into (5.6) we have

$$\sigma_2(v)=\lambda|u|^2.$$

Thus from (5.4) we get

(5.7) $$F_2(v)\geqq \frac{m-1}{4}\lambda k|u|^2.$$

Summing up A_1, A_2 and A_3, and taking account of (5.7) we obtain

$$(\lambda-2k)\{\lambda-(m+1)k\}\,\|u\|^2\geqq 2\int_K |A_{jih}|^2\, d\sigma \geqq 0.$$

Consequently we know that u is F-special with k if λ is the possible minimum value $(m+1)k$. Hence we get taking account of Lemma 5.1 the following

THEOREM 5.2. *Let K^{2m} be a compact Kählerian manifold satisfying* (*). *If K^{2m} admits a coclosed proper* 1-*form u of Δ corresponding to the proper value* $(m+1)k$, *then u is F-special Killing with k.*

REMARK. M. Obata [7] has stated the following theorem without proof.

THEOREM. *Let K^{2m} be a complete simply connected Kählerian manifold. In order for K^{2m} to admit a non-trivial solution ϕ of the differential equation*

(5.8) $$\nabla_j \nabla_i \phi_h+\tfrac{1}{2}c(2\phi_j g_{ih}+\phi_i g_{jh}+\phi_h g_{ji}-\tilde{\phi}_i F_{jh}-\tilde{\phi}_h F_{ji})=0$$

where $\phi_j=\nabla_j \phi$ and c is positive constant, it is necessary and sufficient that K^{2m} be isometric with $CP^m(c)$.

$CP^m(c)$ means the complex projective space with Fubini metric of constant holomorphic curvature c. Let a non-zero Killing 1-form $u=u_i dx^i$ be F-special with c. If we put $\phi=\nabla_i \tilde{u}^i$, then it is easy to see that ϕ is a solution of (5.8) which is non-trival because of $\nabla_j\phi=-(m+1)c\tilde{u}_j$. Hence we have

THEOREM 5.4. *Let K^{2m} be a compact simply connected Kählerian manifold satisfying (*). If K^{2m} admits a coclosed proper 1-form of Δ corresponding to the proper value $(m+1)k$, then K^{2m} is isometric with $CP^m(k)$.*

Bibliography

[1] M. Berger: Sur les variétés à opérateur de courbure positif, C.R. Acad. Sc. Paris, **253** (1963) 2832–2834.

[2] M. Berger: Les variétés kähleriennes compactes d'Einstein de dimension quatre à courbure positif, Tensor **13** (1963) 71–74.

[3] M. Berger, P. Gauduchon et E. Mazet: Le Spectre d'une Variété Riemannienne, Springer-Verlag, **194**, 1971.

[4] D. Meyer: Sur les variétés riemanniennes à opérateur de courbure positif, C.R. Acad. Sc. Paris, **272**, (1971) 482–485.

[5] S. Gallot et D. Meyer: Sur la première valeur propre du p-spectre pour les variétés à opérateur de courbure positif, C.R. Acad. Sc. Paris, **276**, (1973) 1619–1621.

[6] S.I. Goldberg: Curvature and Homology, Academic Press, 1962.

[7] M. Obata: Riemannian manifolds admitting a solution of a certain system of differential equations, Proc. of the United States-Japan seminar in differential geometry, 1965, 101–114.

[8] S. Tachibana: On the Bochner curvature tensor, Nat Sc. Rep. of the Ochanomizu Univ., **18**, (1967) 15–19.

[9] S. Tachibana: On Killing tensors in Riemannian manifolds of positive curvature operator, to appear.

[10] S. Tachibana: A theorem on Riemannian manifolds of positive curvature operator, Proc. Japan Acad., **50**, (1974) 301–302.

[11] Y. Tashiro: On holomorphically projective correspondences in an almost complex spaces, Math. J. Okayama Univ., **6** (1957) 147–152.

[12] K. Yano: Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

[13] K. Yano and S. Bochner: Curvature and Betti numbers, Ann. of Math. Stud., **32**, 1953.