Natural Science Report, Ochanomizu University, Vol. 22, No. 1, 1971 17
お茶の水女子大学 自然科学報告 第22巻 第1号

On the Choquet Boundaries of Locally Compact Spaces

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(Received April 10, 1971)

§1. Preliminaries

In [1], [2], Prof. H. Bauer has proved the existence of the Choquet boundary with respect to a separating family ξ of lower semicontinuous functions on a compact Hausdorff space Ω and proved that particularly for a wedge ξ, the closure of the Choquet boundary is equal to the Silov boundary. Prof. G. Mokobodzki and Prof. D. Sibony have proved, for a locally compact, σ-compact Ω, the existence of the Choquet boundary with respect to a convex cone ξ with some properties and proved that the closure of the Choquet boundary is the smallest closed determining set by applying Choquet's theorem concerning adapted spaces.

In this paper we shall show that the same conclusions will be obtained for a locally compact, paracompact Ω and a wedge ξ, according to Mokobodzki and Sibony [5].

§2. Adapted spaces

Let Ω be a locally compact space and $C(\Omega)$ be the vector space of all continuous real-valued functions on Ω. We say, according to Choquet, that a vector subspace $V\subset C(\Omega)$ is adapted if it satisfies the following conditions;

1° $V=V^+-V^+$ where $V^+=V\cap C^+(\Omega)$,

2° for any $x\in\Omega$, there exists $v\in V$ such that $v(x)>0$,

3° for any $v\in V^+$, there exists $w\in V^+$, $w\geqq v$ such that for any $\varepsilon>0$, there exists a compact $K\subset\Omega$ such that

$$x\in CK\Rightarrow v(x)\leqq\varepsilon w(x)\,.$$

Then, we can state the following theorem according to Choquet.

THEOREM 1. *Let $V\subset C(\Omega)$ be an adapted vector space. For any positive linear from T, there exists a measure $\mu\geqq 0$ on Ω such that*

(i) *any $v\in V$ is μ-integrable*

and (ii) *for all* $v \in V$ $T(v) = \int v d\mu$ *holds.*

Let f be a non-negative function on Ω, g be a function on Ω. We call f dominates g if for any $\varepsilon > 0$, there exists a compact K such that

$$x \in CK \Rightarrow g(x) \leqq \varepsilon f(x) .$$

For any $f \geqq 0$ on Ω, we denote by $0(f)$ the set of all functions dominated by f.

THEOREM 2. *Let Ω be a locally compact, paracompact*[1] *Hausdorff space. Then for any function $g \geqq 0$ on Ω, $0(g) \cap C(\Omega)$ is an adapted space.*

PROOF. Obviously $0(g)$ is a vector space. For any $f \in 0(g)$, it is able to write $f = f^+ - f^-$, where $f^+ \in 0(g)$ and $f \in 0(g)$. Since any continuous function on Ω with compact support is included in $0(g)$, the conditions 2° is satisfied. Let $V(x)$ be a relatively compact neighborhood for $x \in \Omega$. Then holds $\Omega \subset \bigcup_{x \in \Omega} V(x)$. Since Ω is paracompact, there exists a locally finite open covering $\{U_\alpha\}$ of a refiment of $\{V(x)\}_{x \in \Omega}$. For any $f \in 0(g)$, $f \geqq 0$, we shall construct by mathematical induction a sequence $\{K_n\}$ of compact subsets. Since $f \in 0(g)$, there exists a compact K such that

$$x \in CK \Rightarrow f(x) \leqq \frac{1}{2} g(x) .$$

We put $K = K_1$. Assume that we have defined $K_1, \cdots K_n$. Since $f \in 0(g)$ there exists a compact K such that

$$x \in CK \Rightarrow f(x) \leqq \frac{1}{2^{n+1}} g(x) .$$

Then $K_{n+1} = \bigcup \{\bar{U}_\alpha ; U_\alpha \cap (K \cup K_n) \neq \phi\}$ is also compact, $K^i_{n+1} \supset K_n$ and

$$x \in CK \Rightarrow f(x) \leqq \frac{1}{2^{n+1}} g(x) .$$

Put $\Omega_0 = \bigcup_{n=1}^{\infty} K$. Then Ω_0 is closed and open. Indeed, for any $x \in \Omega_0$ there exists a compact $K_n \in x$ and $\Omega_0 \supset K^i_{n+1} \supset K_n \in x$. Hence Ω_0 is open. Further, for any $x_0 \in \bar{\Omega}_0$ there exists a open set U_α with $U_\alpha \ni x$ and $U_\alpha \cap \Omega_0 \neq \phi$. Then there exists a natural number n such that $K_n \cap U_\alpha \neq \phi$. From the construction of $\{K_n\}$, we have $K_{n+1} \supset U_\alpha \ni x$. Hence $x \ni \Omega_0$. Moreover it is clear that

1) We say that a topological space Ω is paracompact if for any open covering $\{U_\alpha\}$ of Ω, we can find a locally finite open covering $\{V_\beta\}$ of Ω to be a refiment of $\{U_\alpha\}$, where we call covering $\{V_\beta\}$ locally finite if for any $x \in \Omega$ there exists a neighborhood V_x such that $\{\beta ; V_x \cap V_\beta\} \neq \phi$ is finite.

$$x \in C\Omega_0 \Rightarrow f(x) = 0 .$$

Let (φ_n) be a sequence of elements of $C^+(\Omega)$ such that

(a) $0 \leqq \varphi_n \leqq 1$
(b) $\varphi_n(x) = 0$ if $x \in K_n \cup C\Omega_0$,
(c) $\varphi_n(x) = 1$ if $x \in CK_n \cap \Omega_0$.

Since Ω is locally compact Hausdorff and Ω_0 is closed-open, we can find (φ_n) satisfying the above conditions. Putting $f' = \sum \varphi_n f$, we find

$$x \in \Omega_0 \cap CK_n \Rightarrow f'(x) \leqq \frac{n+1}{2^n} g(x) .$$

Hence for any $\varepsilon > 0$, there exists a compact $K \subset \Omega_0$ such that

$$x \in CK \Rightarrow f'(x) \leqq \varepsilon g(x) .$$

Hence $f' \in 0(g)$. Since f' is continuous on Ω_0 and Ω_0 is closed-open, we have

$$x \in \Omega_0 \cap CK_{n+1} \Rightarrow nf(x) \leqq f'(x) .$$

Hence $f \in 0(f')$.

We put $0(F) = \bigcup_{f \in F} 0(f)$ for a family F of functions $f \geqq 0$ on Ω.

COROLLARY. *Let Ω be a locally compact, paracompact Hausdorff space and ξ be a family of non-negative functions on Ω satisfying the following conditions;*

$$f, g \in \xi \Rightarrow f + g \in \xi .$$

Then $0(\xi)$ is an adapted space.

§ 3. The Choquet boundaries

Let Ω be a locally compact Hausdorff space and ξ be a family of lower semicontinuous functions from Ω to $\mathbf{R} \cup \{+\infty\}$. We say $x \in \Omega$ a boundary point of Ω with respect to ξ if the following condition is satisfied;

$$(\mu \in \mathfrak{M}^+(\Omega), \int |v| d\mu < +\infty, \int v d\mu \leqq v(x), \forall v \in \xi) \Rightarrow \mu = \varepsilon_x .$$

Then the Choquet boundary with respect to ξ, denoted by $\delta(\xi)$ is the set of all boundary points of Ω with respect to ξ. A closed subset $F \subset \Omega$ is called stable if for any $x \in F$ and any measure $\mu \geqq 0$ on Ω such that

$$\int v d\mu \leqq v(x) \qquad \forall v \in \xi ,$$

μ is supported by F.

EXAMPLE. If $u \in \xi$ has a compact support, the set $\{y ; \inf_{x \in \Omega} u(x) = u(y)\}$ is a non-empty closed stable set.

Let v be a lower semicontinuous non-negative function satisfying

$$\int v d\mu \leqq v(x)$$

for any $\mu \geqq 0$ on Ω and $x \in \Omega$ such that

$$\int |u| d\mu < +\infty \quad \text{and} \quad \int u d\mu \leqq u(x) \quad \forall u \in \xi .$$

Then v is called a ξ-concave function. We denote by $\hat{\xi}$ the set of all ξ-concave functions.

EXAMPLE. If $u \in \hat{\xi}$ and the set $F = \{y : \inf_{x \in \Omega} u(x) = u(y)\}$ is not empty, the F is a closed stable set.

Obviously we have

LEMMA 1. *Any intersection of closed stable sets is closed stable and any compact stable set includes a minimal compact stable set.*

We suppose that for any $x \in \Omega$, there exists $v \in \xi$ such that $0 < v(x) < +\infty$. We denote by Ω^+ the set $\Omega^+ = \bigcap_{v \in \xi} \{x \in \Omega ; v(x) \geqq 0\}$. Then we have evidently

LEMMA 2. *For a point $x \in C\Omega^+$, the following two conditions are equivalent;*

(i) *x is a boundary point with respect to ξ,*

(ii) *the set $\{x\}$ is a minimal compact stable set.*

THEOREM 2. *Let Ω be a locally compact Hausdorff space and ξ be a family of lower semicontinuous functions on Ω with compact supports satisfying the following conditions;*

(i) *for any $x \in \Omega$ there exists a $u \in \xi$ such that $0 < u(x) < \infty$,*

(ii) *ξ separates the point of Ω,*

(iii) $\xi \neq \xi^+$.[1)]

Then $\delta(\xi)$ is not empty and we have

$$(u \in \xi,\ u(x) \geqq 0,\ \forall x \in \delta(\xi)) \Rightarrow (u(x) \geqq 0,\ \forall x \in \Omega) .$$

PROOF. From $\xi \neq \xi^+$, there exists a $u \in \xi$ and $x_0 \in \Omega$ such that $u(x_0) < 0$. Then the set $K = \{y \in \Omega ; \inf_{x \in \Omega} u(x) = u(y)\}$ is a compact stable set. Therefore K includes a minimal compact stable set K_0. Suppose that K_0 is not reduced to one point. Since ξ separates the points of K, there exists $v \in \xi$, such that $K_1 = (y \in K_0 ; \inf_{x \in K_0} v(x) = v(y)\} \subsetneqq K_0$. Then K_1 is also a compact stable set which contradicts that K_0 is minimal. Therefore, we have $K_0 = \{x_1\}$. From $x_1 \in C\Omega^+$, by lemma 2, x_1 is a

1) We denote by ξ^+ the set $\{f \in \xi ; f \geqq 0\}$.

boundary point and $\delta(\xi)\neq\phi$, which shows also that if $u(x)<0$, $\{y\in\Omega, u(y)<0\}\cap\delta(\xi)\neq\phi$. Therefore,

$$(u\in\xi,\ u(x)\geqq 0,\ \forall x\in\delta(\xi))\Rightarrow(u(x)\geqq 0,\ \forall x\in\Omega)\,.$$

We consider the existence of the Choquet boundary for a family ξ of functions with supports not necessarly compact. We denote by $\inf(\xi, 0)$ the set $\{\inf(f, 0)\,;f\in\xi\}$. We say that ξ^+ linearly separates Ω is for any $x\neq y\in\Omega$ and for any $\lambda\geqq 0$, there exists $u\in\xi^+$ such that $u(x)\neq\lambda u(x)$.

THEOREM 3. *Let Ω be a locally compact, paracompact Hausdorff space and ξ be a family of lower semicontinuous functions on Ω satisfying the following conditions;*

(i) $\xi\ni u,\ -\infty<u(x)<+\infty,\ \forall x\in\Omega,$
(ii) $\xi\ni u,\ v\Rightarrow u+v\in\xi$
(iii) ξ^+ *linearly separates* Ω,
(iv) $\inf(\xi, 0)\subset 0(\xi^+)$ *and* $\xi\neq\xi^+$.

Then $\delta(\xi)$ is not empty and satisfies

$$(v\in\xi,\ v(x)\geqq 0,\ \forall x\in\delta(\xi))\Rightarrow(v(x)\geqq 0,\ \forall x\in\Omega)\,.$$

The proof will be established in several steps.

LEMMA 3. *For any $x_0\in\Omega$ there exists a $h\in\hat{\xi}$ such that $h>0$ on Ω and $h(x_0)<+\infty$.*

PROOF. For any $x\in\Omega$ there exists a relatively compact neighbourhood $V(x)$. Since $\Omega\subset\bigcup_{x\in\Omega}V(x)$ and Ω is paracompact, there exists a locally finite open covering $\{U_\alpha\}$ to be a refiment of the covering $\{V(x)\}$. We put Ω_0 the set of all the points of $y\in\Omega$ such that there exists finite subsets $\{U_{\alpha_j}\}_{1\leqq j\leqq n}\subset\{U_\alpha\}$ with $x_0\in U_{\alpha_0}$, $U_{\alpha_{j-1}}\cap U_{\alpha_j}\neq\phi$ $(j=1, 2, \cdots, n)$, $U_{\alpha_n}\in y$. Then Ω_0 is an open and closed subset and we can write $\Omega_0=\bigcup K_n$ where K_n is compact and $K_n^i\subset K_{n+1}$, $\bigcap_{n=1}^{\infty}K_n\ni x$. Since ξ^+ linearly separates Ω, for any $x\in\Omega$, there exists $f\in\xi^+$ such that $0<f(x)<+\infty$. Hence we may find $f_n\in\xi^+$ such that $0<f_n(y)<+\infty$ for any $y\in K_n$. Accordingly we may find $g_n\in\hat{\xi}$ such that $g_n(x_0)<1/2^n$ and $0<g_n(y)$ for any $y\in K_n$. We put

$$h_n(x)=\begin{cases} g_n(x) & \text{if}\quad x\in\Omega_0,\\ +\infty & \text{if}\quad x\in C\Omega_0.\end{cases}$$

Since Ω_0 is a closed and open set, h_n is lower semicontinuous and $h_n\in\hat{\xi}$. Putting $h=\sum_{n=1}^{\infty}h_n$, we have $h\in\hat{\xi}$, $h(x_0)<+\infty$ and $h>0$ on Ω.

LEMMA 4. *For any* $v\in\xi$, $v\notin\xi^+$, *there exists a compact stable* $K\subset\{x\in\Omega\,;\ v(x)<0\}$.

PROOF. Let $v(x_0)<0$. By lemma 3 there exists a $w\in\xi$ such that $w>0$ and $w(x_0)<+\infty$. From inf $(v, 0)\in 0(\xi^+)$, we may suppose inf $(v, 0)\in 0(w)$. Put

$$\lambda=\{\sup\alpha\,;\ \alpha>0,\ -\alpha v\leqq w\}$$

and $u=w+\lambda v$, then $u\in\hat{\xi}$ and the set $u^{-1}(0)$ is not empty and is compact stable. Further

$$u^{-1}(0)\subset\{x\in\Omega\,;\ v(x)<0\}$$

which proves our lemma.

Similarly we shall be able to prove the following lemma ;

LEMMA 5. *Let F be a closed stable subset of Ω such that* $\{x\in\Omega\,;\ v(x)<0\}\cap F\neq\phi$, *with* $v\in\xi$. *Then there exists a compact stable* $K\subset\{x\in\Omega\,;\ v(x)<0\}\cap F$.

LEMMA 6. *Any compact stable minimal K with $K\notin\Omega^+$ is reduced to one point.*

PROOF. Let $x\in K$, $v\in\xi$ such that $v(x)<0$. Suppose that there exists $x_1\in K$, $x_1\neq x$, then we shall show that there exists a function $u\in\xi$ satisfying the following property ;

$$(*)\qquad u(x)>0\ \text{ and }\ u(x_1)\geqq 0\ \text{ or }\ u(x_1)<0\ \text{ and }\ u(x)\geqq 0\,.$$

Indeed, if $v(x_1)\geqq 0$, we may put $u=v$. If $v(x)<0$, there exists a $w\in\xi^+$ such that $w(x)\neq(v(x)/v(x))w(x)$. We may consider the case where $w(x)<(v(x)/v(x))w(x)$. Then we can find positive intergers n, m such that

$$-\frac{w(x)}{v(x)}<\frac{m}{n}<-\frac{w(x_1)}{v(x_1)},$$

and we get $mv(x)+nw(x)<0$, $mv(x_1)+nw(x_1)>0$. Put $u=mv+nw$, we have $u\in\xi$ and $u(x)<0$, $u(x_1)>0$. Therefore there exists a function $u\in\xi$ with the property $(*)$. Accordingly $\{y\,;\ u(y)<0\}\cap K\subsetneqq K$. From lemma 5 there exists a compact stable set $K_1\subset\{y\,;\ u(y)<0\}\cap K$. This contradicts that K is minimal.

From lemma 3, 4, 5, 6, Theorem 3 is proved.

THEOREM 4. *Let Ω be a locally compact, paracompact Hausdorff space, and ξ be a family of lower semicontinuous functions on Ω satisfying the conditions of Theorem 3. Then $\overline{\delta(\xi)}$ is the smallest closed set F having the following condition ;*

$$(*)\qquad (v\in\xi,\ v(x)\geqq 0,\ \forall x\in F)\Rightarrow(v(x)\geqq 0,\ \forall x\in\Omega)\,.$$

PROOF. From theorem 3, $\overline{\delta(\xi)}$ satisfies $(*)$ condition. If we prove the following lemma, for any $x \in CF$ there exists a positive measure μ such that $S_\mu \subset F$ and

$$\int v d\mu \leqq v(x), \ \forall v \in \xi .$$

Since $\mu \neq \varepsilon_x$, we have $x \notin \delta(\xi)$. Hence $\delta(\xi) \subset F$. Therefore $\overline{\delta(\xi)}$ is the smallest closed set satisfying $(*)$ condition.

LEMMA 7. *Under the same condition with theorem 4, there exists, for any $x \in \Omega$, a measure $\mu \geqq 0$ on Ω supported by F such that*

$$\int v d\mu \leqq v(x), \ \forall v \in \xi .$$

PROOF. Let $x_0 \in \Omega$. $H = 0(\xi^+) \cap C(F)$ is an adapted space on F by theorem 1. For any $\varphi \in H$, we put

$$P(\varphi) = \inf \{v(x_0) ; \ v \in \xi, \ v \geqq \varphi \text{ on } F\} .$$

Then, P is a sublinear form on H. By Hahn-Banach's extention theorem there exists a linear form on H such that $T \leqq P$. It is clear that T is positive. Since H is an adapted space, there exists a positive measure μ on F such that

$$\int \varphi d\mu = T(\varphi) \leqq P(\varphi) .$$

For any $v \in \xi$ we have

$$\int v d\mu = \sup_{\substack{\varphi \leqq v \\ \varphi \in H}} \int \varphi d\mu .$$

Hence

$$\int v d\mu \leqq v(x_0), \ \forall v \in \xi .$$

In the same way we may prove also the following characterization of $\delta(\xi)$.

PROPOSIOION. The necessary and sufficient condition for $x \in \Omega$ to be an element of $\delta(\xi)$ is that

$$\varphi(x) = \inf \{f(x) ; \ f \in \xi, \ f \geqq \varphi\}$$

for any continuous function φ on Ω with compact support.

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