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On Perturbations of Discontinuous Solutions of Ordinary Differential Equations

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1. Introduction

The equation

$$\varepsilon d^2u/dt^2+(u^2-1)du/dt+u=0$$

is known as the van der Pol equation. As $\varepsilon\to 0$, solutions of this equation on reaching $u=1+0$ tend to jump to $u=-2$. N. Levinson made clear how to determine the jump curve for solutions of a system of the following type

$$\varepsilon d^2u/dt^2+g(t, u, x, \varepsilon)du/dt+h(t, u, x, \varepsilon)=0 ,$$
$$dx/dt=f(t, u, x, \varepsilon)du/dt+\varphi(t, u, x, \varepsilon) ,$$

where u is a scalar, but x is a vector [3]. In this paper we shall study the case when u and x are both vectors.

2. Definition of a Discontinuous Solutions

Let x and U be m- and m'-dimensional vectors respectively, and let t and ε be a real variable and a real nonnegative parameter respectively. We shall consider a system of equations

$$\varepsilon d^2U/dt^2+\tilde{A}(t, U, x, \varepsilon)dU/dt+\tilde{a}(t, U, x, \varepsilon)=0 ,$$
$$dx/dt=\tilde{B}(t, U, x, \varepsilon)dU/dt+\tilde{b}(t, U, x, \varepsilon) .$$

We shall suppose that every component of the matrices $\tilde{A}$, $\tilde{B}$, and the vectors $\tilde{a}$, $\tilde{b}$, and their first order partial derivatives with respect to (t, U, x) are uniformly continuous and bounded as functions of (t, U, x, ε) when (t, U, x) in an open set R in the (t, U, x)-space and $\varepsilon\geqq 0$ is small.

Putting $\tilde{A}_0(t, V, y)=\tilde{A}(t, V, y, 0)$, etc., we shall write the so-called degenerate system as

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$$(2.1)\qquad \begin{cases} \tilde{A}_0(t,\ V,\ y)dV/dt+\tilde{a}_0(t,\ V,\ y)=0\ , \\ dy/dt=\tilde{B}_0(t,\ V,\ y)dV/dt+\tilde{b}_0(t,\ V,\ y)\ . \end{cases}$$

We shall define a discontinuous solution S_0 of (2.1) as a curve in the $(t,\ V,\ y)$-space. S_0 is also assumed to be in R.

Let a point $P_1=(s_1,\ V_1,\ y_1)$ be in R, and assume that real parts of all characteristic roots of $\tilde{A}_0$ are positive at P_1. Let $V(t)$, $y(t)$ be a solution of (2.1) with initial values at P_1, and it is assumed to be continued in R until $t\rightarrow s_2-0$. Let $V_2=V(s_2-0)$, $y_2=y(s_2-0)$ exist, and assume that the point $P_2=(s_2,\ V_2,\ y_2)$ is in R. Assume that real parts of all characteristic roots of $\tilde{A}_0(t,\ V(t),\ y(t))$ are positive for $s_1\leqq t<s_2$, whereas the matrix $\tilde{A}_0(s_2,\ V_2,\ y_2)$ has 0 as a simple characteristic root, and other characteristic roots of $\tilde{A}_0(s_2,\ V_2,\ y_2)$ have positive real parts. Then without loss of generality we can assume that the first row and the first column vectors of $\tilde{A}_0(s_2,\ V_2,\ y_2)$ are equal to zero. Therefore letting $m'=1+n$ we can write the original system and the degenerate system (2.1) respectively in the following form:

$$(2.2)\qquad \begin{cases} \varepsilon d^2u/dt^2+g(t,\ u,\ w,\ x,\ \varepsilon)du/dt+\beta(t,\ u,\ w,\ x,\ \varepsilon)dw/dt \\ \qquad +h(t,\ u,\ w,\ x,\ \varepsilon)=0\ , \\ \varepsilon d^2w/dt^2+\alpha(t,\ u,\ w,\ x,\ \varepsilon)du/dt+A(t,\ u,\ w,\ x,\ \varepsilon)dw/dt \\ \qquad +a(t,\ u,\ w,\ x,\ \varepsilon)=0\ , \\ dx/dt=f(t,\ u,\ w,\ x,\ \varepsilon)du/dt+B(t,\ u,\ w,\ x,\ \varepsilon)dw/dt \\ \qquad +b(t,\ u,\ w,\ x,\ \varepsilon)\ , \end{cases}$$

and

$$(2.3)\qquad \begin{cases} g_0(t,\ v,\ r,\ y)dv/dt+\beta_0(t,\ v,\ r,\ y)dr/dt+h_0(t,\ v,\ r,\ y)=0\ , \\ \alpha_0(t,\ v,\ r,\ y)dv/dt+A_0(t,\ v,\ r,\ y)dr/dt+a_0(t,\ v,\ r,\ y)=0\ , \\ dy/dt=f_0(t,\ v,\ r,\ y)dv/dt+B_0(t,\ v,\ r,\ y)dr/dt+b_0(t,\ v,\ r,\ y)\ , \end{cases}$$

where u and v are scalars, but w and r are n-dimensional vectors. On the other hand

$$\tilde{A}=\begin{pmatrix} g, & \beta \\ \alpha, & A \end{pmatrix},\qquad \tilde{a}=\begin{pmatrix} h \\ a \end{pmatrix},\qquad \tilde{B}=(f,\ B)\ ,\qquad \tilde{b}=b\ ,$$

where g, h are real scalars, a and α are real n-dimensional column vectors, β is a real n-dimensional row vector, A is a real n by n matrix, f and b are real m-dimensional column vectors, and B is a real m by n matrix. We have the same formulas for $\tilde{A}_0$, etc.

Put $U=(u,\ w)$, $V=(v,\ r)$ and $P_k=(s_k,\ v_k,\ r_k,\ y_k)$ $(k=1,\ 2)$. At P_2 we shall assume

$$(2.4)\qquad \begin{cases} g_0=0\ , & \alpha_0=0\ , \quad \beta_0=0\ , \\ h_0<0\ , & (\partial g_0/\partial y)f_0+\partial g_0/\partial v\neq 0\ , \end{cases}$$

where $\partial g_0/\partial y$ is an m-dimensional row vector whose components are the first order partial derivatives of g_0 with respect to components of y. On the other hand we shall assume

$$(2.5)\qquad g_0-\beta_0 A_0^{-1}\alpha_0>0$$

for $(t, v, r, y)=(t, v(t), r(t), y(t))$ when t $(<s_2)$ is sufficiently near to s_2.

Let $y=\phi(v)$ be a solution of the system

$$(2.6)\qquad dy/dv=f_0(s_2, v, r_2, y)$$

with initial value $\phi(v_2)=y_2$. Let v_3 be the first value of $v>v_2$ such that

$$(2.7)\qquad \int_{v_2}^{v} g_0(s_2, v, r_2, \phi(v))dv=0\ .$$

The curve

$$(2.8)\qquad t=s_2\ , \qquad r=r_2\ , \qquad y=\phi(v)\ , \qquad v_2\leqq v\leqq v_3\ ,$$

is assumed to be well defined in R. Finally along the curve (2.8) we shall assume that

$$(2.9)\qquad \alpha_0=0$$

and that real parts of all characteristic roots of the matrix $A_0-g_0 1_n$ are positive, where 1_k is the k by k unit-matrix.

Put $y_3=\phi(v_3)$ and $P_3=(s_2, v_3, r_2, y_3)$. Assume that all characteristic roots of $\tilde{A}_0(t, v, r, y)$ have positive real parts at P_3. Let $\hat{v}(t)$, $\hat{r}(t)$, $\hat{y}(t)$ be a solution of (2.3) with initial values at P_3, and it is assumed to be continued in R and with $\mathrm{Re}\,\lambda_j(t)>0$ $(j=1, 2, \cdots, n, n+1)$ for $s_2\leqq t\leqq s_3$, where $s_3>s_2$, and the $\lambda_j(t)$ $(j=1, 2, \cdots, n, n+1)$ are characteristic roots of the matrix $\tilde{A}_0(t, \hat{v}(t), \hat{r}(t), \hat{y}(t))$. Put $P_4=(s_3, \hat{v}(s_3), \hat{r}(s_3), \hat{y}(s_3))$.

A discontinuous solution S_0 is defined by connecting two curves

$$v=v(t)\ , \qquad r=r(t)\ , \qquad y=y(t)\ ,$$
$$v=\hat{v}(t)\ , \qquad r=\hat{r}(t)\ , \qquad y=\hat{y}(t)\ ,$$

with the curve (2.8), where $V(t)=(v(t), r(t))$.

3. Main Theorem

We shall denote by $|c|$ the norm of a vector c which is defined by the maximum-value of the absolute values of components. Put

$$|A|=\max\{|Ac|;\ |c|=1\}\ ,$$

where A is a matrix.

THEOREM: *If δ_1, δ_2, δ_3 and ε (>0) are small enough there is a solutions $u(t)$, $w(t)$, $x(t)$ of (2.2) over the interval $s_1 \leqq t \leqq s_3$ for any initial values satisfying*

$$(3.1) \qquad \begin{cases} |u(s_1)-v(s_1)|+|w(s_1)-r(s_1)| \leqq \delta_1 , \\ |du(s_1)/dt-dv(s_1)/dt|+|dw(s_1)/dt-dr(s_1)/dt| \leqq \delta_2/\varepsilon \\ |x(s_1)-y(s_1)| \leqq \delta_3 , \end{cases}$$

and as δ_1, δ_2, δ_3 and ε tend to zero the curve representing the solution $u(t)$, $w(t)$, $x(t)$ in the (t, u, w, x)-space tends to S_0. Moreover for any fixed $\delta > 0$

$$|u(t)-v(t)|+|w(t)-r(t)| , \qquad |x(t)-y(t)| ,$$

and

$$|du(t)/dt-dv(t)/dt|+|dw(t)/dt-dr(t)/dt|$$

tend uniformly to zero over the intervals $s_1 \leqq t \leqq s_2-\delta$ and $s_1+\delta \leqq t \leqq s_2-\delta$ respectively. This is also true for

$$|u(t)-\hat{v}(t)|+|w(t)-\hat{r}(t)| , \qquad |x(t)-y(t)| ,$$
$$|du(t)/dt-d\hat{v}(t)/dt|+|dw(t)/dt-d\hat{r}(t)/dt| ,$$

for the interval $s_2+\delta \leqq t \leqq s_3$.

In order to prove our theorem, we shall use the following lemma: [1]

LEMMA 1. *Let $A(t)$ and $a(t)$ be an n by n matrix and an n-dimensional vector respectively, components of which are continuous for $0 \leqq t \leqq T$ ($T>0$). Letting $\rho_1(t), \cdots, \rho_n(t)$ be characteristic roots of $A(t)$, we shall assume that*

$$(3.2) \qquad \operatorname{Re} \rho_j(t) \leqq -\rho_0 < 0 \qquad (j=1, 2, \cdots, n)$$

for $0 \leqq t \leqq T$, where ρ_0 is a positive constant independent of t. Then for every solution $u(t)$ of the system

$$(3.3) \qquad \varepsilon du/dt = A(t)u + a(t) ,$$

we have the estimate

$$(3.4) \qquad |u(t)| \leqq K\{|u(0)|e^{-(\rho_0/2\varepsilon)t} + \max_{0 \leqq s \leqq t} |a(s)|\} \qquad (0 \leqq t \leqq T) ,$$

where K is a positive constant independent of $u(t)$.

Proof: Let A be a constant matrix such that $|A| \leqq M$, where M is a positive constant. Suppose that real parts of all characteristic roots of A is less than $-\rho_0$. Assume that $0<\rho<\rho_0$. Then we have

$$(3.5) \qquad |\exp(At)| \leqq K_0 e^{-\rho t}$$

for $t \geqq 0$, where K_0 is a positive constant depending only on M, ρ_0 and ρ [2].

Letting $0 \leqq \tau < \tau' \leqq T$, we shall put

$$(3.6) \qquad \hat{B}(\tau, \tau') = \max\{|A(t) - A(t')|;\ \tau \leqq t,\ t' \leqq \tau'\} .$$

On the other hand, let $\Phi(t, \tau)$ be an n by n matrix satisfying

$$(3.7) \qquad \varepsilon d\Phi(t, \tau)/dt = A(t)\Phi(t, \tau) ,$$

and

$$(3.8) \qquad \Phi(\tau, \tau) = 1_n .$$

Since

$$\varepsilon d\Phi/dt = A(\tau)\Phi + \{A(t) - A(\tau)\}\Phi ,$$

we have

$$(3.9) \qquad \Phi(t, \tau) = e^{(1/\varepsilon)A(\tau)(t-\tau)} + \frac{1}{\varepsilon}\int_\tau^t e^{(1/\varepsilon)A(\tau)(t-s)}\{A(s) - A(\tau)\}\Phi(s, \tau)ds .$$

Hence

$$|\Phi(t, \tau)| \leqq K_0\left\{e^{-(\rho/\varepsilon)(t-\tau)} + \frac{1}{\varepsilon}\hat{B}(\tau, \tau')\int_\tau^t e^{-(\rho/\varepsilon)(t-s)}|\Phi(s, \tau)|ds\right\}$$

for $\tau \leqq t \leqq \tau'$, where K_0 is a positive constant depending only on ρ_0, ρ, and $\max\{|A(s)|;\ 0 \leqq s \leqq T\}$. Let

$$\Psi(t) = |\Phi(t, \tau)| e^{(\rho/\varepsilon)(t-\tau)} .$$

Then

$$\Psi(t) \leqq K_0\left\{1 + \frac{1}{\varepsilon}\hat{B}(\tau, \tau')\int_\tau^t \Psi(s)ds\right\} .$$

This implies

$$\int_\tau^t \Psi(s)ds \leqq K_0\int_\tau^t e^{(1/\varepsilon)K_0\hat{B}(\tau,\tau')(t-s)}ds .$$

Hence

$$\Psi(t) \leqq K_0 e^{(1/\varepsilon)K_0\hat{B}(\tau,\tau')(t-\tau)} .$$

Then

$$(3.10) \qquad |\Phi(t, \tau)| \leqq K_0 e^{(1/\varepsilon)\{K_0\hat{B}(\tau,\tau') - \rho\}(t-\tau)}$$

for $\tau \leqq t \leqq \tau'$.

Now let us take ρ so that $\frac{1}{2}\rho_0 < \rho$, and let $\tau' - \tau$ be so small that we have $K_0 \hat{B}(\tau, \tau') - \rho \leqq -\frac{1}{2}\rho_0$. Then we have

$$(3.10') \qquad |\Phi(t, \tau)| \leqq K_0 e^{-(\rho_0/2\varepsilon)(t-\tau)}$$

for $\tau \leqq t \leqq \tau'$.

Since solutions $u(t)$ of (3.3) can be written in the form

$$(3.11) \qquad u(t) = \Phi(t, 0)u(0) + \frac{1}{\varepsilon}\int_0^t \Phi(t, s)a(s)ds$$

and since components of the matrix $A(t)$ are uniformly continuous for $0 \leqq t \leqq T$, our lemma can be easily obtained.

4. Proof of Theorem, Part I

At the outset let us consider the interval $s_1 \leqq t \leqq s_2 - \delta$, where δ is an arbitrarily fixed positive number. Putting

$$(4.1) \qquad U = W + V(t), \qquad x = z + y(t),$$

we have

$$(4.2) \qquad \begin{cases} \varepsilon d^2W/dt^2 + \tilde{A}(t, W + V(t), z + y(t), \varepsilon)dW/dt + c_1(t, W, z, \varepsilon) = 0, \\ dz/dt = \tilde{B}(t, W + V(t), z + y(t), \varepsilon)dW/dt + c_2(t, W, z, \varepsilon), \end{cases}$$

where

$$(4.3) \qquad \begin{aligned} c_1(t, W, z, \varepsilon) = {} & \{\tilde{A}(t, W + V(t), z + y(t), \varepsilon) - \tilde{A}_0(t, V(t), y(t))\}dV/dt \\ & + \{\tilde{a}(t, W + V(t), z + y(t), \varepsilon) - \tilde{a}_0(t, V(t), y(t))\} \\ & + \varepsilon d^2V/dt^2, \end{aligned}$$

$$(4.3') \qquad \begin{aligned} c_2(t, W, z, \varepsilon) = {} & \{\tilde{B}(t, W + V(t), z + y(t), \varepsilon) - \tilde{B}_0(t, V(t), y(t))\}dV/dt \\ & + \{\tilde{b}(t, W + V(t), z + y(t), \varepsilon) - \tilde{b}_0(t, V(t), y(t))\}. \end{aligned}$$

Since $dV/dt = -\tilde{A}_0(t, V(t), y(t))^{-1}\tilde{a}_0(t, V(t), y(t))$, the second order derivative of $V(t)$ with respect to t exists and it is continuous for $s_1 \leqq t \leqq s_2 - \delta$.

Let $U(t)$, $x(t)$ be a solution of (2.2) with initial values satisfying (3.1). Then $W(t) = U(t) - V(t)$, $z(t) = x(t) - y(t)$ satisfy the system (4.2) with initial values satisfying

$$(4.4) \qquad |W(s_1)| \leqq \delta_1, \qquad |dW(s_1)/dt| \leqq \delta_2/\varepsilon, \qquad |z(s_1)| \leqq \delta_3.$$

Let us take positive constants ρ_1, ρ_2 and ρ_3 so small that

$$(4.5) \qquad |c_j(t, W, z, \varepsilon)| \leqq L_0\{|W| + |z|\} + \varphi(\varepsilon) \qquad (j = 1, 2)$$

and

$$(4.6) \qquad |\tilde{B}(t,\ W+V(t),\ z+y(t),\ \varepsilon)|\leqq L_0\ ,$$

when

$$(4.7) \qquad |W|\leqq\rho_1\ , \qquad |z|\leqq\rho_2\ , \qquad 0\leqq\varepsilon\leqq\rho_3\ , \qquad s_1\leqq t\leqq s_2-\delta\ ,$$

where L_0 is a positive constant and $\varphi(\varepsilon)$ is a positive-valued function of ε such that $\varphi(\varepsilon)\to 0$ as $\varepsilon\to 0$. Further we can assume that real parts of all characteristic roots of $\tilde{A}(t,\ W+V(t),\ z+y(t),\ \varepsilon)$ are not less than a positive constant ρ_0 for (4.7).

Now by the use of lemma 1 we have

$$(4.8) \qquad |dW(t)/dt|\leqq K_0\left\{\frac{\delta_2}{\varepsilon}e^{-(\rho_0/2\varepsilon)(t-s_1)}+L_0\max_{s_1\leqq s\leqq t}\{|W(s)|+|z(s)|\}+\varphi(\varepsilon)\right\}$$

and

$$(4.8') \qquad |dz/dt|\leqq L_0\{|dW/dt|+|W(t)|+|z(t)|\}+\varphi(\varepsilon)\ ,$$

when $(t,\ W(t),\ z(t),\ \varepsilon)$ is in (4.7). Let $\delta_1<\rho_1$ and $\delta_3<\rho_2$. Then we have (4.8) and (4.8′) at least in the neighborhood of s_1. Assume that (4.8) and (4.8′) are true. Since

$$|W(t)|\leqq\int_{s_1}^{t}|dW(s)/ds|ds+\delta_1\ ,$$

$$|z(t)|\leqq\int_{s_1}^{t}|dz(s)/ds|ds+\delta_3\ ,$$

and

$$|dW/dt|+|dz/dt|\leqq(L_0+1)|dW/dt|+L_0\{|W(t)|+|z(t)|\}+\varphi(\varepsilon)\ ,$$

we have

$$|W(t)|+|z(t)|\leqq L_1\left\{\delta_1+\delta_3+\varphi(\varepsilon)+\int_{s_1}^{t}|dW(s)/ds|ds\right\}\ ,$$

where L_1 is a positive constant. Hence

$$|dW/dt|\leqq K_0\left\{\frac{\delta_2}{\varepsilon}e^{-(\rho_0/2\varepsilon)(t-s_1)}+L_2\int_{s_1}^{t}|dW/ds|ds+L_3(\delta_1+\delta_3+\varphi(\varepsilon))\right\}\ ,$$

where L_2 and L_3 are positive constants. Finally we get

$$\int_{s_1}^{t}|dW/ds|ds\leqq L_4(\delta_1+\delta_2+\delta_3+\varphi(\varepsilon))\ .$$

This proves our theorem for $s_1\leqq t\leqq s_2-\delta$.

5. Proof of Theorem, Part II

Let us consider the neighborhood of the point P_2. At the beginning we shall remark that

$$dv/dt = -(\beta_0 dr/dt + h_0)/g_0 \ ,$$
$$dr/dt = -A_0^{-1}(\alpha_0 dv/dt + a_0) \ .$$

Hence

$$dv/dt = (\beta_0 A_0^{-1} a_0 - h_0)/(g_0 - \beta_0 A_0^{-1}\alpha_0) \ .$$

α_0, β_0 and $g_0 \to 0$ as $t \to s_2 - 0$. At the point P_2, $h_0 < 0$ and A_0^{-1} exists. Since (2.5) is true for $t < s_2$, we have

$$\lim_{t \to s_2 - 0} dv/dt = +\infty \ .$$

Let $v^* < v_2$ be near enough to v_2 so that as t increases from s_1 toward s_2 there is a value of $t = \tau^*$ near to s_2 such that

$$v(\tau^*) = v^* \ ,$$

and such that dv/dt is large for $\tau^* \leqq t < s_2$. Let us denote the point $(\tau^*, v^*, r(\tau^*), y(\tau^*))$ by Q. We can choose Q as near to P_2 as we wish.

Since $\tau^* < s_2$, by applying the results of the preceding section, we see that, if δ_1, δ_2, δ_3 and ε are small enough, then for some value of $t = t^*$ we have $u(t^*) = v^*$. It should be remarked that t^* tends to τ^* as δ_1, δ_2, δ_3 and $\varepsilon \to 0$. Therefore the point $(t^*, v^*, w(t^*), x(t^*))$ tends to Q as δ_1, δ_2, δ_3 and $\varepsilon \to 0$. We change from t to v as the independent variable. Since $dr/dv = (dr/dt)(dv/dt)^{-1}$, we have $dr/dv \to 0$ as $v \to v_2$. Therefore for any small positive constants δ_4, δ_5, δ_6 and δ_7 we have

$$(5.1) \qquad \begin{cases} |r(v) - r_2| < \delta_4 \ , & |dr(v)/dv| < \delta_5 \ , \\ 0 < dt(v)/dv < \delta_6 \ , & |y(v) - y_2| < \delta_7 \end{cases}$$

for $v^* \leqq v \leqq v_2$, if we choose Q near enough to P_2.

Now changing from t to u as the independent variable, we derive from (2.2) the following system

$$(5.2) \qquad \begin{cases} \varepsilon dp/du = p^2(g + \beta dw/du) + p^3 h \ , \\ \varepsilon d^2w/du^2 + p\{(\alpha + pa) + (A - (g+ph)1_n)dw/du \\ \qquad\qquad -(\beta dw/du)dw/du\} = 0 \\ dx/du = f + B dw/du + pb \ , \\ dt/du = p \ . \end{cases}$$

Let $v_2 < v^{**} \leqq v_3$. Since the curve (2.8) is in R, we can choose positive constants ρ_1, ρ_2 and ρ_3 so small that the domain defined by

$$(5.3)\qquad \begin{cases} v^* \leqq u \leqq v^{**}\ , & |w-r_2| \leqq \rho_1\ , \qquad |t-s_2| \leqq \rho_2\ , \\ \min\{|x-\phi(v)|;\ v_2 \leqq v \leqq v^{**}\} \leqq \rho_3\ , \end{cases}$$

is also in R. Then if $\rho_4 > 0$ is sufficiently small, the components of A, B, etc. and their first order partial derivatives with respect to (t, u, w, x) are uniformly continuous and bounded as functions of $(t, u, w, x, \varepsilon)$ when (t, u, w, x) is in the domain (5.3) and $0 \leqq \varepsilon \leqq \rho_4$. Let L_0 be the upper bound of those quantities in absolute value.

For $v < v_2$, we have

$$\begin{aligned} dg_0/dv &= (\partial g_0/\partial t)dt/dv + (\partial g_0/\partial r)dr/dv \\ &\quad + (\partial g_0/\partial y)(f_0 + B_0 dr/dv + b_0 dt/dv) + \partial g_0/\partial v\ . \end{aligned}$$

On the other hand, we assumed (2.4) and (2.5). Furthermore we got

$$dt/dv \to 0\ , \qquad dr/dv \to 0$$

as $v \to v_2$. Therefore at P_2 we get

$$(\partial g_0/\partial y) f_0 + \partial g_0/\partial v < 0\ .$$

Then we can assume that

$$(\partial g/\partial x) f + \partial g/\partial u \leqq -M < 0$$

when (t, u, w, x) is in the domain (5.3) and $0 \leqq \varepsilon \leqq \rho_4$, where M is a positive constant.

Without loss of generality we can assume that

$$(5.4)\qquad \begin{cases} |w(v^*)-r_2| < \delta_4 < \rho_1\ , & |dw(v^*)/du| < \delta_5\ , \\ 0 < p(v^*) < \delta_6\ , & |t^*-s_2| < 2|\tau^*-s_2| < \rho_2\ , \\ |x(v^*)-y_2| < \delta_7 < \rho_3\ . \end{cases}$$

In general L_0^* stands for a constant depending only on n and L_0. For example the Lipschitz' constants for α and β will be denoted by L_0^*, and on the other hand, we shall assume that

$$|\partial g/\partial t| + |\partial g/\partial w| + |\partial g/\partial y|(|B|+|b|) \leqq L_0^*,\ \text{etc.}$$

in R.

Let N_1, N_2 and N_3 be positive constants such that

$$(5.5)\qquad \begin{cases} L_0^*(N_1+N_2) < \frac{1}{2}M\ , & N_1(v^{**}-v^*) < \rho_1-\delta_4\ , \\ N_2(v^{**}-v^*) < \rho_2 - 2|\tau^*-s_2|\ , & N_3 < \rho_3\ . \end{cases}$$

Without loss of generality we can assume that

$$(5.6)\qquad \begin{cases} \delta_5 < N_1\ , \\ \delta_6 < \min\ \{-2N_1L_0^*(1+L_0^*)(N_1+N_2+1)(v^{**}-v^*)/h_0^*,\ N_2\}\ , \\ \delta_7 < N_3\ , \end{cases}$$

where

$$(5.7)\qquad h_0^* = h_0(s_2,\ v_2,\ r_2,\ y_2)\ .$$

Then the condition

$$(5.8)\qquad |dw/du| \leqq N_1\ , \qquad |p(u)| \leqq N_2\ , \qquad |x(u)-\hat{\phi}(u)| \leqq N_3\ ,$$

is satisfied at least in the neighborhood of v^*, where

$$(5.9)\qquad \hat{\phi}(u) = \begin{cases} y_2 & (u \leqq v_2)\ , \\ \phi(u) & (u \geqq v_2)\ . \end{cases}$$

Assume $v^* \leqq u \leqq v^{**}$. Then as long as (5.8) is satisfied we get

$$(5.10)\qquad |w(u)-r_2| < \rho_1\ , \qquad |t(u)-s_2| < \rho_2\ , \qquad |x(u)-\hat{\phi}(u)| < \rho_3\ .$$

We shall suppose N_2 so small that real parts of all characteristic roots of the matrix $A-(g+ph)1_n$ are not less than $\rho_0 > 0$ as long as (5.8) is satisfied.

Since

$$\begin{aligned} &dg(t(u),\ u,\ w(u),\ x(u),\ \varepsilon)/du \\ &\quad = p\partial g/\partial t + (\partial g/\partial w)dw/du + (\partial g/\partial x)(f + Bdw/du + pb) + \partial g/\partial u \end{aligned}$$

we have

$$(5.11)\qquad dg/du < -\tfrac{1}{2}M < 0$$

as long as (5.8) is satisfied.

Under the assumption (5.8) we shall derive another set of estimates for dw/du, $p(u)$ and $x(u)-\hat{\phi}(u)$.

When $u \leqq v_2$, we have

$$|x(u)-x(v^*)| \leqq L_0^*(N_1+N_2+1)(v_2-v^*)\ .$$

This implies

$$(5.12)\qquad |x(u)-\hat{\phi}(u)| < L_0^*(N_1+N_2+1)(v_2-v^*)+\delta_7$$

when $u \leqq v_2$. If $u \geqq v_2$, $\hat{\phi}(u)$ satisfies

$$d\hat{\phi}/du = f_0(s_2,\ u,\ r_2,\ \hat{\phi})\ .$$

Therefore, letting

$$\tilde{\phi}(u) = x(u) - \hat{\phi}(u)\ ,$$

we have

$$d\tilde{\phi}(u)/du=f(t,\ u,\ w,\ \tilde{\phi}+\hat{\phi},\ \varepsilon)-f_0(s_2,\ u,\ r_2,\ \hat{\phi})+Bdw/du+pb\ .$$

Hence

$$\begin{aligned}|d\tilde{\phi}/du| &\leqq L_0^*\{|\tilde{\phi}|+|t-s_2|+|w-r_2|+N_1+N_2+\psi_1(\varepsilon)\}\\ &\leqq L_0^*\{|\tilde{\phi}|+(N_1+N_2)(1+v^{**}-v^*)+2|\tau^*-s_2|+\delta_4+\psi_1(\varepsilon)\}\ ,\end{aligned}$$

where ψ_1 is a positive-valued function of ε which tends to zero as $\varepsilon\to 0$. Therefore

$$\begin{aligned}(5.13)\qquad |\tilde{\phi}(u)| &\leqq |\tilde{\phi}(v_2)|+\int_{v_2}^{u}|d\tilde{\phi}/du|du<|\tilde{\phi}(v_2)|(1+L_0^*K^*)\\ &\quad +L_0^*K^*\{(N_1+N_2)(1+v^{**}-v^*)+2|\tau^*-s_2|+\delta_4+\psi_1(\varepsilon)\}\ ,\end{aligned}$$

where

$$K^*=\int_{v^*}^{v^{**}}e^{L_0^*(u-\xi)}d\xi\ .$$

The assumption (2.4) implies that we can assume

$$(5.14)\qquad |\beta(t,\ u,\ w,\ x,\ \varepsilon)|\leqq L_0^*\{|t-s_2|+|u-v_2|+|w-r_2|+|x-y_2|+\psi_2(\varepsilon)\}$$

when $(t,\ u,\ w,\ x)$ is in (5.3) and $0\leqq\varepsilon\leqq\rho_4$, where ψ_2 is a function of ε which is similar to ψ_1. On the other hand as long as (5.8) is satisfied we get

$$(5.15)\qquad |x(u)-x(v^*)|\leqq L_0^*(N_1+N_2+1)(v^{**}-v^*)\ .$$

Further under the same situation, g is a decreasing function of u. Therefore

$$\begin{aligned}(5.16)\qquad g+\beta dw/du &< g(t^*,\ v^*,\ w(v^*),\ x(v^*),\ \varepsilon)\\ &\quad +N_1L_0^*\{|t-s_2|+|u-v_2|+|w-r_2|+|x-y_2|+\psi_2(\varepsilon)\}\\ &< g(t^*,\ v^*,\ w(v^*),\ x(v^*),\ \varepsilon)\\ &\quad +N_1L_0^*\{(N_1+N_2+1)(1+L_0^*)(v^{**}-v^*)+|t^*-s_2|+|w(v^*)-r_2|\\ &\quad +|x(v^*)-y_2|+\psi_2(\varepsilon)\}\\ &< g(t^*,\ v^*,\ w(v^*),\ x(v^*),\ \varepsilon)\\ &\quad +N_1L_0^*\{(1+L_0^*)(N_1+N_2+1)(v^{**}-v^*)+\delta_4+\delta_7\\ &\quad +2|\tau^*-s_2|+\psi_2(\varepsilon)\}\ .\end{aligned}$$

It is easily seen that $g(t^*,\ v^*,\ w(v^*),\ x(v^*),\ \varepsilon)$ is positive when δ_1, δ_2, δ_3 and ε are sufficiently small.

Since $h_0^*<0$, we can assume

$$h<\tfrac{1}{2}h_0^*$$

for the domain (5.3) and $0\leqq\varepsilon\leqq\rho_4$. On the other hand

$$0<p(v^*)<-2M_0h_0^*,$$

where

$$M_0=N_1L_0^*\{(1+L_0^*)(N_1+N_2+1)(v^{**}-v^*)+\delta_4+\delta_7 \\ +2|\tau^*-s_2|+\psi_2(\varepsilon)\}+g(t^*,\ v^*,\ w(v^*),\ x(v^*),\ \varepsilon)\ .$$

As long as (5.8) is satisfied, p remains positive. Further if

$$p=-2M_0/h_0^* \tag{5.17}$$

for the first time at a value of u, we have

$$\varepsilon dp/du=p^2(g+\beta dw/du+ph)<p^2(M_0+p\tfrac{1}{2}h_0^*)=0\ .$$

This is impossible, since p reaches the value (5.17) for the first time. Therefore as long as (5.8) is satisfied, we have

$$0<p(u)<-2M_0/h_0^*\ . \tag{5.18}$$

Now we shall give an estimate for dw/du. By virtue of (5.18) we may consider $Y=dw/du$ as a vector-valued function of t. Y satisfies

$$\varepsilon dY/dt+(A-(g+ph)1_n)Y+(\alpha+pa)+\beta Y)Y=0\ . \tag{5.19}$$

By the use of lemma 1 we get

$$|dw/du|\leqq K_0\{\delta_5e^{-(\rho_0/2\varepsilon)(t-t^*)}+L_0^*(N_i^2+N_2) \\ +\max_{v^*\geqq z\geqq u}|\alpha(t(z),\ z,\ w(z),\ x(z),\ \varepsilon)|\}\ . \tag{5.20}$$

The assumption (2.9) implies that

$$\begin{aligned} &|\alpha(t(z),\ z,\ w(z),\ x(z),\ \varepsilon)| \\ &\leqq L_0^*\{|t-s_2|+(v_2-v^*)+|w-r_2|+|x(z)-\hat{\phi}(z)|+\psi_3(\varepsilon)\} \\ &<L_0^*\Big\{N_2(v^{**}-v^*)+N_3+(v_2-v^*)+2|\tau^*-s_2|+\delta_4+\psi_3(\varepsilon) \\ &+\int_{v^*}^{u}|dw/du|du\Big\}\ , \end{aligned} \tag{5.21}$$

where ψ_3 is a function similar to ψ_1. Hence

$$|dw/du|<M_0'+K_0L_0^*\int_{v^*}^{u}|dw/du|du\ ,$$

where

$$M_0'=K_0[\delta_5+L_0^*\{(N_1^2+N_2)+N_2(v^{**}-v^*)+N_3+(v_2-v^*) \\ +2|\tau^*-s_2|+\delta_4+\psi_3(\varepsilon)\}]\ . \tag{5.22}$$

Thus we obtain

$$(5.23)\qquad |dw/du| < M_0' e^{K_0 L_0^*(v^{**}-v^*)} .$$

Therefore under the assumption (5.8) we have the following estimates:

$$(5.24)\qquad \begin{cases} |dw/du| < K_1(N_2+N_3+N_1^2+\varDelta) , \\ 0 < p(u) < K_2 N_1\{(v^{**}-v^*)(N_1+N_2+1)+\varDelta\} + \varDelta , \\ |x(u)-\hat{\phi}(u)| < K_3(K^*+\varDelta)(N_1+N_2)+\varDelta , \end{cases}$$

where K_1, K_2 and K_3 are sufficiently large positive constants, and $\varDelta$ is a positive constant which can be made as small as we wish if we choose δ_1, δ_2, δ_3 and ε sufficiently small.

Let us consider the system of equations

$$(5.25)\qquad \begin{cases} K_1(N_2+N_3+N_1^2+\varDelta) = N_1 , \\ K_2 N_1\{(v^{**}-v^*)(N_1+N_2+1)+\varDelta\} + \varDelta = N_2 , \\ K_3(K^*+\varDelta)(N_1+N_2)+\varDelta = N_3 . \end{cases}$$

If we assume that $(v^{**}-v^*)$ is sufficiently small, then K^* becomes small. Hence there exists a real-valued solution $N_1(\varDelta)$, $N_2(\varDelta)$ and $N_3(\varDelta)$ such that

$$N_1(0)=0 , \qquad N_2(0)=0 , \qquad N_3(0)=0 .$$

This solution is analytic in $\varDelta$, and $N_j(\varDelta)>0$ $(j=1, 2, 3)$ for small positive values of $\varDelta$.

Now choose v^{**} so near to v_2 that $(v^{**}-v^*)$ is small enough. Then keep v^{**} fixed. Take $\varDelta$ so small that $N_j(\varDelta)$ $(j=1, 2, 3)$ satisfy all of conditions given above. Then choose δ_1, δ_2, δ_3 and ε so small that (5.24) is satisfied under the assumption (5.8). As a consequence, we get

$$|dw/du| < N_1 , \qquad 0 < p(u) < N_2 , \qquad |x(u)-\hat{\phi}(u)| < N_3 ,$$

under the assumption (5.8). This proves our theorem for $v^* \leqq u \leqq v^{**}$.

It should be remarked that v^{**} can be kept fixed even when δ_1, δ_2, δ_3 and ε tend to zero.

6. Proof of Theorem, Part III

By using an argument similar to that used in the preceding section we can prove the following lemma:

LEMMA 2. *There exists a positive constant L^* such that, if*

$$v^{**} \leqq u^* \leqq v_3 , \qquad u^* < u^{**} , \qquad u^{**}-u^* \leqq L^* ,$$

then for any positive numbers δ_4, δ_5, δ_6, δ_7, δ_8 and N there exist two *positive numbers N' and N'' such that if*

$$\begin{cases} |w(u^*)-r_2|\leqq\delta_4\,, & |dw(u^*)/du|\leqq\delta_5\,, \\ 0<p(u^*)\leqq\delta_6\,, & |t(u^*)-s_2|\leqq\delta_7\,, \\ |x(u^*)-\phi(u^*)|\leqq\delta_8\,, & \end{cases}$$

and

$$0<p(u)\leqq N$$

for $u^*\leqq u\leqq u^{**}$, then we have

$$|dw/du|\leqq N'\,, \qquad |x(u)-\hat{\hat{\phi}}(u)|\leqq N''$$

for $u^*\leqq u\leqq u^{**}$, where

$$\hat{\hat{\phi}}(u)=\begin{cases} \phi(u) & (u\leqq v_3)\,, \\ y_3 & (u\geqq v_3)\,. \end{cases} \tag{6.1}$$

Further we can make N' and N'' as small as we wish by taking δ_4, δ_5, δ_6, δ_7, δ_8, ε, N and $max\,(u^{**}-v_3,\,0)$ sufficiently small.

From the results obtained in the preceding section, we can derive

$$\begin{cases} |w(v^{**})-r_2|<\delta_4\,, & |dw(v^{**})/du|<\delta_5\,, \\ 0<p(v^{**})<\delta_6\,, & |t(v^{**})-s_2|<\delta_7\,, \\ |x(v^{**})-\phi(v^{**})|<\delta_8\,. & \end{cases} \tag{6.2}$$

Therefore we can prove the following lemma:

LEMMA 3. *Let u^* be a real number sufficiently near to v_3. Then for any positive number N there exist two positive numbers N' and N'' such that if (6.2) is satisfied and*

$$0<p(u)\leqq N \tag{6.3}$$

for $v^{**}\leqq u\leqq u^*$, then we have

$$|dw/du|\leqq N'\,, \qquad |x(u)-\hat{\hat{\phi}}(u)|\leqq N'' \tag{6.4}$$

for $v^{**}\leqq u\leqq u^*$. Further we can make N' and N'' as small as we wish by taking δ_1, δ_2, δ_3, ε, N and $max\,(u^*-v_3,\,0)$ sufficiently small

Let δ_9 be positive and small. Since $g_0(s_2,\,v,\,r_2,\,\phi(v))<0$ when $v>v_2$ is sufficiently near to v_2 and v_3 is the first value of $v>v_2$ such that (2.7) is satisfied, we have

$$\int_{v_2}^{v} g_0(s_2,\,v,\,r_2,\,\phi(v))dv>-\delta_{10} \tag{6.5}$$

for $v^{**} \leqq v \leqq v_3 - \delta_9$ when $\delta_{10} > 0$ is sufficiently small.

Now we shall consider the neighborhood of the curve

$$t = s_2 , \qquad w = r_2 , \qquad x = \phi(u) , \qquad v^{**} \leqq u \leqq v_3 - \delta_9 .$$

We shall prove that

$$0 < p(u) \leqq \delta_6 \tag{6.6}$$

is satisfied for

$$v^{**} \leqq u \leqq v_3 - \delta_9 , \tag{6.7}$$

when δ_1, δ_2, δ_3 and ε are sufficiently small.

Under the assumption (6.6) we have

$$\int_{v_2}^{u} (g + \beta dw/du) du > -\tfrac{1}{2}\delta_{10}$$

when δ_1, δ_2, δ_3, δ_6 and ε are sufficiently small. Therefore we have

$$\varepsilon/p(u) = \varepsilon/p(v_2) - \int_{v_2}^{u} (g + \beta dw/du) du - \int_{v_2}^{u} phdu > \tfrac{1}{2}\delta_{10} - L_0^* \delta_6 .$$

Then assuming $\delta_6 < \delta_{10}/(4L_0^*)$, we get $p(u) < 4(\varepsilon/\delta_{10})$. Choosing ε so small that $4(\varepsilon/\delta_{10}) < \delta_6$, we get

$$0 < p(u) < \delta_6$$

under the assumption (6.6). This result together with lemma 3 proves our theorem for (6.7).

7. Proof of Theorem, Part IV

Put $u^* = v_3 - \delta_9$, and let $u^{**} > v_3$ be so near to v_3 that we have

$$g > \tfrac{1}{2} g_0^* ,$$

for

$$|w - r_2| \leqq \rho_1 , \qquad |t - s_2| \leqq \rho_2 , \qquad u^* \leqq u \leqq u^{**} , \qquad 0 \leqq \varepsilon \leqq \rho_3 ,$$
$$\min \{|x - \phi(u)| ; u^* \leqq u \leqq v_3\} \leqq \rho_4 ,$$

where

$$g_0^* = g_0(s_2, v_3, r_2, y_3) .$$

It should be noted that g_0^* is positive.

Since

$$\varepsilon/p(u^*) = \varepsilon/p(v^*) - \int_{v^*}^{u^*} (g + \beta dw/du + ph) du ,$$

$\varepsilon/p(u^*)$ can be made as small as we wish if we choose δ_1, δ_2, δ_3 and ε sufficiently small.

Now assume that the solution $t(u)$, $w(u)$, $x(u)$ of (5.2) can be continued until $u=u^{**}$ with condition

$$0<p(u)<N.$$

Then we get

$$|dw/du|\leqq N', \qquad |x(u)-\hat{\hat{\phi}}(u)|\leqq N''$$

for $u^*\leqq u\leqq u^{**}$. Since

$$\varepsilon/p(u)=\varepsilon/p(u^*)-\int_{u^*}^{u}(g+ph)du-\int_{u^*}^{u}(\beta dw/du)du>0,$$

the integral

$$\int_{u^*}^{u^{**}}(g+ph)du$$

is made less than

$$\delta_{11}+L_0^*N'(u^{**}-u^*),$$

where

$$0<\varepsilon/p(u^*)<\delta_{11}.$$

Therefore at a value of u in the interval $u^*\leqq u\leqq u^{**}$ we get

$$g+ph<L_0^*N'+\delta_{11}/(u^{**}-v_3).$$

Let δ_1, δ_2, δ_3, ε, N and $u^{**}-v_3$ be so small that the right-hand member of the inequality is less than $g_0^*/4$. We fix N and u^{**} hereafter. Since we got

$$ph<-g_0^*/4,$$

we have $h<0$ and

$$p>g_0^*/(4L_0^*).$$

This implies that there exists a value of u in the interval $u^*\leqq u\leqq u^{**}$ at which we have $p=\min\{g_0^*/(4L_0^*), N\}$. At this point we shall change from u to t as the independent variable. Then we have a situation similar to that in § 4.

Thus the proof of our theorem is completed.

8. Remarks

Along the jump curve (2.8) r remains constant by the assumption

(2.9). However if we consider a system of the type

$$(8.1)\qquad \begin{cases} \varepsilon d^2u/dt^2+gdu/dt+\beta dw/dt+h=0\ , \\ \varepsilon^{1+\sigma}d^2w/dt^2+\alpha du/dt+Adw/dt+a=0\ , \\ dx/dt=fdu/dt+Bdw/dt+b\ , \end{cases}$$

where σ is a positive number, then we can define the jump curve without the condition (2.9), and r need not remain constant along the jump curve. Namely we shall suppose that, at a point $0=(0, 0, 0, 0)$, we have

$$(8.2)\qquad \begin{cases} g_0=0\ , & \alpha_0=0\ , & \beta_0=0\ , \\ h_0<0\ , & (\partial g_0/\partial y)f_0+\partial g_0/\partial v<0\ . \end{cases}$$

Let ϕ_1, ϕ_2 be a solution of the equations

$$(8.3)\qquad \alpha_0+A_0d\phi_1/dv=0\ , \qquad d\phi_2/dv=f_0+B_0d\phi_1/dv$$

with initial conditions

$$(8.4)\qquad \phi_1(0)=0\ , \qquad \phi_2(0)=0\ .$$

Let v_1 be the first value of $v>0$ such that

$$(8.5)\qquad \int_0^v[g_0(0, v, \phi_1(v), \phi_2(v))-\beta_0(\cdots)A_0(\cdots)^{-1}\alpha_0(\cdots)]dv=0\ .$$

Assume that real parts of all characteristic roots of the matrix A_0 are positive along the curve

$$(8.6)\qquad t=0\ , \qquad 0\leqq v\leqq v_1\ , \qquad r=\phi_1(v)\ , \qquad y=\phi_2(v)\ .$$

Finally we shall assume that at the point $(0, v_1, \phi_1(v_1), \phi_2(v_1))$, we have

$$(8.7)\qquad g_0-\beta_0A_0^{-1}\alpha_0>0\ .$$

Then we can prove the following result:

Let δ_1, δ_2, δ_3, δ_4, δ_5 and δ_6 be sufficiently small positive constants. Suppose that

$$(8.8)\qquad \begin{cases} |t^*|\leqq\delta_1\ , & |u(t^*)|\leqq\delta_2\ , & du(t^*)/dt\geqq\delta_3^{-1}\ , \\ |w(t^*)|\leqq\delta_4\ , & |dw(t^*)/dt|\leqq\delta_5du(t^*)/dt\ , & \\ |x(t^*)|\leqq\delta_6\ . & & \end{cases}$$

Then there exists a positive-valued function $T(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \varepsilon)$ of δ_1, δ_2, δ_3, δ_4, δ_5, δ_6, ε, such that, for any initial values satisfying (8.1), there exists a solution $u(t)$, $w(t)$, $x(t)$ of (8.1) over the interval $0\leqq t\leqq T$. Furthermore as δ_1, δ_2, δ_3, δ_4, δ_5, δ_6 and ε tend to zero the curve representing the solution $u(t)$, $w(t)$, $x(t)$ in the (t, u, w, x)-space tends *to the curve (8.6).*

In fact if we put

$$w=\phi_1(u)+\hat{w}\ , \qquad x=\phi_2(u)+\hat{x}\ , \tag{8.9}$$

we get

$$\left\{\begin{aligned} &\varepsilon d^2u/dt^2+(g-\beta A_0^{-1}\alpha_0)du/dt+\beta d\hat{w}/dt+h=0\ ,\\ &\varepsilon^{1+\sigma}d^2\hat{w}/dt^2+(\alpha+Ad\phi_1/du)du/dt+Ad\hat{w}/dt+a\\ &\qquad +\varepsilon^{1+\sigma}(d^2\phi_1/du^2)\ du/dt)^2\\ &\qquad -\varepsilon^{\sigma}\{(g-\beta A_0^{-1}\alpha_0)du/dt+\beta d\hat{w}/dt+h\}\,d\phi_1/du=0\ ,\\ &d\hat{x}/dt=(f+Bd\phi_1/du-d\phi_2/du)du/dt+Bd\hat{w}/dt+b\ . \end{aligned}\right. \tag{8.10}$$

Therefore by changing from t to u as the independent variable, we get

$$\left\{\begin{aligned} &\varepsilon dp/du=p^2(g-\beta A_0^{-1}\alpha_0+\beta d\hat{w}/du)+p^3h\ ,\\ &\varepsilon^{1+\sigma}d^2\hat{w}/du^2+p[Ad\hat{w}/du+\alpha+Ad\phi_1/du+pa\\ &\qquad +\varepsilon^{\sigma}\{(\varepsilon/p)d^2\phi_1/du^2-(d\phi_1/du+d\hat{w}/du)(g-\beta A_0^{-1}\alpha_0\\ &\qquad +\beta d\hat{w}/du+ph)\}]=0\ ,\\ &d\hat{x}/du=(f+Bd\phi_1/du-d\phi_2/du)+Bd\hat{w}/du+pb\ ,\\ &dt/du=p\ . \end{aligned}\right. \tag{8.11}$$

Then by the use of an argument similar to that used in the proof of our theorem we can prove the statement given above.

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