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On a Characterization of Riemannian Spaces Admitting Minimal Subspaces Complementary Orthogonal

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Introduction

Let M be an n dimensional Riemannian space of class C^∞. A tensor field $F_\mu{}^\lambda$ is called an almost product structure if it satisfies

$$F_\mu{}^\alpha F_\alpha{}^\lambda=\delta_\mu^\lambda\,, \tag{0.1}$$

$$g_{\alpha\beta}F_\mu{}^\alpha F_\nu{}^\beta=g_{\mu\nu}\,, \tag{0.2}$$

where $g_{\mu\nu}$ denotes the Riemannian metric, [4].[1)]

By a metric tensor of rank p we shall mean, following to T.Y. Thomas [7], a parallel symmetric tensor $h_{\lambda\mu}$ of rank $(h_{\lambda\mu})=p$.

It is known that the following four propositions **A**$\sim$**D** are equivalent to one another.

A. *M admits a metric tensor of rank p.*

B. *M admits a parallel field of p-plane,* [7].

C. *M is covered with coordinate neighbourhoods $\{U\}$ such that the line element is written as*

$$ds^2=\sum_{i,j} g_{ij}(x^k)dx^i dx^j+\sum_{a,b} g_{ab}(x^c)dx^a dx^b$$

and the coordinates in $U\cap U'\neq\phi$ relate with each other by $x'^i=x'^i(x^j)$ and $x'^a=x'^a(x^b)$, [7].

D. *M admits a parallel almost product structure satisfying $F_\lambda{}^\lambda=2p-n$*, [4].

A differential p-form u is called to be *simple*, if there exist 1-forms v_i in a neighbourhood of any point such that

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1) All tensor fields, p-forms and subspaces in this paper are of class C^∞. We shall use the following ranges of indices.

$$1\leqq\lambda,\mu,\nu,\ldots\leqq n, \qquad 1\leqq A,B,C,\ldots\leqq n,$$

$$1\leqq i,j,k,\ldots\leqq p(<n), \qquad p+1\leqq a,b,c,\ldots\leqq n.$$

Indices $\lambda,\mu,\nu,\ldots$ represent the components of the tensor with respect to the natural base, and the summation convention is assumed for these indices.

$$u=v_1\wedge\cdots\wedge v_p\,.$$

Let consider the following proposition:

E. *M admits a parallel simple p-form.*

When M is orientable, Proposition **E** is equivalent to Proposition **B** in which "*p-plane*" is replaced with "*oriented p-plane*", [**3**], [**5**].

Generalizing **B** and **C**, Y.C. Wong [**8**] has discussed the line element of M which admits complementary subspaces. He proved that the following two propositions are equivalent to each other.

B′. *Through any point of M there are minimal submanifolds of dimensions p and n-p which are orthogonal to each other.*

C′. *M is covered with coordinate neighbourhoods* $\{U\}$ *such that the line element is written as*

$$ds^2=\sum_{i,j} g_{ij}(x^k,\,x^c)dx^i dx^j+\sum_{a,b} g_{ab}(x^k,\,x^c)dx^a dx^b\,,$$

where g_{ij} *and* g_{ab} *satisfy*

$$\frac{\partial}{\partial x^a}\det(g_{ij})=0\,,\qquad \frac{\partial}{\partial x^i}\det(g_{ab})=0$$

and the coordinates in $U\cap U'\neq\phi$ *relate with by* $x'^i=x'^i(x^j)$ *and* $x'^a=x'^a(x^b)$.

Recently, the present author [**5**] has discussed the Riemannian space which admits a harmonic simple p-form, where a harmonic form means a form u satisfying $du=0$ and $\delta u=0$.

If we replace "*submanifolds*" in **B′** by "*orientable submanifolds*", the proposition is equivalent to the following **E′** when M is orientable.

E′. *M admits a harmonic simple p-form of constant length.*

On the other hand, C.J. Hsu [**2**] has discussed the almost product structure and introduced its several special classes. The purpose of this paper is to find the proposition which is equivalent to **B′** and corresponds to **D**. The result is related with one of Hsu's special classes.

1. The coefficients of rotation of an orthogonal ennuple. Consider an n dimensional Riemannian space M and let $v_A=(v_A{}^\lambda)$ be an orthogonal ennuple, i.e., a local orthonormal base field. We define scalar functions γ_{ABC} by

$$\nabla_\lambda v_{A\mu}=\sum_{B,C}\gamma_{ABC}v_{B\lambda}v_{C\mu}\,, \tag{1.1}$$

where $v_{A\mu}=g_{\mu\lambda}v_A{}^\lambda$ and ∇_λ denotes the operator of the Riemannian covariant derivative. γ_{ABC} are called the coefficients of rotation of v_A, [**1**, p. 100]. They satisfy

$$\gamma_{ACB}+\gamma_{BCA}=0\,, \tag{1.2}$$

because of $v_{A\lambda}v_B{}^\lambda=\delta_{AB}$.

We shall identify v_A with the 1-form defined by $v_A = v_{A\lambda} dx^\lambda$ as far as the ambiguity does not occur. If the scalar functions C_{ABC} satisfying

(1.3) $$C_{ABC} + C_{ACB} = 0$$

are introduced by

$$dv_A = \frac{1}{2} \sum_{B,C} C_{ABC} v_B \wedge v_C ,$$

it is easily seen that the following relations

(1.4) $$C_{ABC} = \gamma_{ABC} - \gamma_{ACB}$$

hold good.

In [**5**] we have got the following

THEOREM. *In an n dimensional Riemannian space M, Proposition* $\mathbf{B}'$ *or* $\mathbf{C}'$ *holds good if and only if*

(1.5) $$C_{iab} = C_{aij} = 0 ,$$

(1.6) $$\sum_i C_{iia} = \sum_a C_{aai} = 0$$

are valid for an orthogonal ennuple v_A.

In this case, the field of the p-plane spanned by the vectors v_i is the involutive field in $\mathbf{B}'$, and its maximal integral manifold is given by $x^a = \text{const.}$ in $\mathbf{C}'$. The analogous facts for v_a are valid.

The conditions (1.5) and (1.6) are written in terms of γ_{ABC} as follows:

(1.7) $$\gamma_{iab} = \gamma_{iba} , \qquad \gamma_{aij} = \gamma_{aji} ,$$

(1.8) $$\sum_i \gamma_{aii} = \sum_a \gamma_{iaa} = 0$$

by virtue of (1.4) taking account of (1.2).

2. The almost product structure. Suppose that M admits an almost product structure $F_\mu{}^\lambda$. The Nijenhuis tensor $N(F)$ is defined by

$$N(F)_{\lambda\mu}{}^\kappa = F_\lambda{}^\varepsilon \nabla_\varepsilon F_\mu{}^\kappa - F_\mu{}^\varepsilon \nabla_\varepsilon F_\lambda{}^\kappa - F_\varepsilon{}^\kappa (\nabla_\lambda F_\mu{}^\varepsilon - \nabla_\mu F_\lambda{}^\varepsilon) .$$

Now, we assume that $N(F)_{\lambda\mu}{}^\kappa$ vanish identically. Then

(2.1) $$F_\lambda{}^\varepsilon \nabla_\varepsilon F_{\mu\omega} - F_\mu{}^\varepsilon \nabla_\varepsilon F_{\lambda\omega} - F_\omega{}^\varepsilon (\nabla_\lambda F_{\mu\varepsilon} - \nabla_\mu F_{\lambda\varepsilon}) = 0$$

hold good, where $F_{\mu\omega} = g_{\omega\alpha} F_\mu{}^\alpha$ is symmetric. Transvecting this with $F_\nu{}^\lambda$ we have

$$\nabla_\nu F_{\mu\omega} = F_\nu{}^\lambda F_\mu{}^\varepsilon \nabla_\varepsilon F_{\lambda\omega} + F_\nu{}^\lambda F_\omega{}^\varepsilon (\nabla_\lambda F_{\mu\varepsilon} - \nabla_\mu F_{\lambda\varepsilon}) .$$

Thus it follows by transvection with $g^{\mu\omega}$ that

$$\nabla_\nu F_\lambda{}^\lambda = F_\nu{}^\lambda F^{\omega\varepsilon} \nabla_\varepsilon F_{\lambda\omega} + F_\nu{}^\lambda F^{\omega\varepsilon} (\nabla_\lambda F_{\omega\varepsilon} - \nabla_\omega F_{\lambda\varepsilon}) = 0$$

hold good, by virtue of $F^{\omega\varepsilon} = F^{\varepsilon\omega}$ and $F^{\omega\varepsilon} \nabla_\lambda F_{\omega\varepsilon} = 0$. Hence

LEMMA 1. *If $N(F)_{\lambda\mu}{}^\kappa = 0$, we have trace $(F_\lambda{}^\mu) = constant$.*

By (0.2) the proper values of the matrix $(F_\lambda{}^\mu)$ are ± 1, and Lemma 1 shows that the multiplicities of the proper values ± 1 are constant. If we denote the multiplicity of $+1$ by p, it follows that $F_\lambda{}^\lambda = p-(n\text{-}p) = 2p-n$.

Let v_i and v_a be the proper vectors of $(F_\lambda{}^\mu)$ corresponding to $+1$ and -1 respectively. We can choose them such that they constitute an orthogonal ennuple v_A which will be called adapted to $F_\lambda{}^\mu$. Putting $F_{\lambda\mu} = \sum_{A,B} a_{AB} v_{A\lambda} v_{B\mu}$ and taking account of

$$F_\lambda{}^\mu v_i{}^\lambda = v_i{}^\mu\,, \qquad F_\lambda{}^\mu v_a{}^\lambda = -v_a{}^\mu\,, \tag{2.2}$$

we get

$$F_{\lambda\mu} = \sum_i v_{i\lambda} v_{i\mu} - \sum_a v_{a\lambda} v_{a\mu}\,. \tag{2.3}$$

If we substitute (2.3) into (2.1) and take account of (2.2) and (1.1), it follows after long computations that

$$\sum_{a,i,j} (\gamma_{aij} - \gamma_{aji}) v_{a\omega} v_{i\lambda} v_{j\mu} + \sum_{i,a,b} (\gamma_{iab} - \gamma_{iba}) v_{i\omega} v_{a\lambda} v_{b\mu} = 0$$

hold good by virtue of (1.2). Thus we obtain

LEMMA 2. $N(F)_{\lambda\mu}{}^\kappa = 0$ *if and only if*

$$\gamma_{aij} = \gamma_{aji}\,, \qquad \gamma_{iab} = \gamma_{iba}\,. \tag{2.4}$$

As each of (2.4) is the condition for the distribution of the vectors v_i or v_a to be integrable, Lemma 2 states a well known theorem about the Nijenhuis tensor for the case of the almost product structure, [**2**].

Next, we shall consider $\nabla^\lambda F_{\lambda\mu}$. By (2.3) we have

$$\nabla_\nu F_{\lambda\mu} = \sum_i (\nabla_\nu v_{i\lambda} v_{i\mu} + v_{i\lambda} \nabla_\nu v_{i\mu}) - \sum_a (\nabla_\nu v_{a\lambda} v_{a\mu} + v_{a\lambda} \nabla_\nu v_{a\mu})\,.$$

If we take account of

$$g^{\nu\lambda} \nabla_\nu v_{B\lambda} = \sum_C \gamma_{BCC}\,,$$

$$g^{\nu\lambda} v_{B\lambda} \nabla_\nu v_{B\mu} = -\sum_C \gamma_{CBB} v_C\,,$$

it follows that

$$\nabla^\lambda F_{\lambda\mu} = 2\Big(\sum_{i,a} \gamma_{iaa} v_{i\mu} - \sum_{i,a} \gamma_{aii} v_{a\mu}\Big)\,.$$

Thus we have

LEMMA 3. $\nabla^\lambda F_{\lambda\mu} = 0$ *if and only if*

$$\sum_a \gamma_{iaa} = 0\,, \qquad \sum_i \gamma_{aii} = 0 \tag{2.5}$$

for an adapted v_A.

Now let consider the following proposition:

D′. *There exists an almost product structure* $F_\mu{}^\lambda$ *satisfying*

$$N(F)_{\lambda\mu}{}^\kappa = 0\,, \qquad \nabla^\lambda F_{\lambda\mu} = 0\,,$$

$$F_{\lambda}{}^{\lambda}=2p-n\,, \qquad 0<p<n\,.$$

By the above arguments and Theorem in § **1** we know that Proposition $\mathbf{D}'$ implies $\mathbf{B}'$ and $\mathbf{C}'$.

Conversely, suppose that Proposition $\mathbf{B}'$ is valid. Then we define $F_{\lambda\mu}$ by (2.3) for an orthogonal ennuple v_A in Theorem of § **1**. By Lemma 2 and 3, the $F_{\lambda\mu}$ satisfies the conditions of Proposition $\mathbf{D}'$ because of (1.7) and (1.8).

Consequently we have proved the following

THEOREM. *Proposition* $\mathbf{B}'$, $\mathbf{C}'$ *and* $\mathbf{D}'$ *are equivalent to one another.*

C.J. Hsu has given an example in which $\mathbf{D}'$ holds good for $p=1$, [**2**, Remark 3.4].

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