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# The Bell-Steinberger Sum Rule in the Neutral Kaon Decays

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## § 1. Introduction

In 1964, *CP* violation was proved by the experiment in the two pion decay of the long-lived neutral Kaon. The Bell-Steinberger sum rule,[1] which was derived next year, has played important role for the phenomenological analyses of *CP* violation in the neutral Kaon decays. It was obtained considering the conservation of probability. The original derivation, however, based upon the more or less intuitive considerations.

Several years later, the question[2] was suggested whether the unitarity of scattering matrix, which has neutral Kaons as intermediate states, could produce new sum rules. After some discussions, Stodolsky[3] and Gien[4] could show that the unitarity of the $S$ matrix leads to nothing but the Bell-Steinberger sum rule.

The purpose of this paper is to give the explicit description of the sum rule taking consistently into account both the usual method and $S$ matrix approach. In Section 2, the general analyses of the time dependence of state vectors will be discussed in detail. In Section 3, the Bell-Steinberger sum rule will be derived directly from the conservation of probability. In Section 4, we shall obtain the $S$ matrix which has $K_S$ and $K_L$ as resonances. It will be shown that the unitarity condition for this $S$ matrix reduces to the same relation as derived in Sec. 3.

## § 2. The time dependence

In this section, we begin with the time-dependent Schrödinger equation for the Hamiltonian $H=H_0+H'$. In the neutral Kaon system, $H_0$ which is assumed to include the $C, P$ and $T$ invariant interactions has two discrete and degenerate eigenstates $|i\rangle$ and $|j\rangle$ with the same energy eigenvalue $m$. $H'$ is assumed to be responsible for the decay of the discrete states and to include the *CP* violating interaction as well as the usual weak interactions. $H_0$ has also continuous eigen states $|\phi_\alpha^{(-)}\rangle$ and $|\phi_\alpha^{(+)}\rangle$ which are out and in state with the same energy eigenvalue $E_\alpha$. Either $|\phi_\alpha^{(-)}\rangle$ or $|\phi_\alpha^{(+)}\rangle$ may form a complete set of state vectors together with the discrete states. In the following discussion, we consider that all these states have zero total momentum and under such a condition they are normalized to unity or to the internal momentum delta function.

We will now adopt the standard analyses[5] provided for the decay process of one discrete state to the present case.

The time-dependent Schrödinger equation is

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{2-1}$$

Assuming the Hermicity of $H'$, the wave function at an arbitrary time $t$ can be written in the following form with real $E^{(*)}$

$$|\psi(t)\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}E\, e^{-iE(t-t_0)}\{G_+(E)-G_-(E)\}\,|\psi(t_0)\rangle, \tag{2-2}$$

where $G_\pm(E)$ are the Green's functions for $H$

$$G_\pm(E) = \lim_{\varepsilon\to+0}\frac{1}{E\pm i\varepsilon - H}. \tag{2-3}$$

At a given time $t=0$, let the wave function $|\psi(0)\rangle$ be provided as linear combination of discrete and degenerate states

$$|\psi(0)\rangle = \sum_i a_i(0)|i\rangle. \tag{2-4}$$

By introducing the probability amplitudes, the wave function may be written for $t>0$

$$|\psi(t)\rangle = \sum_i a_i(t)|i\rangle + \sum_\alpha b_\alpha(t)|\phi_\alpha^{(-)}\rangle, \tag{2-5}$$

where the probability amplitudes for the discrete states are given by using Eqs. (2-2), (2-4) and the footnote (**)

$$a_i(t) = \langle i|\psi(t)\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}E\, e^{-iEt}\sum_j \langle i|G_+(E)|j\rangle a_j(0), \tag{2-6}$$

and for the decay channels $|\phi_\alpha^{(-)}\rangle$ by

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(*) Using the following relation,

$$G_+(E) - G_-(E) = \lim_{\varepsilon\to+0}\frac{-2i\varepsilon}{(E-H)^2+\varepsilon^2} = -2\pi i\delta(E-H),$$

we may rewrite $e^{-iH(t-t_0)}|\psi(t_0)\rangle$ as Eq. (2-2).

(**) Here we consider

$$\theta(\pm t \mp t_0)e^{-iH(t-t_0)} = \mp\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}E\, e^{-iE(t-t_0)}\frac{1}{E-H\pm i\varepsilon},$$

where

$$\theta(t-t_0) = \begin{cases} 1 & t-t_0>0 \\ 0 & t-t_0<0. \end{cases}$$

$$b_\alpha(t)=\langle\phi_\alpha^{(-)}|\psi(t)\rangle=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}E\,e^{-iEt}\sum_j\langle\phi_\alpha^{(-)}|G_+(E)|j\rangle a_j(0). \tag{2-7}$$

To calculate these probability amplitudes, it is necessary to obtain the matrix elements of Green's functions.

By the formal evaluation, using the following function

$$R(z)=H'+H'(1-\boldsymbol{P})\frac{1}{z-H_0-(1-\boldsymbol{P})H'(1-\boldsymbol{P})}(1-\boldsymbol{P})H', \tag{2-8}$$

the matrix elements of $G(z)$ in Eq. (2-6) can be obtained as

$$\boldsymbol{P}G(z)\boldsymbol{P}=\frac{1}{z-H_0-\boldsymbol{P}R(z)\boldsymbol{P}}, \tag{2-9}$$

where $\boldsymbol{P}$ is the projection operator onto the two dimensional discrete subspace. On the other hand, the matrix elements in Eq. (2-7) are given by

$$(1-\boldsymbol{P})G(z)\boldsymbol{P}=\frac{1}{z-H_0}(1-\boldsymbol{P})R(z)\boldsymbol{P}G(z)\boldsymbol{P}. \tag{2-10}$$

Now we consider the elementary properties of $R(z)$ given in Eq. (2-9). When we set

$$R_\pm(E)=\lim_{\varepsilon\to+0}R(E\pm i\varepsilon), \tag{2-11}$$

assuming the Hermicity of $H$, it follows that

$$R_\pm(E)-R_\pm^\dagger(E)=\mp 2\pi iR_\pm^\dagger(E)\delta(E-H_0)(1-\boldsymbol{P})R_\pm(E), \tag{2-12-a}$$

$$=\mp 2\pi iR_\pm^\dagger(E)\delta(E-H_0)R_\pm(E)$$

$$\pm R_\pm^\dagger(E)\left(\frac{\boldsymbol{P}}{E-H_0-i\varepsilon}-\frac{\boldsymbol{P}}{E-H_0+i\varepsilon}\right)R_\pm(E). \tag{2-12-b}$$

Particularly in the discrete subspace, we rewrite $R_\pm(E)$ as

$$R_\pm(E)=D(E)\mp\frac{i}{2}\Gamma(E), \tag{2-13}$$

where $D(E)$ and $\Gamma(E)$ are Hermitian. From Eq. (2-12-a), $\Gamma(E)$ may be written as

$$\Gamma(E)=2\pi R_\pm^\dagger(E)\delta(E-H_0)(1-\boldsymbol{P})R_\pm(E). \tag{2-14}$$

Now, to evaluate the integrals (2-6) and (2-7) we must diagonalize $\langle i|G|j\rangle$ such as

$$\langle i|G|j\rangle=\sum_n\langle i|n\rangle\langle n'|G|n\rangle\langle n'|j\rangle, \tag{2-15}$$

where we use the normalization condition in the two dimensional discrete subspace

$$\langle n'|m\rangle=\delta_{nm},$$
$$\sum_i |i\rangle\langle i|=\sum_n |n\rangle\langle n'|. \tag{2-16}$$

If we define the mass matrix in this subspace as

$$M(z)=m+R(z), \tag{2-17}$$

from Eq. (2-9) we may rewrite the Green's function in this subspace

$$G(z)=\frac{1}{z-M(z)}. \tag{2-18}$$

Then $|n\rangle$ and $|n'\rangle$ in Eq. (2-15) turn out to be the eigenvectors of $M(z)$.

In the neutral Kaon system, these eigenvectors are well-known as follows

$$M_+|K_{S,L}\rangle=M_{S,L}|K_{S,L}\rangle, \tag{2-19}$$
$$\langle K_{S,L}'|M_+=M_{S,L}\langle K_S',{}_L|,$$
$$M_-|K_{S,L}'\rangle=M_{S,L}^*|K_{S,L}'\rangle,$$
$$\langle K_{S,L}|M_-=M_{S,L}^*\langle K_{S,L}|,$$

where $M_\pm(E)$ are defined by

$$M_\pm(E)=\lim_{\varepsilon\to+0} M(E\pm i\varepsilon), \tag{2-20}$$

hence

$$M_\pm(E)=m+D(E)\mp\frac{i}{2}\Gamma(E). \tag{2-21}$$

Assuming the usual analytic properties of $R_+(E)$, the integral (2-6) may be divided into two parts, *i. e.*, the contributions from the poles and the residual terms. Utilizing the results estimated already[5] in the weak decay, we get

$$a_i(t)=\sum_j \sum_{n=S,L}\langle i|K_n\rangle\langle K_n'|j\rangle a_j(0)$$
$$\times\left\{e^{-iM_nt}+\frac{\Gamma_n}{4\sqrt{\pi}\,m^2\sqrt{m}}\,e^{\frac{\pi}{4}i}t^{-\frac{3}{2}}\right\}.$$

The second term was evaluated using a crude model. We will, however, estimate the magnitude of the second term in the case of the neutral Kaon system. Using the experimental values, the order of magnitude is given to be $\sim 10^{-35}$. Then the contributions except from the poles are negligible in the neutral Kaon decays.

## § 3. The Bell-Steinberger sum rule

In this section, we will explicitly derive the sum rule by considering the decay properties of the neutral Kaon system.

The Schrödinger equation with a Hermitian Hamiltonian assures the conservation of the probability, that is,

$$\frac{d}{dt}\langle\phi(t)|\phi(t)\rangle=0. \tag{3-1}$$

In the decay process, using Eq. (2-5), the above requirement becomes

$$\frac{d}{dt}\left(\sum_i |a_i(t)|^2+\sum_\alpha\int d\rho_\alpha(E_\alpha)dE_\alpha|b_\alpha(t)|^2\right)=0, \tag{3-2}$$

where $d\rho_\alpha$ is the density of continuous states per unit energy. Omitting the contribution except from the poles, the probability amplitudes for the discrete states are given by

$$a_i(t)=\sum_j\sum_{n=S,L}e^{-iM_nt}\langle i|K_n\rangle\langle K_n'|j\rangle a_j(0). \tag{3-3}$$

Evaluating the integral (2-7), the probability amplitudes for the continuous states are obtained by

$$b_\alpha(t)=\sum_j\sum_n e^{-iE_\alpha t}\frac{1}{E_\alpha-M_n}\{1-e^{i(E_\alpha-M_n)t}\}$$
$$\times\langle\phi_\alpha^{(-)}|R_+(m)|K_n\rangle\langle K_n'|j\rangle a_j(0), \tag{3-4}$$

where we set $R_+(E_\alpha)=R_+(m)$ since we consider the weak decay.

Provided that $|\phi(0)\rangle=|K^0\rangle$, the probability that the system is in the initial state is obtained by (3-3)

$$\sum_i|a_i(t)|^2=\sum_{n,m}e^{i(M_m^*-M_n)t}\langle K_m|K_n\rangle\langle K_n'|K^0\rangle\langle K^0|K_m'\rangle. \tag{3-5}$$

On the other hand, the probability that the system is in the decay channels is given by

$$\sum_\alpha\int d\rho_\alpha(E_\alpha)dE_\alpha|b_\alpha(t)|^2$$
$$=2\pi i\sum_{n,m}\sum_\alpha\int d\rho_\alpha\frac{\langle\phi_\alpha^{(-)}|R_+|K_m\rangle^*\langle\phi_\alpha^{(-)}|R_+|K_n\rangle}{(M_m^*-M_n)}$$
$$\times\{1-e^{i(M_m^*-M_n)t}\}\langle K_n'|K^0\rangle\langle K^0|K_m'\rangle, \tag{3-6}$$

where we set $d\rho_\alpha(E_\alpha)=d\rho_\alpha(m)$ using the smallness of $\Gamma_S$ and employed the following relation

$$\int \mathrm{d}E_\alpha\frac{1}{E_\alpha-M_n}\frac{1}{E_\alpha-M_m^*}\{1-e^{i(E_\alpha-M_n)t}\}\{1-e^{-i(E_\alpha-M_m^*)t}\}$$
$$=\frac{2\pi i}{M_m^*-M_n}\{1-e^{i(M_m^*-M_n)t}\}. \tag{3-7}$$

Substituting these probabilities into Eq. (3-2), we obtain

$$\sum_{n,m}i(M_m^*-M_n)e^{i(M_m^*-M_n)t}\langle K_m|K_n\rangle\langle K_n'|K^0\rangle\langle K^0|K_m'\rangle$$
$$+2\pi\sum_{n,m}\sum_\alpha\int d\rho_\alpha\langle\phi_\alpha^{(-)}|R_+|K_m\rangle^*\langle\phi_\alpha^{(-)}|R_+|K_n\rangle$$
$$\times e^{i(M_m^*-M_n)t}\langle K_n'|K^0\rangle\langle K^0|K_m'\rangle=0. \tag{3-8}$$

By comparing the coefficients for arbitrary time in Eq. (3–8), we obtain

$$2\pi \sum_{\alpha} \int d\rho_{\alpha} |\langle \phi_{\alpha}{}^{(-)} | R_{+} | K_S \rangle|^2 = \Gamma_S, \tag{3-9-a}$$

$$2\pi \sum_{\alpha} \int d\rho_{\alpha} |\langle \phi_{\alpha}{}^{(-)} | R_{+} | K_L \rangle|^2 = \Gamma_L, \tag{3-9-b}$$

$$2\pi \sum_{\alpha} \int d\rho_{\alpha} \langle \phi_{\alpha}{}^{(-)} | R_{+} | K_S \rangle^{*} \langle \phi_{\alpha}{}^{(-)} | R_{+} | K_L \rangle = -i(M_S{}^{*} - M_L) \langle K_S | K_L \rangle, \tag{3-9-b}$$

$$2\pi \sum_{\alpha} \int d\rho_{\alpha} \langle \phi_{\alpha}{}^{(-)} | R_{+} | K_L \rangle^{*} \langle \phi_{\alpha}{}^{(-)} | R_{+} | K_S \rangle = -i(M_L{}^{*} - M_S) \langle K_L | K_S \rangle. \tag{3-9-d}$$

It is stated by Eqs. (3–9–a) and (3–9–b) that the decay width $\Gamma_S$ and $\Gamma_L$ are given in terms of the matrix elements of $R_+$. The other two relations, Eq. (3–9–c) and its complex conjugate Eq. (3–9–d), are the so-called Bell-Steinberger sum rule.

Now we will show that the equivalent relations can be obtained essentially from the property of $R_+$. Using the relations between the mass matrix and its eigenvectors, the Antihermitian part of the mass matrix is expressed as

$$\langle K_m | M_{+} - M_{+}{}^{\dagger} | K_n \rangle = (M_n - M_m{}^{*}) \langle K_m | K_n \rangle. \tag{3-10}$$

Using Eq. (2–12–a), it is obtained that

$$\langle K_m | R_{+}(m) - R_{+}{}^{\dagger}(m) | K_n \rangle = -2\pi i \sum_{\alpha} \int d\rho_{\alpha}(m)$$
$$\times \langle K_m | R_{+}{}^{\dagger}(m) | \phi_{\alpha}{}^{(-)} \rangle \langle \phi_{\alpha}{}^{(-)} | R_{+}(m) | K_n \rangle. \tag{3-11}$$

From the definition of the mass matrix defined in Eq. (2–17), we can obtain the same relations as Eqs. (3–9–a∼d).

## § 4. *S* matrix

The $S$ matrix element with the initial and final state interactions considered may be defined as follows

$$S_{fi} = \lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} \langle \phi_f{}^{(-)} | e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} | \phi_i{}^{(+)} \rangle, \tag{4-1}$$

this may be rewritten as

$$S_{fi} = \sum_{n} \lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} \langle \phi_f{}^{(-)} | e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} | \phi_n{}^{(-)} \rangle \langle \phi_n{}^{(-)} | \phi_i{}^{(+)} \rangle, \tag{4-2}$$

where the complete set formed by $|\phi_n{}^{(-)}\rangle$ includes discrete states. Using the footnote (**), we may have

$$S'_{fn}=\lim_{\substack{t\to+\infty \\ t_0\to-\infty}}\langle\phi_f^{(-)}|e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}|\phi_n^{(-)}\rangle$$

$$=-\frac{1}{2\pi i}\lim\int dE\, e^{iE_ft-iE_nt_0}e^{-iE(t-t_0)}\langle\phi_f^{(-)}|G_+(E)|\phi_n^{(-)}\rangle. \quad (4\text{-}3)$$

By formal calculation, $\langle\phi_f^{(-)}|G_+(E)|\phi_n^{(-)}\rangle$ may be given as the matrix element of

$$\frac{1}{E+i\varepsilon-H_0}+\frac{1}{E+i\varepsilon-H_0}R_+(E)\frac{1}{E+i\varepsilon-H_0}$$

$$+\frac{1}{E+i\varepsilon-H_0}R_+(E)\frac{\boldsymbol{P}}{E+i\varepsilon-M_+(E)}R_+(E)\frac{1}{E+i\varepsilon-H_0}. \quad (4\text{-}4)$$

Using the following relation

$$\lim_{\varepsilon\to+0}\lim_{t\to+\infty}\frac{e^{-ixt}}{x+i\varepsilon}=-2\pi i\delta(x), \quad (4\text{-}5)$$

we obtain explicitly from Eq. (4-3)

$$S'_{fn}=\delta_{fn}-2\pi i\delta(E_f-E_n)T'_{fn}(E), \quad (4\text{-}6)$$

where

$$T'_{fn}(E)=\langle\phi_f^{(-)}|R_+(E)|\phi_n^{(-)}\rangle$$

$$+\langle\phi_f^{(-)}|R_+(E)\frac{\boldsymbol{P}}{E+i\varepsilon-M_+(E)}R_+(E)|\phi_n^{(-)}\rangle. \quad (4\text{-}7)$$

In case that either $|\phi_n^{(-)}\rangle$ or $|\phi_f^{(-)}\rangle$ is the discrete eigenstate of $H_0$, $T'_{fn}(E)$ given by Eq. (4-7) has not $K_S$ and $K_L$ as resonances since the resonance denominators are multiplied by $\boldsymbol{P}R_+(m)\boldsymbol{P}=M_+(m)-m$.

We now consider the unitarity of the $S$ matrix. Using the property of $R_+(E)$ given by Eq. (2-12-b) and the following relation

$$\frac{1}{E+i\varepsilon-H_0}+\frac{1}{E+i\varepsilon-H_0}\boldsymbol{P}R_+(E)\boldsymbol{P}\frac{1}{E+i\varepsilon-H_0-\boldsymbol{P}R_+(E)\boldsymbol{P}}$$

$$=\frac{1}{E+i\varepsilon-H_0-\boldsymbol{P}R_+(E)\boldsymbol{P}}, \quad (4\text{-}8)$$

for $T'(E)$ obtained by Eq. (4-7), it is satisfied that

$$-2\pi iT'^{\dagger}(E)\delta(E-H_0)T'(E)=T'(E)-T'^{\dagger}(E), \quad (4\text{-}9)$$

this implies the unitarity relation of $S'$ given in Eq. (4-6). $S_0$ which $(n, i)$ element is $\langle\phi_n^{(-)}|\phi_i^{(+)}\rangle$ governed by $H_0$ must be unitary. Therefore, $S=S'S_0$ is also unitary. We come to the same conclusion that the unitarity of $S$ matrix starting from an Hermite Hamiltonian reduces to the property of $R_+(E)$ as already obtained in Sec. 3 by considering the decay process of the neutral Kaons.

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