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# Note on Conformally Flat Almost-Kählerian Spaces

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Let $M$ be an $n$ dimensional differentiable manifold with a fixed almost-Kählerian structure given by $(g_{ji}, \varphi_i^{\ h})$.[1] More precisely, if we denote by $g_{ji}$ and $\varphi_i^{\ h}$ the Riemannian metric and the almost-complex structure, i. e. a tensor field such that $\varphi_i^{\ r}\varphi_r^{\ j}=-\delta_i^j$, respectively, then they satisfy the following equations

(1) $$g_{rs}\varphi_j^{\ r}\varphi_i^{\ s}=g_{ji},$$

(2) $$\nabla_k\varphi_{ji}+\nabla_j\varphi_{ik}+\nabla_i\varphi_{kj}=0,$$

where $\varphi_{ji}=\varphi_j^{\ r}g_{ri}$ and $\nabla_i$ denotes the Riemannian covariant derivation.

Since the tensor field $\varphi_{ji}$ is harmonic,[2] the following theorem is a direct consequence of Bochner-Lichnerowicz' one.[3]

**Theorem.** *There does not exist a compact almost-Kählerian space of positive constant curvature.*

In this note we shall obtain an identity in an almost-Kählerian space and generalize the above theorem.

We have from (2)

$$\nabla^r\nabla_r\varphi_{ji}+\nabla^r\nabla_j\varphi_{ir}+\nabla^r\nabla_i\varphi_{rj}=0,$$

where $\nabla^r=g^{rs}\nabla_s$. Since $\varphi_{ji}$ is skew-symmetric, we get

(3) $$\nabla^r\nabla_r\varphi_{ji}=\nabla^r\nabla_j\varphi_{ri}-\nabla^r\nabla_i\varphi_{rj}.$$

On the other hand, taking account of the Ricci's identity

$$\nabla_k\nabla_j\varphi_{hi}-\nabla_j\nabla_k\varphi_{hi}=-R_{kjh}^{\ \ \ r}\varphi_{ri}-R_{kji}^{\ \ \ r}\varphi_{hr}$$

and $\nabla_r\varphi_i^{\ r}=0$, we have

(4) $$\nabla^r\nabla_j\varphi_{ri}=(1/2)\,\varphi^{rt}R_{rtji}+R_j^{\ r}\varphi_{ri},$$

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1) As to the notations, we follow Tachibana, S. [3]. We shall express any quantities in terms of their components with respect to a natural repere $\partial/\partial x^i$, where $\{x^i\}$ is a local coordinate. Indices run over $1,\cdots,n=2m$. Throughout the paper we shall assume that $n>2$ and the differentiability of $M$ is sufficiently high if necessary.

2) Schouten, J. A. and K. Yano [2].

3) Yano, K. and S. Bochner [6].

where

$$R_{kji}{}^{h}=\partial_k\{{}_j{}^h{}_i\}-\partial_j\{{}_k{}^h{}_i\}+\{{}_k{}^h{}_r\}\{{}_j{}^r{}_i\}-\{{}_j{}^h{}_r\}\{{}_k{}^r{}_i\}, \qquad \partial_j=\partial/\partial x^j,$$

and

$$R_{ji}=R_{rji}{}^{r}, \qquad R_{kjih}=R_{kji}{}^{r}g_{rh}.$$

From (3) and (4) it holds that

$$\nabla^r\nabla_r\varphi_{ji}=-R_{jit}{}^{r}\varphi_r{}^t+R_j{}^r\varphi_{ri}-R_i{}^r\varphi_{rj}. \tag{5}$$

Putting

$$R^*_{ji}=(1/2)\varphi^{rt}R_{rtsi}\varphi_j{}^s, \qquad R^*=R^*_{ji}g^{ji}, \qquad R=R_{ji}g^{ji}$$

and transvecting (5) with $\varphi^{ji}=\varphi_s{}^ig^{sj}$, we find

$$\varphi^{ji}\nabla^r\nabla_r\varphi_{ji}=2(R-R^*).$$

If we operate $\nabla^r\nabla_r$ to $\varphi^{ji}\varphi_{ji}=n$, then it follows that

$$\varphi^{ji}\nabla^r\nabla_r\varphi_{ji}=-(\nabla^r\varphi^{ji})\nabla_r\varphi_{ji}.$$

From these two equations, we obtain a final identity

$$R-R^*=-(1/2)(\nabla^r\varphi^{ji})\nabla_r\varphi_{ji}.$$

Since the right hand side is non-positive, we have

**Lemma.** *In an almost-Kählerian space, the relation $R^*\geqq R$ holds. The equality holds if and only if the space is Kählerian.*

Let us consider an almost-Kählerian space such that the curvature tensor takes the form:

$$R_{kjih}=\alpha[(g_{kh}g_{ji}-g_{jh}g_{ki})+(\varphi_{kh}\varphi_{ji}-\varphi_{jh}\varphi_{ki})-2\varphi_{kj}\varphi_{ih}]^{4)}. \tag{6}$$

Then we have $R=R^*=n(n+2)\alpha$. Thus we get

**Theorem 1.** *An almost-Kählerian space such that the curvature tensor takes the form* (6) *is Kählerian.*

Next suppose an almost-Kählerian space to be conformally flat, then the curvature tensor is the following form:[5)]

$$(n-2)R_{kjih}=g_{kh}R_{ji}-g_{jh}R_{ki}+g_{ji}R_{kh}-g_{ki}R_{jh}-\{1/(n-1)\}R(g_{ji}g_{kh}-g_{ki}g_{jh}).$$

Therefore it holds that

$$(n-2)R^*_{ji}=R_{rs}\varphi_j{}^r\varphi_i{}^s+R_{ji}-\{1/(n-1)\}Rg_{ji}, \qquad R^*=\{1/(n-1)\}R,$$

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4) A Kählerian space such that the curvature tensor takes the form (6) is called a space of constant holomorphic sectional curvature [5].

5) For example, Schouten, J. A. [1] p. 306.

from which we find that $(n-1)(R-R^*)=(n-2)R$. Thus we get

**Theorem 2.** *There does not exist a conformally flat almost-Kählerian space if* $R>0$.

**Corollary.** *There does not exist an almost-Kählerian space of positive constant curvature.*

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