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A Characterization of Unit-Boundaries in Finitely Generated Banach Algebras

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§1. Introduction

Let A be a commutative Banach algebra with identity e and $\mathfrak{M}(A)$ the maximal ideal space of A. By definition [2], a unit-boundary F for A is a closed subset of $\mathfrak{M}(A)$ satisfying ;

1) *if an element a in A whose Gelfand transform $\hat{a}$ does not vanish on F, then a is invertible in A, namely, there exists an element b in A such that $ab=e$,*

or equivalently,

2) $\hat{a}(F)=\hat{a}(\mathfrak{M}(A))^*$, *for every element a in A.*

We know that, in general, the unit-boundary for A cannot necessarily be determined uniquely, but that every unit-boundary contains the Šilov boundary.

In this paper we give a description of unit-boundaries which is closely related to the notion of rational convex hulls. Although we treat here finitely generated Banach algebras, it should be understood that the notion of unit-boundaries is a generalization of that of rational convex hulls to general normed algebras.

§2. Ploynomial and rational convex hulls

The polynomial convex hull of a compact subset X of $\boldsymbol{C}^n$, hull($\boldsymbol{X}$), is a closed set which consists of all z in $\boldsymbol{C}^n$ such that

$$|p(z)|\leqq\max_{x\in X}|p(x)|, \text{ for every polynomial } p.$$

When hull$(X)=X$, we say that X is polynomially convex.

Let A be a commutative Banach algebra generated by $f_1, \dots, f_n$. Then, under the natural projection π of $\mathfrak{M}(A)$ into $\boldsymbol{C}^n$ defined by $\pi(m)=(f_1(m), \dots, f_n(m))$ $(m\in\mathfrak{M}(A))$, $\mathfrak{M}(A)$ is homeomorphic to a poly-

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* We denote by $\hat{a}(S)$ the values of Gelfand transform $\hat{a}(a\in A)$ on a subset S of $\mathfrak{M}(A)$.

nomially convex set in C^n which is called the joint spectrum $\sigma(f_1, \dots, f_n)$ of $f_1, \dots, f_n$.

The rational convex hull of a compact subset X of C^n, R-hull(X), is a closed set which consists of all z in C^n such that

$$|f(z)| \leqq \max_{x \in X} |f(x)|, \text{ for every rational function } f \text{ which}$$

is analytic about X,

or alternatively,

1′) *if a polynomial p satisfies that $p(z)=0$, then $p(X) \ni 0$.*

The above definitions are also equivalent to

2′) *$p(z) \in p(X)$, for every polynomial p.*

When R-hull(X)$=X$, we say that X is rationally convex [1] [3].

Note that $X \subset R$-hull(X)$\subset$hull(X).

Let $\partial(A)$ be the Šilov boundary for A. For any closed subset F of $\mathfrak{M}(A)$, hull($\pi(F)$)$=\pi(\mathfrak{M}(A))$ if F contains $\partial(A)$, and conversely. However, in general, R-hull($\pi(F)$) need not coincide with $\pi(\mathfrak{M}(A))$ even if F contains $\partial(A)$. Now, $\pi(\mathfrak{M}(A))$ is polynomially convex set, so a rationally convex set in C^n, and $\mathfrak{M}(A)$ is, by definition, a unit-boundary for A. While, every closed subset of $\mathfrak{M}(A)$ containing $\partial(A)$ need not be a unit-boundary for A. These facts will lead us to a characterization of the unit-boundaries in finitely generated commutative Banach algebras.

§ 3. A characterization of the unit-boundaries

THEOREM 1. *Let A be a commutative Banach algebra generated by $f_1, \dots, f_n$ and $\mathfrak{M}(A)$ the maximal ideal space of A. Then, a closed subset F of $\mathfrak{M}(A)$ is a unit-boundary if and only if R-hull($\pi(F)$)$=\pi(\mathfrak{M}(A))$, where π being the natural projection of $\mathfrak{M}(A)$ into C^n.*

PROOF. Note that a closed subset F of $\mathfrak{M}(A)$ with $\hat{b}(F)=\hat{b}(\mathfrak{M}(A))$ for every element b in a dense subset of A is a unit-boundary for A, and conversely.

Suppose R-hull($\pi(F)$)$=\pi(\mathfrak{M}(A))$. If m is in $\mathfrak{M}(A)$, then, by definitions, $\hat{p}(m)=p(\pi(m))\in p(\pi(F))=\hat{p}(F)$ for every polynomial $p(f_1, \dots, f_n)$, whence follows that F is a unit-boundary for A. The converse is also an immediate consequence of definitions.

COROLLARY. *Let A be the same as in Theorem 1. Then, $\mathfrak{M}(A)$ is the only unit-boundary for A if and only if R-hull($\pi(F)$)$\subsetneqq\pi(\mathfrak{M}(A))$ for any proper closed subset F of $\mathfrak{M}(A)$.*

REMARK. It is well-known that every compact set in C^1 is rationally convex. If in particular A has one generator, then, from Corollary, the maximal ideal space is the only unit-boundary for A.

However in $\boldsymbol{C}^n (n>1)$, since every compact set need not be rationally convex, there may be compact sets which have strictly larger convex hulls. And as Theorem 1 says the unit-boundaries mean the closed subsets of the joint spectrum $\sigma(f_1, \dots, f_n)$ whose rational convex hulls become $\sigma(f_1, \dots, f_n)$. This is why proper unit-boundaries exist mostly in Banach algebras of functions in several complex variables (see examples in [2]).

One can deduce the next theorem as one of the results about unit-boundaries [2].

THEOREM 2. *Let A be the same as in Theorem 1. If $\{K_\alpha\}_{\alpha\in\Lambda}$ is a family of totally ordered closed subset of $\sigma(f_1, \dots, f_n)$ with respect to set-inclusion satisfying R-hull $(K_\alpha)=\sigma(f_1, \dots, f_n)$ for every $\alpha\in\Lambda$, then R-hull $(\bigcap_{\alpha\in\Lambda} K_\alpha)=\sigma(f_1, \dots, f_n)$.*

References

[1] Gamelin, T.W., Uniform algebras, Prentice-Hall (1969).

[2] Nishizawa, K., On a unit-boundary of a function algebra, Nat. Sci. Rep., Ochanomizu Univ. **22** (1971), 9–15.

[3] Stolzenberg, G., Polynomially and rationally convex sets, Acta Math. **109** (1963), 258–289.