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On Spectral Properties of Positive Irreducible Operators in $C(S)$ and a Problem of H. H. Schaefer

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§ **1. Introduction.** In a previous paper [6], the author investigated some spectral properties of semi-non-support operators in an ordered Banach space. This paper shows further precise spectral properties of positive irreducible operators which is equivalent to semi-non-support operators in a special space $C(S)$ where S is a compact Hausdorff space.

Concerning the spectrum of positive operators, the following result was obtained by H. H. Schaefer [7] [9];

THEOREM A. *Let T be a positive irreducible operator in $C(S)$ whose resolvent $R(\lambda, T)$ has a pole at $\lambda = r(T)$*[1]. *Then the point spectrum on the spectral circle*[2] *coincides with the set of all k-th roots ($k \geqq 1$) of unity multiplied by $r(T)$ each of which, proper value of T, is a simple pole of $R(\lambda, T)$.*

Recently, F. Niiro [5] has shown that, in L_p $(1 < p < \infty)$, the spectrum of a similar operator on the spectral circle consists only of simple poles (of course, these are proper values) and coincides with the set of all k-th roots ($k \geqq 1$) of unity multiplied by $r(T)$.

In this paper, we shall prove the following two theorems;

THEOREM 1. *Let T be a positive irreducible operator in $C(S)$ whose resolvent has a pole at $\lambda = r(T)$. Then the residual spectrum of T on the spectral circle is void.*

THEOREM 2. *Let T satisfy the same condition as in theorem* 1. *Then the continuous spectrum of T on the spectral circle is void.*

Combining theorem 1, 2 and theorem A, we shall obtain, in the case of $C(S)$, a result similar to that of F. Niiro mentioned above. At the same time this answers affirmatively to the problem (b) of H. H. Schaefer [8] in the case of positive irreducible operators in $C(S)$.

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1) $r(T)$ denotes the spectral radius of T.
2) The spectral circle is the circle $\{\lambda ; |\lambda| = r(T)\}$.

(Added in proof) Recently the present author and F. Niiro could prove both theorem 1 and theorem 2 in the general case of a Banach lattice. The proof will be given elsewhere.

§ 2. Fundamental notions. We shall denote by $C(S)$ a real (or complex) Banach space of all real (or complex) continuous functions on a compact Hausdorff space S with the usual supremum norm and define

$$|x|(s)=|x(s)| \qquad (s\in S)$$

$$x_+=\frac{|x|+x}{2} \qquad x_-=\frac{|x|-x}{2} \qquad (\text{for real } x).$$

If S is not a finite set, $C(S)$ is a non-reflexive Banach lattice with the positive cone $K=\{x\,;\, x(s)\geqq 0\ (s\in S)\}$. Let $\mathfrak{B}$, f and $v(f, A)$ be the family of all Borel sets in S, a Radon measure on $\mathfrak{B}$ and the total variation of f on a Borel set A respectively. It is well known that the dual space $C(S)^*$ is the Banach lattice of all real (or complex) Radon measures on S where

$$||f||=v(f, S)$$

$$|f|(A)=v(f, A) \qquad (A\in\mathfrak{B})$$

$$f_+=\frac{|f|+f}{2} \qquad f_-=\frac{|f|-f}{2} \qquad (\text{for real } f).$$

We consider also a real (or complex) Banach lattice L_p $(1\leqq p\leqq\infty)$ with the natural norm, order and absolute value. It will be noticed that, in all cases, $|x|$ and $|f|$ are defined for any x and f, real or complex.

We denote by E one of the spaces L_p $(1\leqq p\leqq\infty)$ or $C(S)$, by K the positive cone of E and by E^* and K^* the duals of E and K respectively. An element $x\in E$ is called a *non-support* element of K if $f(x)>0$ for every non-zero $f\in K^*$. In E, the notion of non-support elements of K coincides with that of quasi-interior elements of K defined by H. H. Schaefer [8]. In L_p $(1\leqq p<\infty)$ with σ-finite measure[3], x is a non-support element of K if and only if $x(s)>0$ a. e. and, in $C(S)$ or L_∞, if and only if x is an interior element of K. We denote by $\mathfrak{L}(E)$ the Banach space of all bounded linear operators from E to E with operator norm. For $T\in\mathfrak{L}(E)$, $r(T)$, $R(\lambda, T)$, $\sigma(T)$, $\rho(T)$, $P_\sigma(T)$, $R_\sigma(T)$, $C_\sigma(T)$ and T^* denote the spectral radius, the resolvent, the spectrum, the resolvent set, the point spectrum, the residual spectrum, the continuous spectrum and the conjugate operator of T respectively. Since $\sigma(T)=\sigma(T^*)$ and $\{R(\lambda, T)\}^*=R(\lambda, T^*)$ [2], it is easy to see that $r(T)=r(T^*)$ and that $\lambda=\lambda_0$ is a pole of $R(\lambda, T)$ if and only if $\lambda=\lambda_0$ is a

3) See, for this, footnote 3) in [5].

pole of $R(\lambda, T^*)$.

An operator $T \in \mathfrak{L}(E)$ is said to be *positive* if $TK \subset K$. The following definition of irreducible operators is due to H. H. Schaefer [7] [9];

DEFINITION. *An operator $T \in \mathfrak{L}(E)$ is said to be irreducible if there exists no non-trivial closed ideal invariant under T.*

This notion of positive irreducible operators coincides with that of positive indecomposable operators defined by F. Niiro [5] in L_p $(1 < p < \infty)$ and that of quasi-interior operators in E defined by H.H. Schaefer [8] and that of semi-non-support operators in E defined by the author [6]. Therefore, we can apply theorem 1, 2 and corollary 1 in our paper [6] to irreducible operators in E whose resolvent has a pole at $\lambda = r(T)$.

In the following part of this paper we shall assume that T is a positive irreducible operator in E whose resolvent has a pole at $\lambda = r(T)$ and, for the sake of simplicity, that $r(T)$ is equal to 1 and denote by Γ the spectral circle of T, i.e., $\Gamma = \{\lambda ; |\lambda| = r(T) = 1\}$.

§ 3. The voidness of the residual spectrum of T on Γ. Our main purpose in this section is to prove that, in $C(S)$, the residual spectrum of T on Γ is void. In L_p $(1 < p < \infty)$, the corresponding result was proved by using the fact that T^* is also positive irreducible in L_p^* [5]. However this fact depends too much on the reflexivity of L_p. Namely if, in $C(S)$, T^* is positive irreducible then S is shown to be at most countable (corollary 2). Therefore we shall consider a restriction T^*_1 of T^* to an invariant subspace P_0E^* which is the space of all f_0-absolutely continuous measures where f_0 is a certain positive measure. Then the following properties can be proved:

(1) The spectral circles of T^* and T^*_1 both coincide with Γ (lemma 4(i)),

(2) $P_\sigma(T^*) \cap \Gamma = P_\sigma(T^*_1) \cap \Gamma$ (proposition 1),

(3) $\lambda = \lambda_0$ on Γ is a pole of $R(\lambda, T^*_1)$ if and only if it is a pole of $R(\lambda, T^*)$ (proposition 2(ii)),

(4) T^*_1 is also positive irreducible in P_0E^* (proposition 4).

Since the method used by F. Niiro in L_p [5] may be applied in the case of L_1, by using (4) we obtain

(5) Any proper value λ_0 of T^*_1 on Γ is a pole of $R(\lambda, T^*_1)$ (corollary 3).

Combining (5) with (3), we have

(6) $P_\sigma(T^*_1) \cap \Gamma \subset P_\sigma(T) \cap \Gamma$ (corollary 4).

Therefore, by (2), (6) and the well known fact $R_\sigma(T) \subset P_\sigma(T^*)$, it is easy to see that the residual spectrum of T on Γ is void.

Hereafter, we denote by E only the space $C(S)$, by x_0 the non-support element of K satisfying

$$Tx_0 = x_0 \quad \text{and} \quad \|x_0\| = 1$$

and by f_0 the strictly positive functional in E^* satisfying

$$T^*f_0 = f_0 \quad \text{and} \quad f_0(x_0) = 1$$

where the existence of such x_0 and f_0 is assured by theorem 2 in [6].

Let P_0 be the operator on E^* defined by

$$P_0 f = \bigvee_n (f \wedge nf_0) \qquad (f \in K^*)^{4)}$$

$$P_0 f = P_0 f_+ - P_0 f_- \qquad (\text{real } f \in E)$$

$$P_0 f = P_0(\Re f) + iP_0(\Im f) \qquad (f \in E^*)\,.$$

Then P_0 is obviously a lattice homorphic projection with

$$\|P_0\| = 1, \quad 0 \leqq P_0 \leqq I \quad \text{and} \quad P_0 f_0 = f_0\,.$$

Therefore P_0E^* is a Banach lattice. Let $Q_0 = I - P_0$. Then we have

$$\|Q_0\| = 1 \text{ or } 0, \quad P_0Q_0 = 0 \quad \text{and} \quad 0 \leqq Q_0 \leqq I\,.$$

This decomposition $I = P_0 + Q_0$ is the Lebesgue decomposition where P_0E^* coincides with the set of all f_0-absolutely continuous measures on S and Q_0E^* coincides with the set of all f_0-singular measures on S. Therefore, it is clear that if $|f|$ is an element of P_0E^* then f is also an element of P_0E^*. We begin by proving the following lemma:

LEMMA 1.

(i) $P_0T^*P_0 = T^*P_0$

(ii) $Q_0T^*Q_0 = Q_0T^*$

(iii) $P_0R(\lambda, T^*)P_0 = R(\lambda, T^*)P_0 \qquad (\lambda \in \mathfrak{D})$

(iv) $Q_0R(\lambda, T^*)Q_0 = Q_0R(\lambda, T^*) \qquad (\lambda \in \mathfrak{D})$

where $\mathfrak{D}$ is the unbounded component of $\rho(T^*)$.

PROOF. (i) It is sufficient to prove (i) on K^*. Let f be an arbitrary element of K^*. Then, for every positive integers n and $m \geqq n$ the following relation holds;

$$T^*(f \wedge nf_0) \leqq T^*(nf_0) = nf_0 \leqq mf_0\,,$$

that is,

$$\{T^*(f \wedge nf_0)\} \wedge mf_0 = T^*(f \wedge nf_0)\,.$$

Therefore

$$P_0T^*(f \wedge nf_0) = \bigvee_m \{T^*(f \wedge nf_0) \wedge mf_0\} = T^*(f \wedge nf_0)$$

which implies

4) Since $\{f \wedge nf_0\}$ is increasing and dominated by $f \in E^*$, $f \wedge nf_0$ converges strongly to $P_0 f$ as $n \to \infty$ in $E^* = C(S)^*$.

$$P_0T^*P_0f = T^*P_0f\,.$$

(ii) From (i), $Q_0 = I - P_0$ and $Q_0P_0 = 0$, it follows that

$$Q_0T^*Q_0 = Q_0T^* - Q_0T^*P_0 = Q_0T^* - Q_0P_0T^*P_0 = Q_0T^*\,.$$

(iii) It is well known that $R(\lambda, T^*) = \sum_{n=0}^{\infty} \frac{T^{*n}}{\lambda^{n+1}}$ for $|\lambda|>1$ where the series converges in the sense of operator norm. Combining this with (i), it is easy to see that

$$P_0R(\lambda, T^*)P_0 = P_0\left(\sum_{n=0}^{\infty} \frac{T^{*n}}{\lambda^{n+1}}\right)P_0 = \sum_{n=0}^{\infty} \frac{P_0T^{*n}P_0}{\lambda^{n+1}} = \sum_{n=0}^{\infty} \frac{T^{*n}P_0}{\lambda^{n+1}}$$

$$= R(\lambda, T^*)P_0 \qquad (\text{for } |\lambda|>1)\,.$$

Since $\{\lambda\,;\,|\lambda|>1\}\subset\mathfrak{D}$ and $R(\lambda, T^*)$ is holomorphic in $\mathfrak{D}$, it follows from unicity theorem that

$$P_0R(\lambda, T^*)P_0 = R(\lambda, T^*)P_0 \qquad (\lambda\in\mathfrak{D})\,.$$

(iv) The proof is the same as that of (ii) (use (iii) instead of (i)).

By lemma 1 (i) we see that P_0E^* is a subspace of E^* invariant under T^*. However Q_0E^* is not necessarily invariant under T^*[5]. Therefore we shall consider the restriction of T^* to P_0E^* and that of Q_0T^* to Q_0E^*. Let these restrictions be denoted by T^*_1 and T^*_2 respectively. Then it is clear that T^*_1 and T^*_2 are elements of $\mathfrak{L}(P_0E^*)$ and $\mathfrak{L}(Q_0E^*)$ respectively. It is also clear that $T^* = T^*_1$ holds if and only if $Q_0 = 0$[6]. After these preparations, we prove the following proposition ;

PROPOSITION 1. *On Γ, the point spectrum of T^* coincides with the point spectrum of T^*_1*, i.e.,

$$P_\sigma(T^*)\cap\Gamma = P_\sigma(T^*_1)\cap\Gamma\,.$$

Further the proper space of T^* for $\lambda\in P_\sigma(T^*)\cap\Gamma$ coincides with the proper space of T^*_1 for λ.

PROOF. It is clear that a proper vector of T^*_1 for $\lambda\in P_\sigma(T^*_1)$ is

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5) For example, let

$$E = C([0, 1])\,,$$

$$f_0(x) = \int_0^1 x(s)\,ds \qquad (x\in E)$$

and

$$Tx = f_0(x)\cdot\mathbf{1} \qquad (x\in E)\,.$$

Then T is a positive irreducible compact operator in E and satisfies

$$r(T)=1,\quad T\mathbf{1}=\mathbf{1},\quad T^*f=f(\mathbf{1})f_0 \quad\text{and}\quad T^*f_0=f_0\,.$$

Let s be an arbitrary element in $[0, 1]$ and f_s be the point measure at s. Then f_s is clearly f_0-singular but T^*f_s is f_0-absolutely continuous since $T^*f_s=f_s(\mathbf{1})f_0$. Thus, for such an operator T in E, Q_0E^* is not invariant under T^*.

6) Therefore, proposition 1, 2 and 3 mentioned later are trivial in the case of $Q_0=0$.

also that of T^* for λ. Let f be a proper vector of T^* for $\lambda \in P_\sigma(T^*) \cap \Gamma$. Then $f \neq 0$ and $T^*f = \lambda f$. Since $|f| = |T^*f| \leqq T^*|f|$[7] and x_0 is a non-support proper element of T for 1 it follows that

$$T^*|f| = |f| .$$

Since the proper space of T^* for 1 is one-dimensional by theorem 2 in [6], it follows that

$$|f| = af_0 \in P_0 E^*$$

which implies $f \in P_0 E^*$. Therefore f is a proper vector of T^*_1 for $\lambda \in P_\sigma(T^*_1) \cap \Gamma$. This completes the proof of proposition 1.

For the proof of proposition 2 we need the following lemmas 2, 3 and 4.

LEMMA 2.

(i) *If* $\lambda \in \rho(T^*)$ *and* $R(\lambda, T^*)P_0 = P_0 R(\lambda, T^*)P_0$ *then* $\lambda \in \rho(T^*_1) \cap \rho(T^*_2)$,

$$R(\lambda, T^*_1)P_0 = R(\lambda, T^*)P_0 \quad \textit{and} \quad R(\lambda, T^*_2)Q_0 = Q_0 R(\lambda, T^*)Q_0 .$$

Conversely, if $\lambda \in \rho(T^*_1) \cap \rho(T^*_2)$ *then*

$$\lambda \in \rho(T^*), \qquad R(\lambda, T^*)P_0 = P_0 R(\lambda, T^*)P_0$$

and

$$R(\lambda, T^*) = R(\lambda, T^*_1)P_0 + R(\lambda, T^*_1)P_0 T^* R(\lambda, T^*_2)Q_0 + R(\lambda, T^*_2)Q_0 .$$

(ii) *If* $\lambda \in \rho(T^*) \cap \{\rho(T^*_1) \cup \rho(T^*_2)\}$ *then*

$$R(\lambda, T^*)P_0 = P_0 R(\lambda, T^*)P_0 .$$

(iii) *If* $\lambda \in \rho(T^*) \cap \{\rho(T^*_1) \cup \rho(T^*_2)\}$ *then*

$$\lambda \in \rho(T^*_1) \cap \rho(T^*_2) .$$

(iv) $\rho(T^*) \cap \rho(T^*_1) = \rho(T^*) \cap \rho(T^*_2) = \rho(T^*_1) \cap \rho(T^*_2)$.

PROOF. (i) Let $\lambda \in \rho(T^*)$ and $R(\lambda, T^*)P_0 = P_0 R(\lambda, T^*)P_0$. Then, it is easy to see that

$$R(\lambda, T^*)P_0(\lambda I - T^*_1)P_0 = P_0 ,$$

$$(\lambda I - T^*_1)R(\lambda, T^*)P_0 = P_0$$

and that the restriction of $R(\lambda, T^*)$ to $P_0 E^*$ is an operator in $\mathfrak{L}(P_0 E^*)$. Therefore, it follows that

7) Let $f = \mathfrak{R}f + i\mathfrak{J}f$. Then, for every $\theta \in [0, 2\pi]$ and $A \in \mathfrak{B}$

$$(\cos\theta \mathfrak{R}f + \sin\theta \mathfrak{J}f)A \leqq |(\cos\theta \mathfrak{R}f + \sin\theta \mathfrak{J}f)(A)|$$
$$\leqq |f(A)| \leqq v(f, A) = |f|(A) .$$

It follows from the positivity and linearity of T^* that

$$\{\cos\theta \mathfrak{R}(T^*f) + \sin\theta \mathfrak{J}(T^*f)\}(A) \leqq T^*|f|(A) \qquad (\theta \in [0, 2\pi],\ A \in \mathfrak{B})$$

and hence

$$|T^*f(A)| \leqq T^*|f|(A) \qquad (A \in \mathfrak{B}) .$$

Thus,

$$|T^*f| \leqq T^*|f| .$$

$$\lambda \in \rho(T^*_1) \quad \text{and} \quad R(\lambda, T^*_1)P_0 = R(\lambda, T^*)P_0 .$$

Since $R(\lambda, T^*)P_0 = P_0 R(\lambda, T^*)P_0$ implies $Q_0 R(\lambda, T^*) = Q_0 R(\lambda, T^*)Q_0$, it follows from lemma 1(ii) that

$$Q_0 R(\lambda, T^*)Q_0(\lambda I - T^*_2)Q_0 = Q_0$$

and

$$(\lambda I - T^*_2)Q_0 R(\lambda, T^*)Q_0 = Q_0 .$$

Therefore, it is easy to see that

$$\lambda \in \rho(T^*_2) \quad \text{and} \quad R(\lambda, T^*_2)Q_0 = Q_0 R(\lambda, T^*)Q_0 .$$

Conversely, let $\lambda \in \rho(T^*_1) \cap \rho(T^*_2)$. Then there exist

$$R(\lambda, T^*_1) \in \mathfrak{L}(P_0 E^*) \quad \text{and} \quad R(\lambda, T^*_2) \in \mathfrak{L}(Q_0 E^*) .$$

Put

$$R_1 = R(\lambda, T^*_1), \quad R_2 = R(\lambda, T^*_2) \quad \text{and} \quad R = R_1 P_0 + R_1 P_0 T^* R_2 Q_0 + R_2 Q_0 .$$

Then,

$$R \in \mathfrak{L}(E^*) ,$$

$$\begin{aligned}(\lambda I - T^*)R &= (\lambda I - T^*)R_1 P_0 + (\lambda I - T^*)R_1 P_0 T^* R_2 Q_0 + (\lambda I - T^*)R_2 Q_0 \\ &= P_0 + P_0 T^* R_2 Q_0 + P_0(\lambda I - T^*)R_2 Q_0 + Q_0(\lambda I - T^*)R_2 Q_0 \\ &= P_0 + P_0 T^* R_2 Q_0 - P_0 T^* R_2 Q_0 + Q_0 = I\end{aligned}$$

and

$$\begin{aligned}R(\lambda I - T^*) &= R_1 P_0(\lambda I - T^*)P_0 + R_1 P_0(\lambda I - T^*)Q_0 + R_1 P_0 T^* R_2 Q_0(\lambda I - T^*) \\ &\quad + R_2 Q_0(\lambda I - T^*) \\ &= P_0 - R_1 P_0 T^* Q_0 + R_1 P_0 T^* Q_0 + Q_0 = I .\end{aligned}$$

From this it follows that

$$\lambda \in \rho(T^*) ,$$

$$R(\lambda, T^*) = R(\lambda, T^*_1)P_0 + R(\lambda, T^*_1)P_0 T^* \; R(\lambda, T^*_2)Q_0 + R(\lambda, T^*_2)Q_0$$

and

$$R(\lambda, T^*)P_0 = P_0 R(\lambda, T^*)P_0 .$$

(ii) Let $\lambda \in \rho(T^*) \cap \{\rho(T^*_1) \cup \rho(T^*_2)\}$. Suppose first $\lambda \in \rho(T^*) \cap \rho(T^*_1)$. Then there exist

$$R(\lambda, T^*) \in \mathfrak{L}(E^*) \quad \text{and} \quad R(\lambda, T^*_1) \in \mathfrak{L}(P_0 E^*) .$$

Since $Q_0 R(\lambda, T^*)P_0(\lambda I - T^*)P_0 = 0$ (by lemma 1(i)), it follows that

$$Q_0 R(\lambda, T^*)P_0 = Q_0 R(\lambda, T^*)P_0(\lambda I - T^*)P_0 R(\lambda, T^*_1)P_0 = 0 .$$

Suppose next $\lambda \in \rho(T^*) \cap \rho(T^*_2)$. Then

$$R(\lambda, T^*) \in \mathfrak{L}(E^*) \quad \text{and} \quad R(\lambda, T^*_2) \in \mathfrak{L}(Q_0 E^*) .$$

Since $Q_0(\lambda I-T^*)Q_0R(\lambda, T^*)P_0=0$ (by lemma 1 (ii)), it follows that

$$Q_0R(\lambda, T^*)P_0=R(\lambda, T^*_2)Q_0(\lambda I-T^*)Q_0R(\lambda, T^*)P_0=0 .$$

In any case it was proved that

$$Q_0R(\lambda, T^*)P_0=0$$

and hence

$$P_0R(\lambda, T^*)P_0=R(\lambda, T^*)P_0 .$$

(iii) is clear by (i) and (ii).

(iv) is also clear by (i) and (iii).

LEMMA 3. *If $\lambda=\lambda_0$ ($|\lambda_0|=1$) is a pole of $R(\lambda, T^*)$ then, for each $j=1, 2$, either $\lambda=\lambda_0$ is a pole of $R(\lambda, T^*_j)$ or $\lambda_0\in\rho(T^*_j)$*[8].

PROOF. Let $\lambda_0(|\lambda_0|=1)$ be a pole of $R(\lambda, T^*)$. Then there exist a neighbourhood $U(\lambda_0)$ and a positive integer k such that, for $\lambda\in U(\lambda_0)\ominus\{\lambda_0\}$[9],

$$\lambda\in\rho(T^*) \quad \text{and} \quad R(\lambda, T^*)=\sum_{n=-k}^{\infty}A_n(\lambda-\lambda_0)^n \tag{1}$$

where the series on the right hand side converges in the sense of operator norm. From $|\lambda_0|=1$ follows

$$U(\lambda_0)\ominus\{\lambda_0\}\subset\mathfrak{D}$$

which, along with lemma 1 (iii), lemma 2 (i) and (1), imply

$$U(\lambda_0)\ominus\{\lambda_0\}\subset\rho(T^*_1)\cap\rho(T^*_2) ,$$

$$R(\lambda, T^*_1)P_0=P_0R(\lambda, T^*)P_0=\sum_{n=-k}^{\infty}P_0A_nP_0(\lambda-\lambda_0)^n \quad (\lambda\in U(\lambda_0)\ominus\{\lambda_0\})$$

and

$$R(\lambda, T^*_2)Q_0=Q_0R(\lambda, T^*)Q_0=\sum_{n=-k}^{\infty}Q_0A_nQ_0(\lambda-\lambda_0)^n \quad (\lambda\in U(\lambda_0)\ominus\{\lambda_0\}) .$$

Namely, $R(\lambda, T^*_1)$ and $R(\lambda, T^*_2)$ have above expansions at λ_0 in $\mathfrak{L}(P_0E^*)$ and in $\mathfrak{L}(Q_0E^*)$ respectively. Since $P_0A_nP_0$ or $Q_0A_nQ_0$ may be 0 for all negative integer n, either $\lambda=\lambda_0$ is a pole of $R(\lambda, T^*_j)$ or $\lambda_0\in\rho(T^*_j)(j=1, 2)$.

LEMMA 4.

(i) $$r(T^*_1)=1$$

(ii) $$r(T^*_2)<1 .$$

PROOF. By lemma 2 (i) it is evident that

$$\{\lambda ; |\lambda|>1\}\subset\mathfrak{D}\subset\rho(T^*_1)\cap\rho(T^*_2)$$

whence,

$$r(T^*_j)\leqq 1 \qquad (j=1, 2) .$$

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8) This result will be improved in proposition 2 (ii) and Lemma 4 (ii).

9) $X\ominus Y=\{\lambda ; \lambda\in X \text{ and } \lambda\notin Y\}$.

On the other hand, by proposition 1, $r(T^*_1)\geqq 1$ is clear. Thus

$$r(T^*_1)=1\,.$$

To prove (ii), suppose the contrary, i. e., $r(T^*_2)=1$. It follows from the positivity of T^*_2 that $1\in\sigma(T^*_2)$[10]. From this, lemma 3 and the fact that $\lambda=1$ is a pole of $R(\lambda,\ T^*)$, it follows that $\lambda=1$ is a pole of $R(\lambda,\ T^*_2)$. Therefore there exists a positive proper functional f in Q_0E^*[11], i. e.,

$$f\in Q_0E^*,\ f\geqq 0,\ f\neq 0 \quad\text{and}\quad T^*_2f=f\,.$$

From $T^*f\geqq Q_0T^*f=T^*_2f=f$ and the fact that x_0 is a non-support proper element of T for 1, it follows that

$$T^*f=f\,.$$

From this and proposition 1 follows $f\in P_0E^*$ which contradicts $f\in Q_0E^*$ and $f\neq 0$. Therefore

$$r(T^*_2)<1$$

which completes the proof of lemma 4.

In the following we shall denote $r(T^*_2)$ by r_2.

PROPOSITION 2.

(i) *$\sigma(T^*_1)$ coincides, in $\{\lambda\,;\,|\lambda|>r_2\}$, with $\sigma(T^*)$. In particular,*

$$\sigma(T^*)\cap\Gamma=\sigma(T^*_1)\cap\Gamma\,.$$

(ii) *$\lambda=\lambda_0$ ($|\lambda_0|=1$) is a pole of $R(\lambda,\ T^*_1)$ if and only if it is a pole of $R(\lambda,\ T^*)$.*

PROOF. (i) By lemma 4 (ii), it is clear that

$$\rho(T^*_2)\supset\{\lambda\,;\,|\lambda|>r_2\}\supset\Gamma\,.$$

From this and lemma 2 (iv) follows (i).

(ii) Let $\lambda=\lambda_0$ be a pole of $R(\lambda,\ T^*)$ and $|\lambda_0|=1$. Then it is a pole of $R(\lambda,\ T^*_1)$ or $\lambda_0\in\rho(T^*_1)$ by lemma 3. Since $\lambda_0\in\sigma(T^*)$ it follows from (i) that $\lambda_0\in\sigma(T^*_1)$. Therefore, $\lambda=\lambda_0$ is a pole of $R(\lambda,\ T^*_1)$.

Conversely, let $\lambda=\lambda_0$ be a pole $R(\lambda,\ T^*_1)$ and $|\lambda_0|=1$. Then $\lambda=\lambda_0$ is an isolated singular point of $R(\lambda,\ T^*_1)$ and $|\lambda_0|>r_2$ by lemma 4 (ii). Hence there exists a neighbourhood $U(\lambda_0)$ such that

$$U(\lambda_0)\ominus\{\lambda_0\}\subset\{\lambda\,;\,|\lambda|>r_2\}\cap\rho(T^*_1)\,.$$

From the latter part of lemma 2 (i), it follows that

$$R(\lambda,\ T^*)=R(\lambda,\ T^*_1)P_0+R(\lambda,\ T^*_1)P_0T^*R(\lambda,\ T^*_2)Q_0+R(\lambda,\ T^*_2)Q_0$$

for $\lambda\in U(\lambda_0)\ominus\{\lambda_0\}$. Since $\lambda_0\in\sigma(T^*)$ (by (i)) and $R(\lambda,\ T^*_2)$ is holomorphic in $U(\lambda_0)$, $R(\lambda,\ T^*)$ and $R(\lambda,\ T^*_1)$ have a pole with the same order at

10) This is an important and well known property of positive operators in general ordered Banach spaces [1]; Th. 1, [4]; Th. 4, [8]; Prop. 1.

11) The existence of such f is assured by, for example, theorem 5 in S. Karlin's paper [4].

$\lambda=\lambda_0$. Thus, the proof of this proposition is completed.

The following proposition will be needed later for the proof of theorem 2 in § 4.

PROPOSITION 3. *Let λ be in $\{\lambda; |\lambda|>r_2\}$. If the range of $\lambda I-T^*_1$ is dense in P_0E^* then the range of $\lambda I-T^*$ is also dense in E^*.*

PROOF. Let $(\lambda I-T^*_1)P_0E^*$ be dense in P_0E^* and f be an arbitrary positive functional in E^* and ε be an arbitrary positive number. By $|\lambda|>r_2$, λ belongs to $\rho(T^*_2)$ and hence $R(\lambda, T^*_2)Q_0f$ is an element of Q_0E^*. Since $(\lambda I-T^*_1)P_0E^*$ is dense in P_0E^* and $P_0f+P_0T^*R(\lambda, T^*_2)Q_0f\in P_0E^*$ it follows that there exists a functional $h\in P_0E^*$ such that

$$\|(\lambda I-T^*_1)h-\{P_0f+P_0T^*R(\lambda, T^*_2)Q_0f\}\|<\varepsilon. \tag{1}$$

It is easy to see that, by a simple calculation, the functional on the left hand side of (1) is equal to the functional

$$(\lambda I-T^*)\{h+R(\lambda, T^*_2)Q_0f\}-f.$$

Therefore, there exists a functional $h+R(\lambda, T^*_2)Q_0f\in E^*$ such that

$$\|(\lambda I-T^*)\{h+R(\lambda, T^*_2)Q_0f\}-f\|<\varepsilon.$$

Consequently, $(\lambda I-T^*)E^*$ is dense in E^* which completes the proof of proposition 3.

We shall next prove that T^*_1 is a positive irreducible operator with the resolvent having a pole at $\lambda=1$. To do this, we require the following lemma.

LEMMA 5. *f_0 is a non-support element of P_0K^*.*

PROOF. Since P_0E^* is the space of all f_0-absoluyely continuous Radon measures on S, it follows from Radon-Nikodym's theorem that P_0E^* is isomorphic to the space $L_1(S, f_0)$ as a Banach lattice and f_0 corresponds to $\mathbf{1}$ in $L_1(S, f_0)$. Since $\mathbf{1}$ is a non-support element of the positive cone, f_0 is also a non-support element of P_0K^*.

PROPOSITION 4. *T^*_1 is a positive irreducible operator in P_0E^* and $\lambda=1$ is a pole of $R(\lambda, T^*_1)$.*

PROOF. $\lambda=1$ is clearly a pole of $R(\lambda, T^*)$ by the assumption for T. It follows from proposition 2(ii) that $\lambda=1$ is also a pole of $R(\lambda, T^*_1)$. Since the proper space for 1 of T^* is a one-dimensional subspace containing f_0, the proper space for 1 of T^*_1 is also a one-dimensional subspace containing $f_0\in P_0E^*$. Further, f_0 is a non-support element of P_0K^* by lemma 5. By theorem 1 in [6], the proper space for 1 of $(T^*_1)^*$ is easily seen to be also one-dimensional in $(P_0E^*)^*$. Since x_0 is a non-support element of K with $Tx_0=x_0$, x_0 is strictly positive as a linear functional defined on P_0E^*. Therefore

(1) $\lambda=1$ is a pole of $R(\lambda, T^*_1)$,

(2) the proper space of T^*_1 for 1 is a one-dimensional subspace passing through the non-support element f_0 of P_0K^*,

(3) the proper space of $(T^*_1)^*$ for 1 is a one-dimensional subspace passing through the strictly positive functional $x_0 \in (P_0E^*)^*$.
Thus, by theorem 2 in [6] and above (1), (2) and (3), T^*_1 is a positive irreducible operator in P_0E^* which completes the proof of this proposition.

COROLLARY 1. *The following statements are equivalent to each other.*

(i) $Q_0=0$

(ii) f_0 *is a non-support element of* K^*.

(iii) T^* *is a positive irreducible operator.*

PROOF. If $Q_0=0$ then $T^*=T^*_1$ which implies (iii) by proposition 4. If (iii) holds true then (ii) is clear by theorem 2 in [6]. If (ii) holds true then $\varphi(f_0)>0$ for every non-zero $\varphi \in K^{**}$. Let $Q_0 \neq 0$. Then there exists ψ_0 such that

$$\psi_0 \in (Q_0E^*)^*, \quad \psi_0 \neq 0 \quad \text{and} \quad \psi_0 \geqq 0 .$$

Defining

$$\varphi_0(f) = \psi_0(Q_0f) \qquad (f \in E^*) ,$$

φ_0 is an element of K^{**} and $\varphi_0(f_0)=0$ which is a contradiction. Therefore (ii) implies (i).

COROLLARY 2. *If* f_0 *is a non-support element of* K^* *then* S *is at most countable.*

PROOF. By corollary 1, $Q_0E^*=\{0\}$ holds. Hence every point measure f_s at $s \in S$ is f_0-absolutely continuous. Therefore, it follows that

$$f_0(\{s\})>0 \qquad \text{for any} \quad s \in S .$$

Since $f_0(S)=v(f_0, S)=||f_0||<\infty$, S must be at most countable.

Now, we shall show the following proposition in the case of L_1.

PROPOSITION 5. *Let* U *be a positive irreducible operator in* L_1 *whose resolvent has a pole at* $\lambda=r(U)=1$. *Then any proper value of* U *on the spectral circle is a simple pole of* $R(\lambda, U)$.

As the method used by F. Niiro to prove the similar result in L_p $(1<p<\infty)$ [5][12] may be applied also to this case, the proof will be omitted[13].

COROLLARY 3. *Any proper value of* T^*_1 *on* Γ *is a pole of* $R(\lambda, T^*_1)$.

This is clear by proposition 4, 5 and the fact that P_0E^* is isomorphic to $L_1(S, f_0)$ as a Banach lattice.

COROLLARY 4. *The point spectrum of* T^*_1 *on* Γ *is contained in that of* T *on* Γ, i. e.,

12) See corollary 2, theorem 5 and the begining part of § 4 in [5].

13) Proposition 5 may be proved also by the method used to prove theorem 3.3 and 3.4 in [9].

$$P_\sigma(T^*_1) \cap \Gamma \subset P_\sigma(T) \cap \Gamma .$$

PROOF. Let λ be in $P_\sigma(T^*_1) \cap \Gamma$. Then, by corollary 3, λ is a pole of $R(\lambda, T^*_1)$ and hence, by proposition 2 (ii), λ is also a pole of $R(\lambda, T^*)$. Therefore λ is a pole of $R(\lambda, T)$. This implies that λ is a proper value of T on Γ.

As we mentioned at the begining of this section the following theorem 1 is a consequence of proposition 1, corollary 4 and the well known fact $R_\sigma(T) \subset P_\sigma(T^*)$.

THEOREM 1. *Let T be a positive irreducible operator in E and $\lambda = r(T) = 1$ be a pole of its resolvent. Then the residual spectrum of T is void on Γ.*

§ 4. The voidness of the continuous spectrum of T on Γ.

Our purpose in this section is to prove that the continuous spectrum of T is void on Γ. We shall sketch the outline of the proof.

First, by using the fact that $\{x_n\}$ converges weakly to 0 if and only if $\{|x_n|\}$ converges weakly to 0, we shall prove

(1) $\quad C_\sigma(T) \cap \Gamma \subset \{P_\sigma(T^*_1) \cup R_\sigma(T^*_1)\} \cap \Gamma \qquad$ (proposition 6).

Next, we shall derive

(2) $\quad R_\sigma(T^*_1) \cap \Gamma = \phi \qquad$ (proposition 7)

from positive irreducibility of $(T^*_1)^*$ (lemma 7) and theorem A by H.H. Schaefer mentioned in § 1. Combining (1), (2) and corollary 4 we have

$$C_\sigma(T) \cap \Gamma \subset P_\sigma(T^*_1) \cap \Gamma \subset P_\sigma(T) \cap \Gamma$$

from which follows

$$C_\sigma(T) \cap \Gamma = \phi .$$

Now we shall go in the detailed proof with following lemma 6:

LEMMA 6. *Let $x_n \in E$ $(n=1, 2, \cdots)$ and a complex number λ $(|\lambda|=1)$ satisfy the following conditions;*

$$\lim_{n\to\infty} \| \lambda x_n - T x_n \| = 0 \qquad \text{and} \qquad \| x_n \| = 1 .$$

Then

$$\underline{\lim}_{n\to\infty} f_0(|x_0|) \geqq 1 .$$

In this proof, we shall make use of an argument analogous to that used in the proof of theorem 2 by F. Niiro [5].

PROOF. Let P be the projection to the proper space of T for 1 and Q be $I-P$. Then,

$$PT = TP = P, \quad QT = TQ \quad \text{and} \quad Px = f_0(x)x_0 .$$

Put $y_n = \lambda x_n - T x_n$. Then

$$\lim_{n\to\infty} ||y_n||=0 \quad \text{and} \quad T|x_n|\geqq|x_n|-|y_n| .$$

Since $(T-I)Q=T-I$ it follows that

$$TQ|x_n|\geqq Q|x_n|-|y_n| .$$

From this and the positivity of $R(\xi, T)$ for $\xi>1$ follows that for every $\xi>1$

$$(\xi-1)R(\xi, T)Q|x_n|+R(\xi, T)|y_n|\geqq Q|x_n|$$

which implies

$$P|x_n|+(\xi-1)R(\xi, T)Q|x_n|+R(\xi, T)|y_n|\geqq|x_n|\geqq 0 .$$

Then

$$f_0(|x_n|)+(\xi-1)||R(\xi, T)||_{QE}||Q|x_n|||+||R(\xi, T)|y_n|||\geqq||x_n||=1$$

where $||\ ||_{QE}$ means the operator norm in $\mathfrak{L}(QE)$. Since $R(\xi, T)$ is a bounded operator in E for arbitrary fixed $\xi>1$, it follows that

$$\varliminf_{n\to\infty} f_0(|x_n|)+(\xi-1)||R(\xi, T)||_{QE}||Q||\geqq 1 .$$

Since ξ may be made arbitrarily close to 1 and $||R(\xi, T)||_{QE}$ is uniformly bounded in a neighbourhood of 1[14], it follows that

$$\varliminf_{n\to\infty} f_0(|x_n|)\geqq 1 .$$

PROPOSITION 6.

$$C_\sigma(T)\cap\Gamma\subset\{P_\sigma(T^*{}_1)\cup R_\sigma(T^*{}_1)\}\cap\Gamma$$

PROOF. Let λ be in $C_\sigma(T)\cap\Gamma$. Then, there exists a sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} ||\lambda x_n-Tx_n||=0 \quad \text{and} \quad ||x_n||=1 . \tag{1}$$

Hence, from lemma 6, it follows that

$$\varliminf_{n\to\infty} f_0(|x_n|)\geqq 1 . \tag{2}$$

From $\sigma(T)=\sigma(T^*)$ and proposition 2 (i) follows

$$\lambda\in\{P_\sigma(T^*{}_1)\cup R_\sigma(T^*{}_1)\cup C_\sigma(T^*{}_1)\}\cap\Gamma .$$

Let $\lambda\in C_\sigma(T^*{}_1)\cap\Gamma$. Then $(\lambda I-T^*{}_1)P_0E^*$ is dense in P_0E^* and therefore $(\lambda I-T^*)E^*$ is also dense in E^* by proposition 3. Since

$$\{(\lambda I-T^*)f\}(x_n)=f\{(\lambda I-T)x_n\} ,$$

it follows from (1) that

14) By corollary 1 in [6] $R(\lambda, T)$ has a simple pole at $\lambda=1$, and hence the leading coefficient of its expansion is equal to P. Therefore the restriction of $R(\lambda, T)$ to QE is holomorphic in a neighbourhood of 1.

$$\lim_{n\to\infty}\{(\lambda I-T^*)f\}(x_n)=0 \qquad \text{for any } f \text{ in } E^* .$$

Therefore $\{x_n\}$ converges weakly to 0 as $n\to\infty$ and hence $\{|x_n|\}$ also converges weakly to 0 by the well known property of E[15]. This contradicts (2). Thus, $\lambda \notin C_\sigma(T^*_1)$ and hence

$$\lambda\in\{P_\sigma(T^*_1)\cup R_\sigma\{T^*_1)\}\cap\Gamma$$

which completes the proof of proposition 6.

LEMMA 7. *$(T^*_1)^*$ is a positive irreducible operator in $(P_0E^*)^*$ and $\lambda=1$ is a pole of $R(\lambda, (T^*_1)^*)$.*

PROOF. It is clear by proposition 4 that $\lambda=1$ is a pole of $R(\lambda, (T^*_1)^*)$ and the proper space of $(T^*_1)^*$ for 1 is a one-dimensional subspace passing through x_0 considered as an element of $(P_0E^*)^*$. It is well known that $(P_0E^*)^*$ is isomorphic to $L_\infty(S, f_0)$ as an ordered Banach space and $x_0\in(P_0E^*)^*$ corresponds to $x_0\in E\subset L_\infty(S, f_0)$. Since x_0 is an interior point of K, $x_0\in L_\infty(S, f_0)$ is also an interior point of the positive cone of $L_\infty(S, f_0)$. Therefore $x_0\in(P_0E^*)^*$ is a non-support point of $(P_0K^*)^*$. By theorem 1 in [6], it is easy to see that the proper space of $(T^*_1)^{**}$ for 1 is one-dimensional in $(PE^*)^{**}$. Since f_0 is a non-support element of P_0K^*, $f_0\in(P_0E^*)^{**}$ is strictly positive. Thus, by theorem 2 in [6], $(T^*_1)^*$ is a positive irreducible operator.

PROPOSITION 7. *The residual spectrum of T^*_1 on Γ is void.*

PROOF. To prove this proposition suppose the contrary, i.e., $\lambda_0\in R_\sigma(T^*_1)\cap\Gamma$. Then $\lambda=\lambda_0$ is a proper value of $(T^*_1)^*$ on Γ. Since $(P_0E^*)^*$ is an abstract (M)-space with the order unit and therefore can be considered as $C(S_0)$ for a certain S_0 [3], it follows from lemma 7 and theorem A that $\lambda=\lambda_0$ is a simple pole of $R(\lambda, (T^*_1)^*)$. Therefore $\lambda=\lambda_0$ is a simple pole of $R(\lambda, T^*_1)$ and hence it is a proper value of T^*_1. This yields a contradiction

$$\lambda_0\in R_\sigma(T^*_1)\cap P_\sigma(T^*_1)=\phi .$$

Thus, $R_\sigma(T^*_1)\cap\Gamma$ is void which completes the proof of this proposition.

As we have mentioned in the begining of this section, combining proposition 6, 7 and corollary 4 we have

THEOREM 2. *Let T be a positive irreducible operator in E and $\lambda=r(T)=1$ be a pole of its resolvent. Then the continuous spectrum on the spectral circle Γ is void.*

Finally, combining theorem 1, 2 and theorem A, we have

THEOREM 3. *Let T satisfy the same condition as theorem 2. Then the spectrum of T on Γ consists only of simple poles of its resolvent and*

15) See, for example, [2].

coincides with the set of all k-th roots of unity, $k\geqq 1$.

REMARK. Theorem 3 answers affirmatively to the problem (b) of H.H. Schaefer [8] in the case of irreducible operators in $C(S)$.

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