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# A note on a Riemannian space with Sasakian 3-structure

**Toyoko Kashiwada** (柏 田 豊 子)

Department of Mathematics, Faculty of Science, Yamagata University
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## Introduction

Recently, S. Tachibana and W.N. Yu studied the Riemannian space admitting a Sasakian 3-structure. S. Tanno has conjectured that the space is one of constant curvature. In this paper, we shall prove that a Riemannian space with Sasakian 3-structure is an Einstein one.

## §1. Preliminaries.

Let $M$ be an $n$ dimensional Riemannian space whose metric tensor is given by $g_{ij}$. A unit Killing vector field $\xi$ in $M$ is called a Sasakian structure if it satisfies

(1.1) $$\nabla_i\phi_{jk}=\xi_j g_{ik}-\xi_k g_{ij},$$

where we have put

$$\phi_{jk}=\nabla_j\xi_k.$$

A Sasakian space is a Riemannian space which admits a Sasakian structure. In such a space we have

$$\phi_i{}^r\phi_r{}^j=-\delta_i^j+\xi_i\xi^j,$$

$$R_{mij}{}^r\xi_r=\xi_m g_{ij}-\xi_i g_{mj}.$$

Differentiating covariantly (1.1) and making use of Ricci's identity, we get

(1.2) $$-R_{mij}{}^r\phi_{rk}-R_{mik}{}^r\phi_{jr}=\phi_{mj}g_{ik}-\phi_{mk}g_{ij}-\phi_{ij}g_{mk}+\phi_{ik}g_{mj}.$$

On the other hand, the following equation is known [2].

(1.3) $$\phi^{il}R_{ijkl}=\phi_j{}^l R_{lk}-(n-2)\phi_{jk}.$$

## §2. Sasakian 3-structure.

A Sasakian 3-structure in a Riemannian space is a structure consisting of three Sasakian structures $\xi$, $\eta$, $\zeta$ which are orthogonal to each other.

If we put

$$\nabla_i\xi_j=\phi_{ij},\qquad \nabla_i\eta_j=\psi_{ij},\qquad \nabla_i\zeta_j=\theta_{ij},$$

the following relations are valid [1], [3].

$$\xi^i = \eta^r \theta_r^{\ i} = -\zeta^r \phi_r^{\ i} ,$$

$$\eta^i = \zeta^r \phi_r^{\ i} = -\xi^r \theta_r^{\ i} ,$$

$$\zeta^r = \xi^r \psi_r^{\ i} = -\eta^r \phi_r^{\ i} ,$$

$$\phi_i^{\ h} = \psi_i^{\ r} \theta_r^{\ h} - \eta_i \zeta^h = -\theta_i^{\ r} \psi_r^{\ h} + \zeta_i \eta^h ,$$

$$\psi_i^{\ h} = \theta_i^{\ r} \phi_r^{\ h} - \zeta_i \xi^h = -\phi_i^{\ r} \theta_r^{\ h} + \xi_i \zeta^h ,$$

$$\theta_i^{\ h} = \phi_i^{\ r} \psi_r^{\ h} - \xi_i \eta^h = -\psi_i^{\ r} \phi_r^{\ h} + \eta_i \xi^h .$$

Transvecting (1.2) with $\psi^{kj}$, we have

$$\theta^{jr} R_{mijr} + 2\theta_{mi} = 0 ,$$

i.e.

$$\theta^{jr} R_{jmir} - \theta_{mi} = 0 .$$

Hence, by means of an equation analogous to (1.3), we obtain

$$\theta_m^{\ r} R_{ri} - (n-1)\theta_{mi} = 0$$

which implies that

$$R_{ji} = (n-1) g_{ij} .$$

Thus the following theorem is proved.

THEOREM. *A Riemannian space with Sasakian 3-structure is an Einstein space.*

REMARK: Since the section spanned by $\eta$ and $\zeta$ is $\phi$-section and the sectional curvature is *1*, we know that if a Riemannian space with Sasakian *3*-structure is constant $\phi$-holomorphic sectional curvature corresponding to one of the three Sasakian structures, then it is a space of constant curvature.

## Bibliography

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