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On Linear Measure of Projections of Two-dimensional Sets to Arbitrary Straight Line

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§1. Introduction. In connection with the paper by H. Federer [1], in which surface area is discussed, Professor K. Iseki has recently submitted the following problem:

Let p_θ be the orthogonal projection of the plane R^2 onto the straight line l_θ passing through the origin and with the slope $\tan\theta$, and denote by m the linear measure on l_θ for any θ. Let E be any bounded set in R^2. Then, what kind of regularities (e. g. upper or lower semi-continuity etc.) are satisfied by the function: $f_E(\theta)=m(p_\theta(E))$? (The condition '$l_\theta$ passes through the origin' is no essential restriction in this problem, but the formulation seems to become a little simpler by this condition.)

If E is a rectangular set or a circular disk, then $f_E(\theta)$ is clearly continuous. But, in general, the continuity of $f_E(\theta)$ does not hold. In the present paper, we shall give some partial answers to the above problem. Main results are as follows;—

I. If G is an open set, then $f_G(\theta)$ is lower semi-continuous (§2, Theorem 1), but not necessarily upper semi-continuous (§3, Example 1).

II. If K is a bounded and closed (consequently compact) set, then $f_K(\theta)$ is upper semi-continuous (§2, Theorem 2), but not necessarily lower semi-continuous (§3, Example 2).

Therefore, neither upper semi-continuity nor lower semi-continuity of $f_E(\theta)$ necessarily holds for general bounded set E in R^2.

We wish to express our hearty thanks to Professor K. Iseki who suggested us the problem and the contents of H. Federer's paper [1].

§2. Theorems. In this section, we shall prove the following theorems.

THEOREM 1. *The function $f_G(\theta)$ is lower semi-continuous for any open set $G \subset R^2$.*

THEOREM 2. *The function $f_K(\theta)$ is upper semi-continuous for any bounded closed (consequently compact) set $K \subset R^2$.*

To prove Theorem 1, we introduce the following notations ;—

T_θ = the orthogonal transformation in R^2 defined by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$\Phi_\theta = T_\theta^{-1} \circ p_\theta$ (= the composed mapping of T_θ^{-1} with p_θ).

Clearly T_θ^{-1} maps l_θ onto l_0 ($\equiv x$-axis) in one-to-one way, accordingly Φ_θ maps any subset of R^2 into l_0. Furthermore, we may easily see that

$$\Phi_\theta(E_1 \cup E_2) = \Phi_\theta(E_1) \cup \Phi_\theta(E_2) \text{ for any } E_1, E_2 \subset R^2 \tag{2.1}$$

and

$$m(\Phi_\theta(E)) = f_E(\theta) \text{ for any } E \subset R^2 . \tag{2.2}$$

If we put

$$\boldsymbol{C} = \{E \subset R^2 ; \lim_{\theta \to \theta_0} m(\Phi_\theta(E) \ominus \Phi_{\theta_0}(E)) = 0 \text{ for any } \theta_0\} \tag{2.3}$$

($A \ominus B$ denotes the symmetric difference of the sets A and B), then

LEMMA 1. *If both E_1 and $E_2 \in \boldsymbol{C}$, then $E_1 \cup E_2 \in \boldsymbol{C}$.*

PROOF. Since, by (2.1),

$$\begin{aligned}\Phi_\theta(E_1 \cup E_2) \ominus \Phi_{\theta_0}(E_1 \cup E_2) &= \{\Phi_\theta(E_1) \cup \Phi_\theta(E_2)\} \ominus \{\Phi_{\theta_0}(E_1) \cup \Phi_{\theta_0}(E_2)\} \\ &\subset \{\Phi_\theta(E_1) \ominus \Phi_{\theta_0}(E_1)\} \cup \{\Phi_\theta(E_2) \ominus \Phi_{\theta_0}(E_2)\} ,\end{aligned}$$

and since E_1 and $E_2 \in \boldsymbol{C}$, we have, by (2.3),

$$\begin{aligned}& m(\Phi_\theta(E_1 \cup E_2) \ominus \Phi_{\theta_0}(E_1 \cup E_2)) \\ & \leqq m(\Phi_\theta(E_1) \ominus \Phi_{\theta_0}(E_1)) + m(\Phi_\theta(E_2) \ominus \Phi_{\theta_0}(E_2)) \to 0 \ (\text{as } \theta \to \theta_0) ,\end{aligned}$$

which means: $E_1 \cup E_2 \in \boldsymbol{C}$.

LEMMA 2. *$f_E(\theta)$ is continuous for any $E \in \boldsymbol{C}$.*

PROOF. This is clear from (2.2), (2.3) and the inequality:

$$| m(\Phi_\theta(E)) - m(\Phi_{\theta_0}(E)) | \leqq m(\Phi_\theta(E) \ominus \Phi_{\theta_0}(E)) .$$

LEMMA 3. *If E is the union of a finite number of circular disks, then $f_E(\theta)$ is continuous.*

PROOF. We may put $E = D_1 \cup \cdots \cup D_n$ where D_ν's are circular disks. It may easily be shown that each D_ν belongs to $\boldsymbol{C}$. Hence $E \in \boldsymbol{C}$ by Lemma 1 and accordingly $f_E(\theta)$ is continuous by Lemma 2.

PROOF OF THEOREM 1. Any open set $G \subset R^2$ is expressible in the form

$$G = \bigcup_{\nu=1}^{\infty} D_\nu \qquad \text{where } D_\nu\text{'s are open circular disks.}$$

If we put $G_n = D_1 \cup \cdots \cup D_n$, then $\{f_{G_n}(\theta)\}$ is monotone increasing in n, $\lim_{n\to\infty} f_{G_n}(\theta) = f_G(\theta)$ for any θ, and each $f_{G_n}(\theta)$ is continuous in θ by Lemma 3. Hence $f_G(\theta)$ is lower semi-continuous.

PROOF OF THEOREM 2. It is sufficient to prove the upper semi-continuity of $f_K(\theta)$ at $\theta=0$ under the assumption that $K \subset S \equiv \{(x, y)\,;\ 0<x<1,\ 0<y<1\}$. Since K is compact and the projection p_0 is continuous mapping of R^2 onto l_0 ($\equiv x$-axis), $p_0(K)$ is a compact subset of l_0. Hence, for any $\varepsilon>0$, there exists a one-dimensional open set $U \subset l_0$ such that $U \supset p_0(K)$ and $m(U) < f_K(0)+\varepsilon$. For any $x \in p_0(K)$, there exists an open interval I_x such that $x \in I_x \subset U$, and accordingly there exists a finite number of mutually disjoint intervals $I_1, \cdots, I_n$ such that $p_0(K) \subset \sum_{\nu=1}^{n} I_\nu \subset U$. If we put

$$G_\nu = p_0^{-1}(I_\nu) \cap S \quad (\nu=1, \cdots, n) \quad \text{and} \quad G = \sum_{\nu=1}^{n} G_\nu\,,$$

then G is an open set in R^2 and contains K, and $p_0(G_\nu) = I_\nu$ $(\nu=1, \cdots, n)$. Since G_ν's are open rectangular sets, we may choose a sufficiently small $\delta>0$ such that $m(p_\theta(G_\nu)) < m(p_0(G_\nu)) + \varepsilon/n$ for any ν whenever $|\theta|<\delta$, whence

$$f_K(\theta) \leqq m(p_\theta(G)) \leqq \sum_{\nu=1}^{n} m(p_\theta(G_\nu)) < \sum_{\nu=1}^{n} m(p_0(G_\nu)) + \varepsilon$$

$$= \sum_{\nu=1}^{n} m(I_\nu) + \varepsilon \leqq m(U) + \varepsilon < f_K(0) + 2\varepsilon\,.$$

Hence $f_K(\theta)$ is upper semi-continuous at $\theta=0$.

§ 3. **Counter examples.**

We first give an example of an open set $G \subset R^2$ for which $f_G(\theta)$ is not upper semi-continuous.

EXAMPLE 1. Let $R_0 = \{r_1, r_2, \cdots, r_n, \cdots\}$ be the totality of rational numbers in the open interval $(0, 1)$, and put

$$I_n = \{x\,;\ r_n - 2^{-(n+2)} < x < r_n + 2^{-(n+2)}\} \quad (n=1, 2, \cdots)\,.$$

Then the set

$$G = \{(x, y)\,;\ x \in \bigcup_{n=1}^{\infty} I_n,\ 0<y<1\}$$

is an open set in R^2, and

$$f_G(0) = m(\bigcup_{n=1}^{\infty} I_n) \leqq \sum_{n=1}^{\infty} 2^{-(n+1)} = 2^{-1}\,;$$

while $f_G(\theta) \geqq 1$ whenever $0<\theta<\pi/2$ since G contains the set

$$\{(x, y)\,;\ x \in R_0,\ 0<y<1\}$$

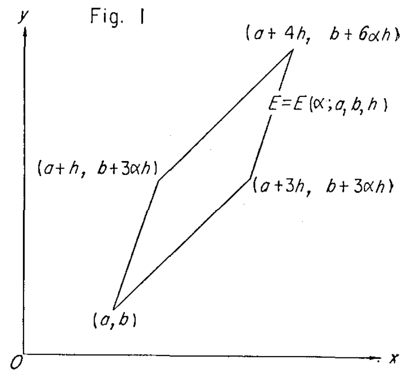
and R_0 is dense in $(0, 1)$. Hence $f_G(\theta)$ is not upper semi-continuous at $\theta=0$.

PRELIMINARIES FOR EXAMPLE 2. In order to construct an example of a compact set K for which $f_K(\theta)$ is not lower semi-continuous, we shall prove some preliminary lemmas.

We define

$$\theta(\alpha)=\text{Arctan}\,\alpha \quad \text{(the principal value)}$$

for any $\alpha>0$, and



$$E(\alpha\ ;\ a, b, h)=\left\{(x, y)\ ;\ \begin{matrix}\alpha(x-a)\leqq y-b\leqq\alpha(x-a+2h)\\ 3\alpha(x-a-2h)\leqq y-b\leqq 3\alpha(x-a)\end{matrix}\right\}\quad \text{(see Fig. 1)}$$

for any real numbers a, b and any positive numbers h and α, and put

$$\boldsymbol{E}_\alpha=\left\{E=E(\alpha, a, b, h)\ ;\ \begin{matrix}a \text{ and } b \text{ are any real numbers}\\ \text{and } h \text{ is any positive number}\end{matrix}\right\}$$

for any fixed $\alpha>0$.

LEMMA 1. *For any* $E\in\boldsymbol{E}_{2\alpha}$ *(α being fixed) and any* $\varepsilon>0$, *there exist*

$$E_1, \cdots, E_n\in\boldsymbol{E}_\alpha \quad (n<\infty)$$

such that

i) $E_j\subset E$ $(j=1, \cdots, n)$ *and* $E_j\cap E_k=\phi$ *for* $j\neq k$,

ii) $[p_{\frac{\pi}{2}+\theta}(E_j)]^\circ\cap[p_{\frac{\pi}{2}+\theta}(E_k)]^\circ=\phi$ *whenever* $0\leqq\theta\leqq\theta(\alpha)$ *and* $j\neq k$,

where $[A]^\circ$ *denotes the interior of* A *as a subset of the straight line* $l_{\frac{\pi}{2}+\theta}$,

iii) $m(p_{\frac{\pi}{2}+\theta(\alpha)}(E_1+\cdots+E_n))<\left(1-\dfrac{\cos\theta(\alpha)}{3}\right)m(p_{\frac{\pi}{2}}(E))$,

iv) $m(p_{\frac{\pi}{2}}(E_1+\cdots+E_n))>(1-\varepsilon)m(p_{\frac{\pi}{2}}(E))$.

PROOF. The set $E\in\boldsymbol{E}_{2\alpha}$ is expressible in the form

$$E=E(2\alpha, a_0, b_0, h_0) \qquad \text{(see Fig. 2)}$$

for some a_0, b_0 (real numbers) and $h_0>0$. Then, since the slope of the 'diagonal' expressed by the broken line in Fig. 2 is 3α, there exist real numbers a, b, a positive number h and an integer n such that the set

$$F=\{(x, y)\ ;\ b\leqq y\leqq b+6n\alpha h,\ 3\alpha(x-a-2h)\leqq y-b\leqq 3\alpha(x-a)\}$$

satisfies that (see Fig. 2)

$$F\subset E \text{ and } m(p_{\frac{\pi}{2}}(F))>12\alpha h_0(1-\varepsilon)\,.$$

(For example, let n be a natural number such that $\dfrac{1}{n+2}<\dfrac{\varepsilon}{2}$, and

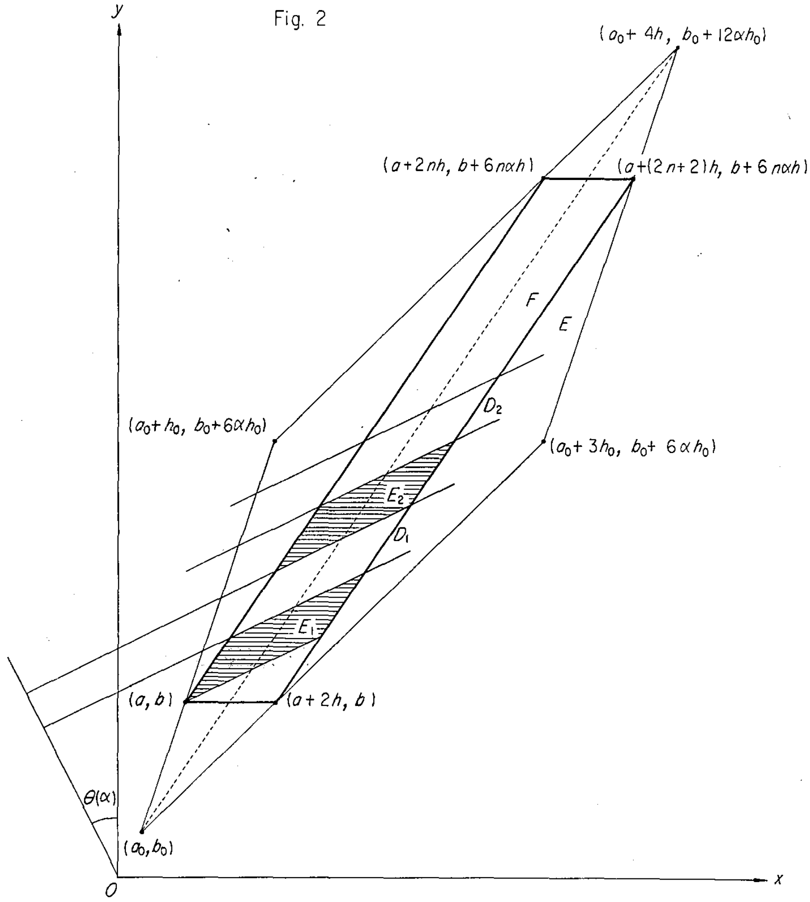


Fig. 2

put $h=\dfrac{2h_0}{n+2}$, $a=a_0+h$ and $b=b_0+6\alpha h$.) Next we put, for $j=1, \cdots, n$,

$$E_j=\left\{(x, y)\ ;\ \begin{array}{c}\alpha(x-a-2(j+2)h)\leqq y-b-6j\alpha h\leqq\alpha(x-a-2(j+1)h),\\ 3\alpha(x-a-2h)\leqq y-b\leqq 3\alpha(x-a)\end{array}\right\}$$

and

$$D_j=\{(x, y)\ ;\ \alpha(x-a-2(j+1)h)<y-b-6j\alpha h<\alpha(x-a-2jh)\}\ ,$$

Then $E_j\in\boldsymbol{E}_\alpha$ $(j=1, \cdots, n)$, and E_j's satisfy i) and ii). Furthermore

$$p_{\frac{\pi}{2}}(E_1+\cdots+E_n)=p_{\frac{\pi}{2}}(F)>12\alpha h_0(1-\varepsilon)=(1-\varepsilon)m(p_{\frac{\pi}{2}}(E))\ ;$$

this shows the condition iv). Finally the condition iii) is proved as follows. Since

$$m(p_{\frac{\pi}{2}+\theta(\alpha)}(F))\leqq 6n\alpha h=m(p_{\frac{\pi}{2}}(F))<m(p_{\frac{\pi}{2}}(E))\ ,$$

$$m(p_{\frac{\pi}{2}+\theta(\alpha)}(D_j))=2\alpha h\cos\theta(\alpha)\qquad (j=1, \cdots, n)$$

and

$$p_{\frac{\pi}{2}+\theta(\alpha)}(D_j)\cap p_{\frac{\pi}{2}+\theta(\alpha)}(D_k)=\phi\quad \text{for}\quad j\neq k\ ,$$

we have

$$m\Big(\sum_{j=1}^{n} p_{\frac{\pi}{2}+\theta(\alpha)}(D_j)\Big)=2n\alpha h\cos\theta(\alpha)=\frac{\cos\theta(\alpha)}{3}m(p_{\frac{\pi}{2}}(F))$$

and hence

$$m\Big(p_{\frac{\pi}{2}+\theta(\alpha)}(E_1+\cdots+E_n)\Big)\leqq m\Big(p_{\frac{\pi}{2}+\theta(\alpha)}(F)-\sum_{j=1}^{n}p_{\frac{\pi}{2}+\theta(\alpha)}(D_j)\Big)$$

$$=\Big(1-\frac{\cos\theta(\alpha)}{3}\Big)m\Big(p_{\frac{\pi}{2}}(F)\Big)<\Big(1-\frac{\cos\theta(\alpha)}{3}\Big)m\Big(p_{\frac{\pi}{2}}(E)\Big)\,;$$

this shows the condition iii).

LEMMA 2. *Let θ be any fixed real number, and $\{K_\nu\}$ be a monotone decreasing sequence of compact sets in R^2. Then*

$$p_\theta(\bigcap_{\nu=1}^{\infty}K_\nu)=\bigcap_{\nu=1}^{\infty}p_\theta(K_\nu)\,.$$

PROOF. The relation $p_\theta(\bigcap_{\nu=1}^{\infty}K_\nu)\subset\bigcap_{\nu=1}^{\infty}p_\theta(K_\nu)$ is evident; we shall prove the converse inclusion relation. For any point $P\in\bigcap_{\nu=1}^{\infty}p_\theta(K_\nu)$, we put

$$L_P=\{Q\in R^2\,;\quad p_\theta(Q)=P\}\,.$$

Then $L_P\cap K_\nu\neq\phi$ for any ν, and $\{L_P\cap K_\nu\,;\ \nu=1,2,\cdots\}$ is a monotone decreasing sequence of compact sets in R^2. Hence $\bigcap_{\nu=1}^{\infty}(L_P\cap K_\nu)\neq\phi$, and accordingly $L_P\cap(\bigcap_{\nu=1}^{\infty}K_\nu)$ contains at least one point Q. Hence

$$P=p_\theta(Q)\in p_\theta(\bigcap_{\nu=1}^{\infty}K_\nu)\,.$$

Thus we see that $\bigcap_{\nu=1}^{\infty}p_\theta(K_\nu)\subset p_\theta(\bigcap_{\nu=1}^{\infty}K_\nu)$.

EXAMPLE 2. We put $\theta_\nu=\theta(2^{-\nu})\equiv\mathrm{Arctan}\,2^{-\nu}$ $(\nu=1,2,\cdots)$, and construct a monotone decreasing sequence $\{K_\nu\}$ of compact sets in R^2 such that the following conditions a_ν), b_ν), c_ν) and d_ν) are satisfied for each ν;—

a_ν) $K_\nu=E_{\nu1}+\cdots+E_{\nu l_\nu}$, $E_{\nu j}\in \boldsymbol{E}_{1/2^\nu}$ $(j=1,\cdots,l_\nu)$,

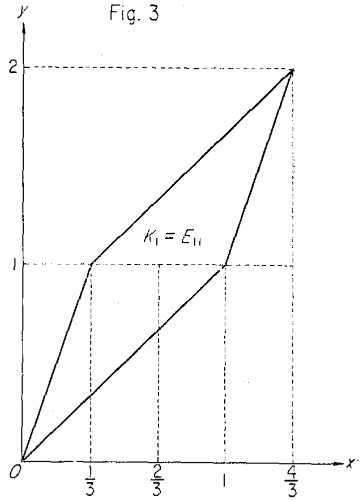
b_ν) $[p_{\frac{\pi}{2}+\theta}(E_{\nu j})]^\circ\cap[p_{\frac{\pi}{2}+\theta}(E_{\nu k})]^\circ=\phi$ whenever $0\leqq\theta\leqq\theta_\nu$ and $j\neq k$,

c_ν) $$m\Big(p_{\frac{\pi}{2}+\theta_\nu}(K_\nu)\Big)<\begin{cases}1-\dfrac{\cos\theta_1}{3} & \text{for}\quad \nu=1\,,\\[2mm] \Big(1-\dfrac{\cos\theta_\nu}{3}\Big)m\Big(p_{\frac{\pi}{2}}(K_{\nu-1})\Big) & \text{for}\quad \nu\geqq 2\,,\end{cases}$$

d_ν) $m\Big(p_{\frac{\pi}{2}}(K_\nu)\Big)<\prod_{\kappa=1}^{\nu}(1-2^{-\kappa})$.

Such sequence $\{K_\nu\}$ is constructed by mathematical induction as follows. For $\nu=1$, we put

$$K_1=E_{11}=E\left(1, 0, 0, \frac{1}{3}\right)$$



(see Fig. 3). Then a_1), b_1), c_1) and d_1) are obviously satisfied. Assume that K_ν is obtained and satisfies a_ν), b_ν), c_ν) and d_ν). K_ν is expressible in the form:

$$K_\nu=E_{\nu 1}+\cdots+E_{\nu l_\nu},$$
$$E_{\nu l}\in \boldsymbol{E}_{1/2^\nu} \ (l=1, \cdots, l_\nu).$$

Applying Lemma 1 to each $E_{\nu l}$ ($1\leqq l\leqq l_\nu$), we obtain

$$E_{\nu l,1}, \cdots, E_{\nu l, n_{\nu l}}\in \boldsymbol{E}_{1/2^{\nu+1}}$$

such that

i) $E_{\nu l,j}\subset E_{\nu l}$ ($j=1, \cdots, n_{\nu l}$) and $E_{\nu l,j}\cap E_{\nu l,k}=\phi$ for $j\neq k$,

ii) $[p_{\frac{\pi}{2}+\theta}(E_{\nu l,j})]^\circ\cap[p_{\frac{\pi}{2}+\theta}(E_{\nu l,k})]^\circ\neq\phi$ whenever $0\leqq\theta\leqq\theta_{\nu+1}$ and $j\neq k$,

iii) $m\left(p_{\frac{\pi}{2}+\theta_{\nu+1}}(E_{\nu l,1}+\cdots+E_{\nu l,n_{\nu l}})\right)<\left(1-\frac{\cos\theta_{\nu+1}}{3}\right)m\left(p_{\frac{\pi}{2}}(E_{\nu l})\right)$,

iv) $m(p_{\frac{\pi}{2}}(E_{\nu l,1}+\cdots+E_{\nu l,n_{\nu l}}))>(1-2^{-(\nu+1)})m(p_{\frac{\pi}{2}}(E_{\nu l}))$.

We reorder the sets $E_{\nu l,j}$ ($j=1, \cdots, n_{\nu l}$; $l=1, \cdots, l_\nu$) and denote them by $E_{\nu+1,1}, \cdots, E_{\nu+1,l_{\nu+1}}$, and put

$$K_{\nu+1}=\sum_{l=1}^{l_{\nu+1}}E_{\nu+1,l}\equiv\sum_{l=1}^{l_\nu}\sum_{j=1}^{n_{\nu l}}E_{\nu l,j} \tag{3.1}$$

Then $K_{\nu+1}$ is compact, $K_{\nu+1}\subset K_\nu$ by a_ν) and i), and $a_{\nu+1}$) also holds. $b_{\nu+1}$) follows from b_ν) and ii). Since each $p_{\frac{\pi}{2}}(E_{\nu l})$ ($1\leqq l\leqq l_\nu$) is an interval, a_ν) and b_ν) imply that

$$\sum_{l=1}^{l_\nu}m(p_{\frac{\pi}{2}}(E_{\nu l}))=m(p_{\frac{\pi}{2}}(K_\nu)). \tag{3.2}$$

Hence

$$\begin{aligned} m\left(p_{\frac{\pi}{2}+\theta_{\nu+1}}(K_{\nu+1})\right) &\leqq\sum_{l=1}^{l_\nu}m\left(p_{\frac{\pi}{2}+\theta_{\nu+1}}(E_{\nu l,1}+\cdots+E_{\nu l,n_{\nu l}})\right) && \text{(by (3.1))}\\ &<\sum_{l=1}^{l_\nu}\left(1-\frac{\cos\theta_{\nu+1}}{3}\right)m\left(p_{\frac{\pi}{2}}(E_{\nu l})\right) && \text{(by iii))}\\ &=\left(1-\frac{\cos\theta_{\nu+1}}{3}\right)m\left(p_{\frac{\pi}{2}}(K_\nu)\right) && \text{(by (3.2))}\end{aligned}$$

$c_{\nu+1}$) is thus obtained. $d_{\nu+1}$) is proved as follows;

$$m\left(p_{\frac{\pi}{2}}(K_{\nu+1})\right)=\sum_{l=1}^{l_\nu}\sum_{j=1}^{n_{\nu l}} m\left(p_{\frac{\pi}{2}}(E_{\nu l,j})\right) \qquad \text{(by (3.1) and } \mathrm{b}_{\nu+1}))$$

$$>\left(1-\frac{1}{2^{\nu+1}}\right)\sum_{l=1}^{l_\nu} m\left(p_{\frac{\pi}{2}}(E_{\nu l})\right) \qquad \text{(by ii) and iv))}$$

$$=\left(1-\frac{1}{2^{\nu+1}}\right)m\left(p_{\frac{\pi}{2}}(K_\nu)\right)>\prod_{\kappa=1}^{\nu+1}\left(1-\frac{1}{2^\kappa}\right) \qquad \text{(by (3.2) and } \mathrm{d}_\nu)).$$

Thus the desired sequence $\{K_\nu\}$ is obtained. Now we put

$$K=\bigcap_{\nu=1}^{\infty} K_\nu .$$

Then K is a non-empty compact set. By Lemma 2 and d_ν), we have

$$m(p_{\frac{\pi}{2}}(K))=m(\bigcap_{\nu=1}^{\infty} p_{\frac{\pi}{2}}(K_\nu))=\lim_{\nu\to\infty} m(p_{\frac{\pi}{2}}(K_\nu)) \tag{3.3}$$

$$\geqq \prod_{\nu=1}^{\infty}\left(1-\frac{1}{2^\nu}\right)>0 .$$

On the other hand, for any ν,

$$m\left(p_{\frac{\pi}{2}+\theta_\nu}(K)\right)\leqq m\left(p_{\frac{\pi}{2}+\theta_\nu}(K_\nu)\right)\leqq\left(1-\frac{\cos\theta_\nu}{3}\right)m\left(p_{\frac{\pi}{2}}(K_{\nu-1})\right)$$

by c_ν). Hence

$$\varliminf_{\theta\to\frac{\pi}{2}+0} m\left(p_\theta(K)\right)\leqq\varlimsup_{\nu\to\infty} m\left(p_{\frac{\pi}{2}+\theta_\nu}(K)\right) \tag{3.4}$$

$$\leqq\lim_{\nu\to\infty}\left(1-\frac{\cos\theta_\nu}{3}\right)m\left(p_{\frac{\pi}{2}}(K_{\nu-1})\right)=\frac{2}{3}m\left(p_{\frac{\pi}{2}}(K)\right).$$

(3.3) and (3.4) imply that the function $f_K(\theta)=m(p_\theta(K))$ is not lower semi-continuous at $\theta=\frac{\pi}{2}$.

References

[1] H. Federer: Surface area I, Trans. Amer. Math. Soc. 55 (1944), 420-437.
[2] S. Saks: Theory of the integral, Warszawa (1937), Chap. V.