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An Integral Representation of sin x/x and Its Applications

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Use of an integral representation of $\sin x/x$ and its related formulas simplifies greatly the derivation of the solutions of wave equations.

§ 1. Initial value problems in wave propagation

In an infinite space of dimension n, the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}=\sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_k^2} \tag{1}$$

with the initial condition

$$(u)_{t=0}=f(x_1,\cdots,x_n)=f(\mathbf{r}), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x_1,\cdots,x_n)=g(\mathbf{r}) \tag{2}$$

is given, as is well known, by the superposition of two Fourier integrals

$$u(\mathbf{r},t)=u_1(\mathbf{r},t)+u_2(\mathbf{r},t) \tag{3}$$

$$u_1(\mathbf{r},t)=\frac{1}{(2\pi)^n}\int d\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{r}}\cos qt\int d\mathbf{r}'f(\mathbf{r}')e^{-i\mathbf{p}\cdot\mathbf{r}'} \tag{4}$$

$$u_2(\mathbf{r},t)=\frac{1}{(2\pi)^n}\int d\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{r}}\frac{\sin qt}{q}\int d\mathbf{r}'g(\mathbf{r}')e^{-i\mathbf{p}\cdot\mathbf{r}'} \tag{5}$$

where each of the integrations ranges over the whole space $\mathbf{r}$ or the whole momentum space $\mathbf{p}$, q standing for the square root of $\mathbf{p}^2$,

$$q=(\mathbf{p}^2)^{1/2}.$$

The assumption of reversibility of the order of integrations leads to the integrals

$$\frac{1}{(2\pi)^n}\int\cos qt\cdot e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{p} \tag{6}$$

$$\frac{1}{(2\pi)^n}\int\frac{\sin qt}{q}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{p}\,. \tag{7}$$

The evaluation of these integrals is difficult.

A usual method consists of splitting the integration into one of the angular part and one of the radial part, performing them successively. To overcome the difficulty encountered at the integration of the radial part, one needs to rely on the Dirichlet integral and certain devices. Since (6) is derived by differentiating (7) with respect to t, it suffices to evalute the integral (7).

§ 2. An integral representation of $\sin x/x$

The difficulty encountered at the evaluation of the integral (7) lies in the function $\sin qt/q$, $q=(\mathbf{p}^2)^{1/2}$, where lurks the square root $(\mathbf{p}^2)^{1/2}$. If one could factorize the function $\sin qt/q$ into the product of n functions, each depending on one variable p_k only, the evaluation would be immediate.

However it is impossible. To achieve the same effect, one may use the integral representation of $\sin x/x$

$$\frac{\sin x}{x}=\frac{\Gamma(3/2)}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\left(z-\frac{x^2}{4z}\right)}\cdot\frac{dz}{z^{3/2}},\qquad c>0\,. \tag{8}$$

The substitution of qt for x factorizes $\exp\left(-\frac{q^2}{4z}t^2\right)$ into the product of $\exp(-p_k^2t^2/4z)$, $k=1, 2, \cdots, n$.

It is more convenient to use the formula

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left(Az-\frac{B}{4z}\right)\cdot\frac{dz}{z^{3/2}}=\frac{\sin\sqrt{AB}}{\sqrt{B}}\cdot\frac{1}{\Gamma(3/2)}\epsilon(A)$$

or, more generally

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp\left(Az-\frac{B}{4z}\right)\cdot\frac{dz}{z^{\nu+1}}=\frac{J_\nu(\sqrt{AB})}{(AB)^{\nu/2}}(2A)^\nu\epsilon(A) \tag{9}$$

where the function $\epsilon(x)$ is defined by

$$\epsilon(x)=\begin{cases}1 & x>0\\ 0 & x<0\,.\end{cases}$$

One gets then the representation

$$\frac{\sin qt}{q}=\frac{\Gamma(3/2)}{2\pi i}\int\exp\left(zt^2-\frac{\mathbf{p}^2}{4z}\right)\cdot\frac{dz}{z^{3/2}} \tag{10}$$

by replacing A and B by t^2 and $\mathbf{p}^2$ respectively. With $\mathbf{s}$ standing for $\mathbf{r}'-\mathbf{r}$, the integral (7) becomes then

$$(7)=\frac{1}{(2\pi)^n}\int\frac{\sin qt}{q}e^{-i\mathbf{p}\cdot\mathbf{s}}d\mathbf{p}$$

$$=\frac{\Gamma(3/2)}{(2\pi)^n}\frac{1}{2\pi i}\int\exp(zt^2)\cdot\frac{dz}{z^{3/2}}\int\exp\left(-\frac{1}{4z}\mathbf{p}^2-i\mathbf{p}\cdot\mathbf{s}\right)d\mathbf{p}$$

$$=\frac{\Gamma(3/2)}{(2\pi)^n}\frac{1}{2\pi i}\int \exp z(t^2-s^2)\cdot(4\pi)^{n/2}z^{(n-3)/2}dz$$

$$=\frac{\Gamma(3/2)}{\pi^{n/2}}\frac{1}{2\pi i}\int \exp z(t^2-s^2)\cdot z^{(n-3)/2}dz$$

$$=\frac{\Gamma(3/2)}{\pi^{n/2}}\delta^{(n-3)/2}(t^2-s^2)\,, \qquad s=|\mathbf{s}| \tag{11}$$

where the function $\delta^\nu(x)$ is defined by

$$\delta^\nu(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp zx\cdot z^\nu dz\,, \qquad c>0 \tag{12}$$

$\delta^0(x)$ being the usual delta function of Dirac.

According as ν is an integer m or a half integer $m-\frac{1}{2}$, one sees that

$$\delta^m(x)=\left(\frac{\partial}{\partial x}\right)^m\delta(x) \tag{13}$$

$$\delta^{m-\frac{1}{2}}(x)=\left(\frac{\partial}{\partial x}\right)^m\delta^{-1/2}(x)=\left(\frac{\partial}{\partial x}\right)^m\frac{\epsilon(x)}{\sqrt{x}}\frac{1}{\Gamma(1/2)} \tag{14}$$

since

$$\delta^{-1/2}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zx}\frac{dz}{z^{1/2}}=\frac{\epsilon(x)}{\sqrt{x}}\frac{1}{\Gamma(1/2)}\,. \tag{15}$$

The differentiation of (9) gives

$$(6)=\frac{1}{(2\pi)^n}\int\exp i\mathbf{p}\cdot\mathbf{s}\cdot\cos qtd\mathbf{p}=\frac{\Gamma(1/2)}{\pi^{n/2}}t\delta^{(n-1)/2}(t^2-s^2)\,. \tag{16}$$

§ 3. Solutions of initial value problems

The substitution of (11) into (5) leads to

$$u_2(\mathbf{r},t)=\frac{\Gamma(3/2)}{\pi^{n/2}}\int\delta^{(n-3)/2}(t^2-s^2)\,q(\mathbf{r}+\mathbf{s})\,d\mathbf{s}$$

$$=\frac{\omega_n\Gamma(3/2)}{\pi^{n/2}}\int_0^\infty\delta^{(n-3)/2}(t^2-s^2)\,Q(s)\,s^{n-1}ds \tag{17}$$

where one uses $Q(s)$ and ω_n defined by

$Q(s)=$ the mean value of $g(\mathbf{r}+\mathbf{s})$ at the spherical surface $|\mathbf{s}|=s$

$\omega_n=$ the area of the spherical surface of radius 1

$$=\frac{2\pi^{n/2}}{\Gamma(n/2)}\,.$$

For an odd n, the application of the formula

$$\delta(t^2-s^2)=\frac{1}{2t}(\delta(t-s)+\delta(t+s))$$

yields

$$u_2(\mathbf{r},\, t)=\frac{2\Gamma(3/2)}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-3)/2}\int_0^\infty \delta(t^2-s^2)\,Q(s)\,s^{n-1}\,ds$$
$$=\frac{2\Gamma(3/2)}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-3)/2}\int_0^\infty \frac{1}{2t}\,(\delta(t-s)+\delta(t+s))\,Q(s)\,s^{n-1}ds$$
$$=\frac{\Gamma(3/2)}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-3)/2}(t^{n-2}Q(t))\,. \tag{19}$$

For an even n, the application of the formula (14)

$$\delta^{(n-3)/2}(t^2-s^2)=\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2}\frac{\epsilon(t^2-s^2)}{\sqrt{t^2-s^2}}\,\frac{1}{\Gamma(1/2)}$$

gives

$$u_2(\mathbf{r},\, t)=\frac{1}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2}\int_0^t \frac{Q(s)s^{n-1}}{\sqrt{t^2-s^2}}\,ds \tag{20}$$

which may be rewritten

$$u_2(\mathbf{r},\, t)=\frac{t}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2}\left(t^{n-2}\int_0^t \frac{Q(s)sds}{\sqrt{t^2-s^2}}\right)\,. \tag{21}$$

§ 4. Klein Gordon equation

The solution to the Klein Gordon equation

$$\frac{\partial^2 u}{\partial t^2}=\sum\frac{\partial^2 u}{\partial x_k^2}-c^2 u \tag{22}$$

with the initial condition

$$(u)_{t=0}=f(\mathbf{r})\,,\qquad \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(\mathbf{r}) \tag{23}$$

may be expressed as in (3), (4) and (5), except that

$$q=\sqrt{\mathbf{p}^2+c^2}\,.$$

Replacing A and B in (9) by t^2 and $\mathbf{p}^2+c^2$ respectively one gets

$$\frac{1}{(2\pi)^n}\int\frac{\sin qt}{q}\,e^{-i\mathbf{p}\cdot\mathbf{s}}d\mathbf{p}$$
$$=\frac{\Gamma(3/2)}{(2\pi)^n}\,\frac{1}{2\pi i}\iint \exp\left(zt^2-\frac{c^2+\mathbf{p}^2}{4z}\right)\cdot\frac{dz}{z^{3/2}}\cdot e^{-i\mathbf{p}\cdot\mathbf{s}}d\mathbf{p}$$
$$=\frac{\Gamma(3/2)}{\pi^{n/2}}\,\frac{1}{2\pi i}\int \exp\left[z(t^2-s^2)-\frac{c^2}{4z}\right]\cdot z^{(n-3)/2}dz\,. \tag{24}$$

For n=odd one gets

$$(24)=\frac{\Gamma(3/2)}{\pi^{n/2}}\left(\frac{\partial}{\partial t^2}\right)^{(n-1)/2}\frac{1}{2\pi i}\int \exp\left[z(t^2-s^2)-\frac{c^2}{4z}\right]\cdot\frac{dz}{z}$$
$$=\frac{\Gamma(3/2)}{\pi^{n/2}}\left(\frac{\partial}{\partial t^2}\right)^{(n-1)/2}\{J_0(c\sqrt{t^2-s^2})\,\epsilon(t^2-s^2)\}$$

by virtue of (9). Hence

$$u_2(\mathbf{r},\ t)=\frac{\Gamma(1/2)}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-1)/2}\int_0^t Q(s)s^{n-1}J_0(c\sqrt{t^2-s^2})ds\,. \tag{25}$$

For $n=$even one gets

$$(24)=\frac{\Gamma(3/2)}{\pi^{n/2}}\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2}\frac{1}{2\pi i}\int \exp\left[z(t^2-s^2)-\frac{c^2}{4z}\right]\frac{dz}{\sqrt{z}}$$

$$=\frac{\Gamma(3/2)}{\pi^{n/2}}\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2}\left\{\frac{\cos c\sqrt{t^2-s^2}}{\sqrt{t^2-s^2}}\frac{\epsilon(t^2-s^2)}{\Gamma(1/2)}\right\}.$$

Hence

$$u_2(\mathbf{r},\ t)=\frac{1}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2}\int_0^t Q(s)s^{n-1}\frac{\cos c\sqrt{t^2-s^2}}{\sqrt{t^2-s^2}}ds\,. \tag{26}$$

Comparison of the derivations of (19), (20), (25), (26) in this paper and those in the book by Courant-Hilbert[1] makes obvious the usefulness of the integral representation of $\sin x/x$.

§ 5. The initial value problem for elastic waves

The solution to the equation of elastic waves

$$\frac{\partial^2\mathbf{u}}{\partial t^2}=c_2^2\Delta\mathbf{u}+(c_1^2-c_2^2)\ \mathrm{grad\ div}\ \mathbf{u}$$

with the initial condition

$$\mathbf{u}(\mathbf{r},\ 0)=\mathbf{f}(\mathbf{r})\,,\qquad \mathbf{u}_t(\mathbf{r},\ 0)=\mathbf{g}(\mathbf{r})$$

is expressed

$$\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$$

$$\mathbf{u}_1=\frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\cdot\mathbf{r}}\Big\{\cos c_2pt$$

$$+\frac{D}{p^2}(\cos c_1pt-\cos c_2pt)\Big\}d\mathbf{p}\int e^{-i\mathbf{p}\cdot\mathbf{r}'}\mathbf{f}(\mathbf{r}')d\mathbf{r}'$$

$$\mathbf{u}_2=\frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\cdot\mathbf{r}}\Big\{\frac{\sin c_2pt}{c_2p}$$

$$+\frac{D}{p^2}\left(\frac{\sin c_1pt}{c_1p}-\frac{\sin c_2pt}{c_2p}\right)\Big\}d\mathbf{p}\int e^{-i\mathbf{p}\cdot\mathbf{r}'}\mathbf{g}(\mathbf{r}')d\mathbf{r}'$$

with the same convention as in § 1, except that $p=|\mathbf{p}|$ and

$$D=\begin{pmatrix} p_1^2 & p_1p_2 & p_1p_3 \\ p_1p_2 & p_2^2 & p_2p_3 \\ p_1p_3 & p_2p_3 & p_3^2 \end{pmatrix}.$$

The evaluation of the integral

$$\frac{1}{(2\pi)^3}\int\frac{D}{p^2}\left(\frac{\sin c_1pt}{c_1p}-\frac{\sin c_2pt}{c_2p}\right)\cdot e^{-i\mathbf{p}\cdot\mathbf{s}}d\mathbf{p}$$

$$=\begin{pmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial x_3}\end{pmatrix}\times\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3}\right)H\,,$$

$$H=\frac{1}{(2\pi)^3}\int\frac{1}{p^2}\left(\frac{\sin c_2pt}{c_2p}-\frac{\sin c_1pt}{c_1p}\right)\cdot e^{-i\mathbf{p}\cdot\mathbf{s}}d\mathbf{p}$$

needs that of H.

The integral representation leads to the expression

$$H=\frac{\Gamma(3/2)}{(2\pi)^3}\frac{1}{2\pi i}\iint e^{zt^2}(e^{-c_2{}^2p^2/4z}-e^{-c_1{}^2p^2/4z})\frac{e^{-i\mathbf{p}\cdot\mathbf{s}}}{p^2}d\mathbf{p}\frac{dz}{z^{3/2}}$$

where one may use an integral formula

$$(e^{-c_2{}^2p^2/4z}-e^{-c_1{}^2p^2/4z})/p^2$$

$$=\frac{1}{4z}\int_{c_2{}^2}^{c_1{}^2}e^{-\lambda p^2/4z}d\lambda\,.$$

One gets then

$$H=\frac{\Gamma(3/2)}{(2\pi)^3}\frac{1}{2\pi i}\int_{c_2{}^2}^{c_1{}^2}\frac{d\lambda}{\lambda^{3/2}}\int e^{z\left(t^2-\frac{s^2}{\lambda}\right)}\frac{dz}{z}\cdot 2\pi^{3/2}$$

$$=\frac{\Gamma(3/2)}{4\pi^{3/2}}\int_{c_2{}^2}^{c_1{}^2}\frac{d\lambda}{\lambda^{3/2}}\,\epsilon\left(t^2-\frac{s^2}{\lambda}\right)$$

$$=\frac{1}{4\pi}\int_{1/c_1}^{1/c_2}\epsilon(t^2-s^2\mu^2)d\mu\,,\qquad(\lambda=1/\mu^2)\,.$$

Hence one sees that

$$\int\mathbf{g}(\mathbf{r}+\mathbf{s})Hd\mathbf{s}=\frac{1}{4\pi}\int_{1/c_1}^{1/c_2}d\mu\int_{|\mathbf{s}|<t/\mu}\mathbf{g}(\mathbf{r}+\mathbf{s})d\mathbf{s}$$

$$=\left(\frac{1}{c_2}-\frac{1}{c_1}\right)\frac{1}{4\pi}\int_{|\mathbf{s}|<c_2t}\mathbf{g}(\mathbf{r}+\mathbf{s})d\mathbf{s}+\frac{1}{4\pi}\int_{c_2t<|\mathbf{s}|<c_1t}\left(\frac{t}{s}-\frac{1}{c_1}\right)\mathbf{g}(\mathbf{r}+\mathbf{s})d\mathbf{s}$$

$=K(\mathbf{r},\ t)$ say.

Hence $\mathbf{u}_2$ may be expressed as

$$\mathbf{u}_2=\frac{1}{4\pi c_2^2t}\int_{|\mathbf{s}|=c_2t}\mathbf{g}(\mathbf{r}+\mathbf{s})d\mathbf{s}+\text{grad div }K\,.$$

The present solution is far simpler than that of Courant.[2)]

References

1) R. Courant: *Methods of Mathematical Physics* Vol. II, p. 682. Interscience Publishers, Inc. New York, 1962.
2) Ibidem p. 706.