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On a Unit-boundary of a Function Algebra

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§ 1. Introduction

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex valued continuous functions on X with the sup-norm. A subalgebra A contained in $C(X)$ is called "*a function algebra on* X" if A satisfies the following three conditions:

1) The constant functions are in A,
2) A separates points on X,
3) A is closed under uniform convergence.

Let $M(A)$ be the space of maximal ideals of A, i.e. the space of all multiplicative linear functionals on A, with Gelfand's topology. Then $M(A)$ becomes a compact Hausdorff space and X is homeomorphically embedded in $M(A)$ as a closed subset. ([1][*1])

Let $Cho(A)$ be the set of all x in X which is an exteme point of $\{L \in A^* : L(1) = \|L\| = 1\}$ as a multiplicative linear functional. The set $Cho(A)$ is called the *Choquet boundary* of A and the closure of $Cho(A)$ in X is called the *Silov boundary* of A and is denoted by $\Gamma(A)$. ([2])

§ 2. Definition and examples

DEFINITION. *A closed subset F in $M(A)$ is called a unit-boundary of A if F satisfies the following condition: for a function f in A which does not attain the value 0 on F, there is a function g in A with* $f \cdot g = 1$.

According to the definition, $M(A)$ is a unit-boundary for every function algebra, A.

REMARK. The definition that F is a unit-boundary is also described as follows; *F is a closed subset of $M(A)$ with* $\{f(y) : y \in F\} = \{f(x) : x \in M(A)\}$ *for all f in A.*

We shall denote by $\mathfrak{F}$ the set of all unit-boundaries of A, then $\mathfrak{F}$ is not empty and becomes, as will be shown, the inductively ordered set

[*1]: the number in brackets refer to the paper in Reference.

for set-inclusion i.e. when $\{F_\alpha\,;\,\alpha\in\mathfrak{A}\}$ is any totally ordered set in $\mathfrak{F}$, then the set $\bigcap_{\alpha\in\mathfrak{A}} F_\alpha$ belongs to $\mathfrak{F}$.

Now to prove this, we assume that a function f in A does not attain the value 0 on $\cap F_\alpha$. Then the following two cases will occur.

The 1'st case: There is an index $\beta\in\mathfrak{A}$: $F_\beta\cap z^{*2}(f)=\phi$.

Since $F_\beta\supset\cap F_\alpha$ and $f(F_\beta)=\{f(y)\,;\,y\in F_\beta\}\not\ni 0$, f is invertible.

The 2'nd case: $F_\alpha\cap Z(f)\neq\phi$ for any index α in $\mathfrak{A}$.

We denote by $Z_\alpha(f)$ the set $Z(f)\cap F_\alpha$, and P the set of all cluster point of $\bigcup_{\alpha\in\mathfrak{A}} Z_\alpha(f)$. Since $P\cap F_\alpha\neq\phi$ and $\{F_\alpha\}$ is a totally ordered set, $\{P\cap F_\alpha\}_{\alpha\in\mathfrak{A}}$ has the finite intersection property. Therefore $P\cap(\cap F_\alpha)\neq\phi$, i.e. f attains the value 0 on $\cap F_\alpha$. This contradicts the assumption $f(\cap F_\alpha)\not\ni 0$. Thus we know that the second case can not occur. From the first case the set $\cap F_\alpha$ is in $\mathfrak{F}$.

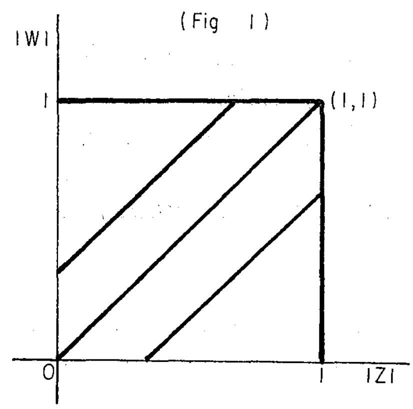
In general there is not the minimum unit-boundary in $\mathfrak{F}$, which will be shown in the following example.

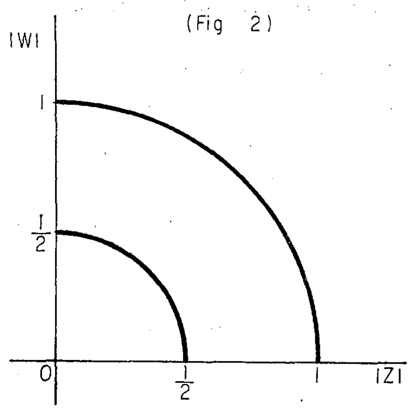
EXAMPLE 1. Let $X=\left\{(z,w):|z|\leqq 1,\,|w|\leqq 1\,;\,||z|-|w||\leqq\frac{1}{3}\right\}$ (Fig. 1)

$A=\{f\in C(X):$ f is holomorphic in the interior of $X(=X')$ and continuous on $X\}$.

Then A is a function algebra on X. Every function in A can be extended holomorphically to the set $\tilde{X}=\{(z,w):|z|\leqq 1,\,|w|\leqq 1\}$. ([3])([4])

$M(A)$ is the set $\tilde{X}$ and X is a proper subset in $M(A)$. Now if a function f in A is not 0 on X, then $1/f$ is also holomorphic in X. Therefore X is a unit-boundary. One of the minimal unit-boundary contained in X is the set $\{(z,w):|z|\leqq 1,\,|w|\leqq 1\,;\,|z|=|w|\}$. The other hand unit-boundary which is not contained in X is the set $\{(z,w):|z|=1$ or $|w|=1\}\cap\tilde{X}$.





On the other hand we can show by the following example that there is the minimum unit-boundary of A.

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*2: the set $Z(f)=\{x\in M(A)\,;\,f(x)=0\}$.

EXAMPLE 2. Let $X=\left\{(z, w) : \frac{1}{2}\leqq|z|^2+|w|^2\leqq 1\right\}$ (Fig. 2)

$A=\{f\in C(X) : f$ is holomorphic in X and continuous on $X\}$.

As in the example 1, any function in A can be extended holomorphically to the set $\tilde{X}=\{(z, w) : |z|^2+|w|^2\leqq 1\}$ ([3]) ([4]). $M(A)$ is $\tilde{X}$ and X is contained in $M(A)$ properly. As in the example 1, X is a unit-boundary. Let F_δ be $\{(z, w) : \delta\leqq|z|^2+|w|^2\leqq 1\}$ $(0\leqq\delta<1)$. For each δ, F_δ is a unit-boundary. Therefore $\bigcap F_\delta=\{(z, w) : |z|^2+|w|^2=1\}$ is also a unit-boundary, and $\bigcap F_\delta$ is the Silov boundary of A by the theorem of maximum modulus principle. By the theorem in § 3 of this paper that every unit-boundary contains the Silov boundary, $\bigcap F_\delta$ is the minimum unit-boundary.

The following example shows us that there is a function algebra of which $M(A)$ is the only unit-boundary.

EXAMPLE 3. Let $X=\{z : |z|=1\}$

$$A=\{f\in C(X) : \int_{-\pi}^{\pi} f(e^{i\theta})e^{in\theta}\,d\theta=0,\ n=1, 2, \cdots\}$$

This function algebra is often called *disk algebra*, and any function in A can be extended holomorphically to the set $\{z : |z|\leqq 1\}$. So $M(A)$ is the set $\{z : |z|\leqq 1\}$ and $M(A)$ is the only unit-boundary, because the function z is contained in A, regarded as a subalgebra $C(M(A))$.

§ 3. Some properties of a unit-boundary

THEOREM. *Every unit-boundary of a function algebra A on X always contains the Silov boundary $\Gamma(A)$.*

PROOF. Let x_0 be an element of $\Gamma(A)$. By the definition, the Silov boundary is the closure of the Choquet boundary of A. The following theorem is due to E. Bishop and K. de Leeuw ([5]):

"Let x be an element of X. Then the following conditions are equivalent;

(1) x is an element of the Choquet boundary of A,

(2) for each nbhd U of x and each positive number $\varepsilon>0$, there is some function f in A with $|f|\leqq 1$, $|f(x)|>1-\varepsilon$, $|f(y)|<\varepsilon$ for all y in $U-X$."

For each nbhd U of x_0, there is some x_1 in the Choquet boundary. Then for x_1 and nbhd U, there is some f in A with $|f|\leqq 1$, $|f(x)|>1-\varepsilon$, $|f(y)|<\varepsilon$ for all y in $X-U$. Now $1-f$ is an element of A and $(1-f)(y)\neq 0$ for all y in $X-U$, then $X-U$ is not contained in a unit-boundary F i.e. $U\cap F\neq\phi$. Therefore any nbhd U of x_0 intersects with F, i.e. x_0 is in F.

COROLLARY 1. *Let A be a function algebra on X. If A is a maximal subalgebra of $C(M(A))$, then $M(A)$ is the only unit-boundary of A.*

PROOF. If there is a unit-boundary F except $M(A)$, then we choose a point x_0 in $M(A)-F(\neq\phi)$. As $M(A)$ is normal, there exists some nbhd $U(x_0)$ such that $U(x_0)\cap F=\phi$.

Now we can consider the following two cases:

The 1'st case; the point x_0 is an isolated point. Then we know that there is a function g in A with $g\in C(M(A))$, $g(x_0)=0$, $g(y)=1$ for all y in $M(A)-\{x_0\}$ ([6]). The existence of the function g contradicts the assumption that F is a unit-boundary.

The 2'nd case; the point x_0 is a cluster point. Then in the nbhd $U(x_0)$ there is a point x_1 different from x_0. As $M(A)$ is a Hausdorff space, there are some nbhd $V(x_0)$, $V(x_1)$ with $V(x_0)\cap V(x_1)=\phi$, $V(x_0)\subset U(x_0)$.

Now we can find the function f in $C(M(A))$ with $\|f\|\leqq 1$, $f(x_0)=1$, $f(y)=0$ for all y in $M(A)-V(x_0)$. Since $M(A)-V(x_0)$ contains F and the unit-boundary F contains the Silov boundary by the theorem proved above, the function f can not belong to A. As A is a maximal subalgebra of $C(M(A))$, the generated function algebra B by f and A coincides with $C(M(A))$, i.e.

$$B=\ll f, A\gg=\{\sum_{n=1}^{\infty}a_nf^n\,;\,a_n\in A\}=C(M(A)).$$

Again we can find a function h in $C(M(A))$ with $\|h\|\leqq 1$, $h(x_1)=1$, $h(y)=0$ for all y in $M(A)-V(x_1)$.

As $B=C(M(A))$, the function h is represented as $a_0+\sum_{n=1}^{\infty}a_nf^n$.

Restricting h to $M(A)-V(x_0)$, $h|M(A)-V(x_0)$ is equal to $a_0|M(A)-V(x_0)$.

By $F\subset M(A)-V(x_0)$, h is identical with a_0 on E. So a_0 is 0 on E.

Since a_0 is a function in A and F contains $\Gamma(A)$, a_0 is the constant 0 on $M(A)$. Consequently h is 0 on $M(A)-V(x_0)$, but this contradicts the constraction of the function h.

Therefore $M(A)$ is the only unit-boundary of A.

The inverse of the corollary 1 "if $M(A)$ is the only unit-boundary of A, then A is maximal in $C(M(A))$" fails to hold in general. The example 3 is a counter example for this.

COROLLARY 2. *If a function algebra A on X is a log-modular (or Dirichlet) algebra on $M(A)$, then $M(A)$ is the only unit-boundary of A.*

To give the proof, we require some concepts; ([2], [7])

* Let $C_R(X)$ be the set of all functions which are continuous real valued functions in A, $Re(A)$ the set of all functions in $C_R(A)$ which are

real parts of some functions in A, A^{-1} the set of all functions in A which are invertible.

** A function algebra A is called *Dirichlet algebra* on X if $Re(A)$ is dense in $C_R(X)$ under uniform norm.

*** A function algebra A on X is called *log-modular algebra* on X if the set $\{\log |f| ; f \in A^{-1}\}$ is dense in $C_R(X)$ under uniform norm. According to this a Dirichlet algebra is a log-modular algebra.

PROOF. Since by K. Hoffman ([7]), in a log-modular algebra on X, the representing measure on X of a point p in $M(A)$ is unique and by applying the above result to the Chouet boundary of A, ([5]), X is the Silov boundary. As $X=M(A)$ in this corollary, $M(A)$ is the Silov boundary. By our theorem, $M(A)$ is the only unit-boundary.

In the corollary 2, the assumption "on $M(A)$" is necessary. In fact we can show by an example that a function algebra A which is a Dirichlet algebra on X may have a unit-boundary except $M(A)$.

EXAMPLE 4. Let $S^2 = \boldsymbol{C}^{*3} \cup \{\infty\}$ be the extended plane. A function defined on S^2 is called analytic at ∞ if the function $z \to f(1/z)$ is analytic at the origin 0, or equivalently, if f is holomorphic and bounded on some deleted nbhd $\{z ; |z| > \frac{1}{\varepsilon}\ \varepsilon > 0\}$ of ∞.

Let $X = \{z ; |z| \leqq 1\}$, $A = \{f \in C(X) ; f$ can be extended holomorphically to $S^2 - X\}$.

Then A is a function algebra on X.

Now we show two facts; the first is that A is a Dirichlet algebra on X and the second is that X is proper subset of $M(A)$ and a unit-boundary of A.

The 1'st: It is well known that the "disk algebra" in the example 3 is a Dirichlet algebra. ([7]) Thus the restriction A to $C_1 = \{z ; |z| = 1\}$ is a Dirichlet algebra under the transformation: $z \to 1/z$.

Let f be a function in $C_R(X)$. As $A|C_1$ is a Dirichlet algebra, there is a function g in A such that $|Re(g)(y) - f(y)| < \varepsilon$ on C_1.

Since $Re(g)$ and f are uniformly continuous on X, there is some nbhd $U_\delta = \{z : 1 - |z| < \delta\}$ on which $|Re(g)(y) - f(y)| < \varepsilon$.

There is a function h in A with $h \in C(X)$, $h(y) = 0$ on C_1, $Re(y) = (f - Re(g))(y)$ in $U_{\frac{\delta}{2}}$ and $0 \leqq |h(y)| \leqq \varepsilon$ in $U_\delta - U_{\frac{\delta}{2}}$. Then $h+g$ belongs to A and $|Re(h+g)(y) - f(y)| < 2\varepsilon$ for all y in X. Therefore $Re(A)$ is dense in $C_R(X)$ i.e. A is a Dirichlet algebra on X.

*3: $\boldsymbol{C}$ is the plane of complex numbers.

The 2'nd: Since every function f in A has the holomorphic extension f in S^2-X, S^2 is contained in $M(A)$. Accordingly X is a proper subset of $M(A)$. Now we shall show that X is a unit-boundary.

To do this, we shall make use of the concept of the variation of the logarithm of a continuous function along a closed curve.

For a continuous function ϕ on a closed interval $[a, b]$, which does not vanish on that interval, a *continuous logarithm* of ϕ is defined as a function ν, continuous on $[a, b]$, with $\phi=e^{\nu}$. Because of the uniform continuity of ϕ and the fact that the exponential function has a local continuous inverse, we know that for any continuous function on X such that $[a, b]\rightarrow \boldsymbol{C}-\{0\}$, the logarithm function is continuous. If γ is a continuous function on $[a, b]$ and f is continuous and nowhere 0 on the curve $C=\gamma\ ([a, b])$, then the *variation of the logarithm of f along C* is defined to be $\nu(b)-\nu(a)$, where ν is any continuous logarithm of f.

LEMMA. *Let $\varphi: \boldsymbol{C}\rightarrow \boldsymbol{C}-\{0\}$ be continuous. For each $r>0$, let $V(r)$ be the variation of the logarithm of φ along the circle $C_r=\{z: |z|=r\}$. Then $V(r)=0$.*

For the proof, see G.M. Leibowitz [8].

By this lemma, we can prove that X is a unit-boundary. Suppose there is a function f in A which is nowhere 0 on X and at some points in S^2-X, f is 0, regarding f as a function on S^2.

$Z(f)$ in S^2 is a non empty finite set i.e. $Z(f)=\{z_0, z_1, \cdots, z_n\}$ (repeated according to their multiplicities), since $Z(f)\cap X=\phi$ and f is holomorphic in S^2-X.

Let g be the function $(z-z_0)^{-1}(z-z_1)^{-1}(z-z_2)^{-1}\cdots(z-z_n)^{-1}f(z)$. Then g has no zero on $\boldsymbol{C}$, $g(\infty)=0$ and is holomorphic in S^2-X.

By our lemma, the variation of log g along each C_r is 0. Since g is analytic at ∞, by simple calculation we see $V(r)=2n\pi i$ for all sufficiently large r (the integer n is the order of the zero of g at ∞), which is a contradiction. Now we know that a function f which is nowhere 0 on X is nowhere 0 on S^2, so that f is invertible i.e. X is a unit-boundary. (in the example 4 we owe the 2nd part completely to G. Leibowitz [8]).

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