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Some Remarks on Compactness of Measure-spaces

Sachio Ohba (大 庭 幸 雄)

Department of Mathematics, Faculty of Science
Ochanomizu University, Tokyo, (Researcher)
(Kohnodai Joshi Gakuin)

1. Introduction

The relations between the topology of a fixed space S and that of the space $\boldsymbol{M}$ of the measures on S have been studied by several authors,[1)2)3)] among which in 1951 by J. H. Blau the following result was obtained :[2)]

"*If S statisfies the second axiom of countability and is a countably compact, Hausdorff's space, then the space* $\boldsymbol{M}$ *of all normalized positive regular outer measures for which open sets are measurable is compact for the weak topology of* $\boldsymbol{M}$."

From this proposition the condition of countability may be removed as is established for example in the text book of N. Bourbaki,[1)] in which the condition of countably-compactness is replaced by that of compactness.

Now the theorem we are going to establish is :

Theorem 1. *If S is a countably compact, normal Hausdorff's space, then the space* $\boldsymbol{M}$ *becomes compact.*

By a measure on a topological space S, we shall mean a set-function (outer measure) m satisfying the following conditions (1)–(5) :

(1) $+\infty > m(A) \geqq 0$ for any $A \subset S$ and $m(\phi)=0$ (ϕ denoting the null set)

(2) $A \subset B \Rightarrow m(A) \leqq m(B)$

(3) $A \subset \bigcup_{j=1}^{\infty} A_j \Rightarrow m(A) \leqq \sum_{j=1}^{\infty} m(A_j)$

(4) $A \subset S \Rightarrow m(A) = \inf_{G \subset A} m(G)$ (G being open)

(5) open sets are measurable with respect to m.

Let us denote by $\boldsymbol{M}$ and $MC(S)$ respectively the set of all measures with total measure 1 and the set of all real valued bounded continuous functions defined on a topological space S, of which the latter becomes by the usual supremum norm, a Banach space.

Then, every function $f \in MC(S)$ is measurable with respect to any measure $m \in \boldsymbol{M}$, so that the integral $\int_S f\,dm$ is well defined.

By a neighbourhood of $m_0 \in \boldsymbol{M}$, we shall mean, for each $\varepsilon > 0$ and for any finite number of $f_1, \ldots, f_n$ ($f_j \in MC(S)$ for $1 \leqq j \leqq n$), the set

$$U(m_0\,;\, f_1, \ldots, f_n\,;\, \varepsilon) = \left\{ m \mid m \in \boldsymbol{M}, \left| \int_S f_j\, dm_1 - \int_S f_j\, dm_0 \right| < \varepsilon \quad \text{for } j = 1, \ldots, n \right\}$$

which assigns to $\boldsymbol{M}$ what is so called the weak topology.

A topological space is countably compact if and only if any countable open covering of it contains a finite sub-covering.

From this definition it is evident that any continuous image of a countably compact space is itself countably compact so that every real valued continuous function on a countably compact space attains its extremum values since, in a space with countable base, countably compactness is equivalent to compactness, and if S is countably compact, $MC(S)$ is identical with $C(S)$, the set of real valued continuous functions defined on S.

2. The proofs and some results

Let us now begin with establishing the following 3 lemmas.

Lemma 1. *Let S be a countably compact, normal, Hausdorff's space. Suppose a linear functional Lf on $C(S)$ satisfy the following conditions* (1), (2) *and* (3) ;

(1) $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ (α, β *being real*)

(2) $L(1) = 1$ (1 *denoting also the constant valued function with the value* 1)

(3) $f \geqq 0 \Rightarrow Lf \geqq 0$.

Then there exists a measure $m \in \boldsymbol{M}$ such that, for all $f \in C(S)$, holds

$$Lf = \int_S f\, dm\,.$$

Proof. Only the outline of the proof will be given here, since further complementation is quite easy.

Given an open set G in S, let us denote by c_G the characteristic function of G. Let us put now

$$\boldsymbol{F}(G) = \{ f \mid 0 \leqq f(x) \leqq c_G(x), f \in C(S) \}\,.$$

Evidently $\boldsymbol{F}(G)$ contains the constant function 0, and if $G \neq \phi$, by the normality of S, it also contains non-constant functions.

Now for any two open sets G_1 and G_2, writing

$$\boldsymbol{F}(G_1) + \boldsymbol{F}(G_2) = \{ f + g \mid f \in \boldsymbol{F}(G_1), g \in \boldsymbol{F}(G_2) \}\,,$$

it follows that

(4) $\boldsymbol{F}(G_1) + \boldsymbol{F}(G_2) \supset \boldsymbol{F}(G_1 \cup G_2)$

(5) $G_1 \cap G_2 = \phi \Rightarrow \boldsymbol{F}(G_1) + \boldsymbol{F}(G_2) \subset \boldsymbol{F}(G_1 \cup G_2)$

in the proof of which the assumption of normality is necessary.

Define now for any open set G, a non-negative number

$$m(G)=\sup_{f\in \boldsymbol{F}(G)} Lf\,.$$

It follows that

(6) $m(S)=1, m(\phi)=0$

(7) $G_1\subset G_2 \Rightarrow m(G_1)\leqq m(G_2)$

(8) $m(G_1\cup G_2)\leqq m(G_1)+m(G_2)$

(9) $G_1\cap G_2=\phi \Rightarrow m(G_1\cup G_2)=m(G_1)+m(G_2)$

(10) $G\subset \bigcup_{j=1}^{\infty} G_j \Rightarrow m(G)\leqq \sum_{j=1}^{\infty} m(G_j)$

in which (6), (7), (8) and (9) are almost immediate and (10) will be proved replacing the condition of compactness in the usual proof of the corresponding proposition by that of countably-compactness.

To define an element of $\boldsymbol{M}$ by making use of m obtained already, let us put for an arbitrary subset A of S

$$m^*(A)=\inf_{G\supset A} m(G)\,.$$

Then

(11) $m^*(G)=m(G)$ for any open set G;

(12) $m^*(S)=1,\ m^*(\phi)=0$;

(13) $A\supset B \Rightarrow m^*(A)\geqq m^*(B)$;

(14) $A\subset \bigcup_{j=1}^{\infty} A_j \Rightarrow m^*(A)\leqq \sum_{j=1}^{\infty} m^*(A_j)$.

Thus m^* is a regular measure belonging to $\boldsymbol{M}$, for which open sets are measurable.

By the well defined integral with respect to the measure m^* constructed above, the functional Lf is so represented that

$$f\in C(S) \Rightarrow Lf=\int_S f\, dm^{*}\text{[4]}.$$

To prove this equality, as is well known it is only to show that for every positive $f\in C(S)$ the following inequality holds:

$$Lf\geqq \int_S f\, dm^*.$$

Let n_0 be a natural number such that for all $x\in S$

$$(0<)f(x)\leqq n_0\,.$$

Then we have

$$(*)\qquad \int_S f\, dm^*=\lim_{n\to\infty}\sum_{k=1}^{n_0 2^{n_0}} \frac{k}{2^n} m^*\left(\left\{x \mid \frac{k-1}{2^n}<f(x)\leqq \frac{k}{2^n}\right\}\right)$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n_0 2^{n_0}-1} \frac{1}{2^n} m^*(G_{nk})=\lim_{n\to\infty}\sum_{k=1}^{n_0 2^{n_0}-1} \frac{1}{2^n} m^*(G_{nk})$$

where $G_{nk}=\left\{x \mid f(x)>\frac{k}{2^n}\right\}$ $(k=0, 1, \dots, n_0 2^{n_0}-1)$.

Since every G_{nk} is open, there exists for each $\varepsilon>0$ a function $f_{nk}\in \boldsymbol{F}(G_{nk})$ such that

(**) $$Lf_{nk}+\varepsilon>m(G_{nk})=m^*(G_{nk})\,.$$

But for functions $f_{nk}\in \boldsymbol{F}(G_{nk})$ we have obviously

$$\frac{1}{2^n}\sum_{k=1}^{n_0 2^{n_0}-1} f_{nk}(x)\leqq f(x)$$

so that

$$\frac{1}{2^n}\sum_{k=1}^{n_0 2^{n_0}-1} Lf_{nk}\leqq Lf\,.$$

From this and (*) and (**), since ε is independent of n, we have

$$Lf\geqq\frac{1}{2^n}\sum_{k=1}^{n_0 2^{n_0}-1} m^*(G_{nk})\to\int_S f\,dm^* \quad (n\to\infty)$$

whence

$$Lf\geqq\int_S f\,dm^*$$

which completes our proof.

Lemma 2. *Let S be a normal Hausdorff's space. If, for all non-negative $f\in MC(S)$,*

$$\int_S f\,dm_1=\int_S f\,dm_2 \quad (m_1\in \boldsymbol{M},\ m_2\in \boldsymbol{M})$$

then we have $m_1=m_2$.

Proof. It is only to show that for any open set G we have $m_1(G)=m_2(G)$, since m_1 and m_2 are regular measures.

From the definition of measure there exist for any $\varepsilon>0$ two closed sets $F_1(\subset G)$ and $F_2(\subset G)$ such that

$$m_1(G-F_1)<\varepsilon,\ m_2(G-F_2)<\varepsilon\,.$$

Writing $F=F_1\cup F_2$, it follows

(15) $m_1(G-F)<\varepsilon,\ m_2(G-F)<\varepsilon.$

F being closed and contained in G, there exists, by Urysohn's lemma,[5] a continuous function $f(x)$ defined on S satisfying $0\leqq f(x)\leqq 1$ for $x\in S$, $f(x)=1$ for $x\in F$ and $f(x)=0$ for $x\in G^c$, the complementary set of G.

Hence $f\in MC(S)$ and from (15) we have

$$\int_S f\,dm_1\leqq m_1(G)<\int_S f\,dm_1+\varepsilon,\ \int_S f\,dm_2\leqq m_2(G)<\int_S f\,dm_2+\varepsilon$$

which shows $|m_1(G)-m_2(G)|<\varepsilon$ and consequently we have

$$m_1(G)=m_2(G)$$

as ε is arbitrary.

Lemma 3. *If S is a mormal Hausdorff's space, then the space $\boldsymbol{M}$ becomes Hausdorff's space.*

Proof. From Lemma 2, we may find, for $m_1 \in \boldsymbol{M}, m_2 \in \boldsymbol{M}, m_1 \neq m_2$, a non-negative function $f \in MC(S)$ satisfying

$$\int_S f\,dm_1 \neq \int_S f\,dm_2 .$$

Putting $\left|\int_S f\,dm_1 - \int_S f\,dm_2\right| = 2\varepsilon$, it follows

$$U(m_1 ; f ; \varepsilon) \cap U(m_2 ; f ; \varepsilon) = \phi$$

which shows that $\boldsymbol{M}$ satisfies the separation axiom of Hausdorff.

Remark. m is determined uniquely by the mapping : $f \to \int_S f\,dm$.

Proof of Theorem 1. It is obvious that the functional

$$m(f) = \int_S f\,dm$$

defined on $C(S) = MC(S)$ satisfies the conditions (1), (2) and (3) of Lemma 1. Since the set $I_f = \{y \mid -\|f\| \leqq y \leqq \|f\|\}$ is compact in the real number space R for any $f \in C(S)$ where $\|f\| = \sup_{x \in S} |f(x)|$, by the well known theorem of Tychonoff,[5] the topological product

$$\boldsymbol{E} = \prod_{f \in C(S)} I_f$$

becomes compact, in which, as a neighbourhood of a point $\mu_0 \in \boldsymbol{E}$, we may choose for each $\varepsilon > 0$ and for each finite number of $f_j \in C(S)$ $(j = 1, \ldots, n)$ the subset

$$V(\mu_0 ; f_1, \ldots, f_n ; \varepsilon) = \{\mu \mid \mu \in \boldsymbol{E}, |\mu(f_j) - \mu_0(f_j)| < \varepsilon \quad \text{for } 1 \leqq j \leqq n\} .$$

Since $m \in \boldsymbol{M}$ is determined by the mapping : $f \to m(f)$ as remarked at the end of Lemma 3 and $m(f) \in I_f$ for any $f \in C(S)$, m may be considered as a point in $\boldsymbol{E}$ with the "f-component" $m(f)$ for $f \in C(S)$. Thus the set $\boldsymbol{M}$ together with its topology may be regarded as a subspace (not as a linear space) of $\boldsymbol{E}$. Let m' be a point of $\boldsymbol{E}$ belonging to the closure of $\boldsymbol{M}$. Then, for each $\varepsilon > 0$ and for three functions f, g and $\alpha f + \beta g$ belonging to $C(S)$, a neighbourhood of m'

$$V(m' ; f, g, \alpha f + \beta g ; \varepsilon)$$

intersects with $\boldsymbol{M}$, that is to say, there exists an $m \in \boldsymbol{M}$ such that

$$|m'(\alpha f + \beta g) - m(\alpha f + \beta g)| < \varepsilon, \; |m'(f) - m(f)| < \varepsilon, \; |m'(g) - m(g)| < \varepsilon .$$

From these inequalities we have the following equality

(16) $\quad m'(\alpha f + \beta g) = \alpha m'(f) + \beta m'(g),$

since ε is arbitrary and

$$m(\alpha f+\beta g)=\int_S(\alpha f+\beta g)\,dm=\alpha\int_S f\,dm+\beta\int_S g\,dm=\alpha m(f)+\beta m(g)\,.$$

In a similar way, it is easy to show that first $m'(1)=1$ and secondly $m'(f)\geqq 0$ for any non-negative function $f\in C(S)$.

Thus, $m'(f)$ satisfying all the conditions of Lemma 1, there exists a measure $m\in \boldsymbol{M}$ such that for any $f\in C(S)$ we have

$$m'(f)=\int_S f\,dm$$

which assures that m' is an element of $\boldsymbol{M}$, showing that $\boldsymbol{M}$ is closed in $\boldsymbol{E}$, and consequently compact.

Remark. The results proved above contains the result of J. H. Blau mentioned in § 1, though the neighbourhood in his theorem are considered only for non-negative functions belonging to $C(S)$.

In an analogous way as above, the set $\boldsymbol{M}'$ of all measures m satisfying $m(S)\leqq 1$ may also be topologized so that the following lemma may hold, the proof of which is almost the same as that of Lemma 1.

Lemma 4. *Let S be a countably compact, normal, Hausdorff's space. Suppose a linear functional Lf defined on C(S) satisfy the conditions of* (1) *and* (3) *of Lemma 1 and the following condition* (2′) (*which is weaker than that of* (2) *of Lemma 1*):

(2′) $f\in C(S)\Rightarrow |Lf|\leqq ||f||$.

Then, there exists a measure $m\in\boldsymbol{M}'$ *such that*

$$f\in C(S)\Rightarrow Lf=\int_S f\,dm\,.$$

From this, it is able to derive as above the following:

Proposition. *If S is a countably compact, normal, Hausdorff's space, then,* $\boldsymbol{M}'$ *becomes a compact, Hausdorff's space.*

Now let us establish a theorem which may be regarded as a generalization of one of the results obtained by Kryloff & Bogoliouboff.[3)]

Theorem 2. *Let S be a normal Hausdorff's space which may be represented as a limit of an ascending sequence of sets* $\{S_n\}$ (*in a strict sense*) *satisfying*

(17) *each* S_n *is closed and countably compact in S*

(18) $S_n\subset S^i_{n+1}$ $(n=1, 2, \ldots)$ (S^i_{n+1} *denoting the open kernel of* S_{n+1}).

Then, a necessary and sufficient condition for $\boldsymbol{M}'$ *to be compact is that for each* $\varepsilon>0$, *there exists an integer* $n_0=n_0(\varepsilon)$ *such that*

(19) $m\in\boldsymbol{M}'\Rightarrow m(S-S_{n_0})<\varepsilon$.

Proof. Let us first remark that just like Lemma 3 $\boldsymbol{M}'$ becomes a Hausdorff's space.

Necessity. Supposing the contrary, for some $\varepsilon_0>0$ and for any natural number n there existed a measure $m=m_n\in\boldsymbol{M}'$ satisfying

(20) $m(S-S_n)\geqq\varepsilon_0$.

The set of m_n thus obtained must be an infinite set since if the set were finite, there would exist a sequence of measures

$$m_{n_1}, m_{n_2}, \ldots \quad (n_1<n_2<\ldots)$$

such that $m_{n_1}=m_{n_2}=\ldots$, while by (20)

$$m_{n_j}(S-S_{n_j})\geqq\varepsilon_0 \quad \text{for } j=1,2,\ldots$$

so that $m_{n_k}(S-S_{n_j})\geqq\varepsilon_0$ for $j=1,2,\ldots$ and for $k=1,2,\ldots$ which would be a contradiction since for a fixed k

$$m_{n_k}(S-S_{n_j})\to 0 \quad (j\to\infty)\,.$$

$\boldsymbol{M}'$ being compact, the infinite set $\{m_n\}$ has a point $m_0\in\boldsymbol{M}'$ of accumulation.

For a fixed k, there exists by the normality of S a function $f_k\in MC(S)$ with $0\leqq f_k(x)\leqq 1$ such that

$$x\in S_k\Rightarrow f_k(x)=1,\ x\notin S^i_{k+1}\Rightarrow f_k(x)=0\,.$$

Then the function $g_k(x)=1-f_k(x)$ also belongs to $MC(S)$ and satisfies for $n>k$

(21) $\int_S g_k\, dm_n\geqq m_n(S-S_{k+1})\geqq m_n(S-S_n)\geqq\varepsilon_0$

while

(22) $\int_S g_k\, dm_0\leqq m_0(S-S_k)\,.$

Since m_0 is an accumulation point of $\{m_n\}$, there exists an $m\in\{m_n\,|\,n>k\}$ such that

(23) $\left|\int_S g_k\, dm-\int_S g_k\, dm_0\right|<\frac{1}{2}\varepsilon_0$.

From (21), (22) and (23) it follows

$$m_0(S-S_k)\geqq\frac{1}{2}\varepsilon_0\,,$$

which, making $k\to\infty$, contradicts $\lim_{k\to\infty} m_0(S_k)=m_0(S)$.

Sufficiency. Since, like the proof of Theorem 1, by making use of Tychonoff's theorem, we can conclude that the topological product

$$\boldsymbol{E}=\prod_{f\in MC(S)} I_f \quad (I_f=\{y\,|\,-\|f\|\leqq y\leqq\|f\|\})$$

is a compact space, in which the set $\boldsymbol{M}'$ may be considered to be contained, it is to be shown that $\boldsymbol{M}'$ is closed in $\boldsymbol{E}$.

Let L be an element of $\boldsymbol{M}'^a$. Then it is obvious that

$$L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)\,;\ |Lf|\leqq\|f\|\,;\ Lf\geqq 0 \quad \text{for } f\geqq 0\,,$$

whence it is only to show that for some $m_0\in\boldsymbol{M}'$ holds

$$f\in MC(S)\Rightarrow Lf=\int_S f\, dm_0\,.$$

To prove the existence of m_0, we have only to follow the construction in the proof of Lemma 1, by putting $m_0(G)=\sup_{f\in\boldsymbol{F}(G)} Lf$ for every open set G. It is easy to verify that $m=m_0$ fulfils the conditions (6)-(9).

To show $m=m_0$ satisfies also the condition (10), let us first remark that there exists by the normality of S a continuous function $h_n\in MC(S)$ such that

$$x\in S_n\Rightarrow h_n(x)=1\ ;\ x\in S_{n+1}^{ic}\Rightarrow h_n(x)=0\ ;\ 0\leqq h_n(x)\leqq 1 \quad \text{for } x\in S\,.$$

By the condition of our theorem, there exists for each $\varepsilon>0$ a natural number n_0 such that

$$m\in\boldsymbol{M}'\Rightarrow m(S-S_{n_0})<\varepsilon\,.$$

By the definition of $L\in\boldsymbol{M}'^a$, there exists for each $f\in MC(S)$, $f\geqq 0$, and for each $\varepsilon>0$ a measure $m\in\boldsymbol{M}'$ such that

$$\left|\int_S f\,dm-Lf\right|<\varepsilon,\ \left|\int_S fh_{n_0}\,dm-L(fh_{n_0})\right|<\varepsilon\,,$$

where fh_{n_0} evidently belongs to $MC(S)$.

Since

$$\left|Lf-L(fh_{n_0})\right|\leqq\left|\int_S f\,dm-Lf\right|+\left|\int_S fh_{n_0}\,dm-L(fh_{n_0})\right|$$

$$+\left|\int_S f\,dm-\int_S fh_{n_0}\,dm\right|\leqq 2\varepsilon+\left\|f-fh_{n_0}\right\|m(S-S_{n_0})$$

$\leqq 2\varepsilon+\varepsilon\|f\|$, we have $\lim_{n\to\infty} L(fh_n)=Lf$.

Choosing a function $f\in\boldsymbol{F}(G)$ for which

(24) $Lf+\varepsilon\geqq m_0(G)$,

we shall put $f_1(x)=\max\{f(x)-\varepsilon, 0\}$, from which follows $|f_1(x)-f(x)|<\varepsilon$, $\sup_{x\in S}|f_1(x)-f(x)|\leqq\varepsilon$, $f_1-f\in MC(S)$ and $|L(f_1-f)|\leqq\|f_1-f\|<\varepsilon$ whence after all we have

(25) $Lf_1\geqq Lf-\varepsilon$.

But since

$$\{x\,|\,f_1(x)\cdot h_n(x)>0\}\subset\{x\,|\,f(x)\geqq\varepsilon\}\cap S_{n+1}$$

$$=F_n\subset\{x\,|\,f(x)>0\}\cap S_{n+1}\subset\bigcup_{k=1}^{\infty}G_k\cap S_{n+1}$$

in which F_n is a closed set of S_{n+1} and S_{n+1} is a countably compact set, there exists a finite number of open sets G_{k_i} $(i=1,\dots,l)$ such that

$$F_n\subset\bigcup_{i=1}^{l}G_{k_i}\cap S_{n+1}\subset\bigcup_{i=1}^{l}G_{k_i}\,,$$

which also shows that for $\varepsilon\leqq 1$ $f_1h_n\in\boldsymbol{F}(G_{k_1}\cup\dots\cup G_{k_l})\subset\boldsymbol{F}(G_{k_1})\cup\dots\cup\boldsymbol{F}(G_{k_l})$ or we may write:

$$f_1h_n=g_1+\dots+g_l\ ;\ g_1\in\boldsymbol{F}(G_{k_1}),\ \dots,\ g_l\in\boldsymbol{F}(G_{k_l})\,.$$

From this follows

$$L(f_1 h_n) = \sum_{i=1} L g_i \leqq \sum_{i=1} m_0(G_{k_i}) \leqq \sum_{n=1}^{\infty} m_0(G_n)$$

in which making $n \to \infty$, we have

$$Lf_1 = \lim L(f_1 h_n) \leqq \sum_{n=1}^{\infty} m_0(G_n) .$$

Hence by (24), (25) we have

$$m_0(G) \leqq Lf + \varepsilon \leqq Lf_1 + 2\varepsilon \leqq \sum_{n=1}^{\infty} m_0(G_n) + 2\varepsilon$$

which shows that $m = m_0$ satisfies also (10).

Next putting $m_0^*(A) = \inf_{A \subset G} m_0(A)$, it is easily seen that m^* is an outer measure for which open sets are measurable, $m_0^*(S) \leqq 1$, $m_0^*(G) = m_0(G)$ for open G and

$$f \in MC(S) \Rightarrow Lf = \int_S f \, dm_0^*$$

which completes our proof.

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