Some Remarks on the Definition of Integrals of Vector-Valued Functions

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There are many who study the generalization of the notion of integrals, where vector-valued functions are concerned with, among whom special mentions are made here of G. Birkhoff [2], B. J. Pettice [7], R. S. Phillips [8] and C. E. Rickart [9]. Adopting the method due to S. Kametani [4] who, giving stress upon the additivity of integrals, has defined an integral of real-valued functions, we shall define integrals of vector-valued functions, investigate some properties of the integrals thus obtained, and consider the relations to some other integrals.

To begin with, let us make a brief mention of the definition due to S. Kametani [4]. Let f be real-valued, bounded functions defined on a fixed set S, $\mathbf{B}$ a σ-ring of sets $\subset S$ to which S belongs as a member, and m a positive measure defined on $\mathbf{B}$. To each f, measurable or not, there exists an additive set-function $F(\sigma)$ defined on $\mathbf{B}$ which satisfies for every $\sigma \in \mathbf{B}$

$$m(\sigma) \inf_{s \in \sigma} f(s) \leqq F(\sigma) \leqq m(\sigma) \sup_{s \in \sigma} f(s).$$

If such a set-function F is uniquely determined, then $F(S)$, by definition, is an integral of f over S. This definition is identical with that of Lebesgue's type.

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1. Terminology, fundamental notion and notations.

$\{x ; P\}$ denotes the set of all the elements which have a certain property P.

By $\mathbf{E}$, we shall mean a *locally convex topological vector space* introduced by J. von Neumann [6] which is as follows:

I. $\mathbf{E}$ *is a vector space over real numbers* [1, p. 26].

II. *There is a complete system* $\mathfrak{V}$ *of neighbourhoods about the zero vector* θ *which satisfy the following conditions* II_1—II_6.

II_1 $\theta \in V$ *for every* $V \in \mathfrak{V}$.

II_2 $x \in V$ *for every* $V \in \mathfrak{V}$ *implies* $x=\theta$.

II_3 $V_1, V_2 \in \mathfrak{V}$ *implies the existence of* $V_3 \in \mathfrak{V}$ *such that* $V_3 \subset V_1 \cap V_2$.

II_4 $V \in \mathfrak{V}$ *implies the existence of* $V' \in \mathfrak{V}$ *such that* $V'+V' \subset V$ *where* $V'+V' = \{x_1+x_2 ; x_1, x_2 \in V'\}$.

II_5 $V \in \mathfrak{V}$ *implies the existence of* $V' \in \mathfrak{V}$ *such that* $\alpha V' \subset V$ *for all* $|\alpha| \leqq 1$ *where* $\alpha V' = \{\alpha x ; x \in V'\}$

II_6 $x\in\mathbf{E}$ and $V\in\mathfrak{B}$ *imply the existence of a real number* α *such that* $x\in\alpha V$.

III. $V\in\mathfrak{B}$ *implies* $V+V=2V$.

From the above conditions II_1—II_6, we see at once that the operations of vector sum and scalar multiplication are continuous. Also it is obvious that scalar multiplication αx gives a continuous mapping : $(\alpha, x)\rightarrow\alpha x$ from $R\times\mathbf{E}$ to $\mathbf{E}$ where R is the totality of real numbers. We may suppose evidently that every neighbourhood of θ is *symmetric* : $V=-V$. It is also evident that $\mathbf{E}$ is a regular Hausdorff's space. From III and the continuity of scalar multiplication, we can conclude with ease that every neighbourhood V is a *convex set* : $\lambda V+(1-\lambda)V=V$ for $0\leqq\lambda\leqq 1$.

A^a denotes the closure of A. $A\pm B=\{a\pm b\,;\,a\in A, b\in B\}$. $\alpha A=\{\alpha a\,;\,a\in A\}$ $(\alpha\in R)$. *Co* A, the *convex cover* of A, is the smallest convex set which contains A. For later use, we shall summarize here some of the wellknown relations without proof :

$(A+B)^a\supset A^a+B^a$ specially $(A+x_o)^a=A^a+x_o$,

$\alpha A^a=(\alpha A)^a$, $A^a=\bigcap\limits_{V\in\mathfrak{B}}(A+V)$,

$Co\,A=\{\sum\limits_{j=1}^{n}\lambda_j a_j\,;\,\sum\limits_{j=1}^{n}\lambda_j=1,\ \lambda_j\geqq 0,\ a_j\in A\}$, $Co\,(Co\,A)=Co\,A$, $Co\,\alpha A=\alpha\,Co\,A$, $Co\,(A\pm B)=Co\,A\pm Co\,B$.

Let $\mathbf{B}$ be a σ-ring of subsets of a fixed set S, which is and will be assumed without specific mention to belong to $\mathbf{B}$. The set $\sigma\in\mathbf{B}$ is called *measurable* as usual. We shall denote by $m(\sigma)$ a non-negative, completely additive measure defined on $\mathbf{B}$ with $m(S)<\infty$. For any point-function $x(s)$ with range in $\mathbf{E}$, $x(\sigma)$ denotes the set of the values taken by x for $s\in\sigma$, that is

$$x(\sigma)=\{t\,;\,t=x(s),\ s\in\sigma\}.$$

It is evident that $x(\sigma)\supset x(\sigma')$ when $\sigma\supset\sigma'$. The *characteristic function* of σ, denoted by $\chi_\sigma(s)$, is a real valued function defined by setting $\chi_\sigma(s)=1$ for $s\in\sigma$ and $\chi_\sigma(s)=0$ otherwise. Then by a *simple funtion* we mean a vector-valued function defined by

$$x(s)=\sum_{j=1}^{n}\chi_{\sigma_j}(s)\,c_j$$

where $\sigma_j\in\mathbf{B}\ (j=1,2,\cdots,n)$, n being a natural number, $\sigma_i\cap\sigma_j=0$ if $i\neq j$, and $c_j\in\mathbf{E}\ (j=1,\cdots,n)$. $x(s)$ is *bounded* if and only if $x(\sigma)$ is *bounded* as a set in $\mathbf{E}$, which means, to each $V\in\mathfrak{B}$, there exists a $\lambda>0$ satisfying the condition $\lambda x(\sigma)\subset V$.

We shall often speak of a *division* of σ, denoted by $\varDelta$ or $\varDelta=\{\sigma_j\}$, as a partition of σ into finite or countable disjoint sets $\sigma_j\in\mathbf{B}$. We say a division $\varDelta_1=\{\sigma_j^1\}$ is a *subdivision* of a division $\varDelta_2=\{\sigma_j^2\}$ when to each σ_j^1 of $\varDelta_1$ there exists a σ_i^2, of $\varDelta_2$, containing σ_j^1, which will be denoted by $\varDelta_2\leqq\varDelta_1$. To any pair $\varDelta_1$, $\varDelta_2$ of divisions of σ, we may associate a third division, denoted by $\varDelta_1\cdot\varDelta_2=\varDelta_2\cdot\varDelta_1$, defined by $\varDelta_1\cdot\varDelta_2=\{\sigma_i^1\cap\sigma_j^2\,;\ i=1,2,\cdots,\ j=1,2,\cdots\}$. We may remark here $\varDelta_1\cdot\varDelta_2\geqq\varDelta_1$, $\varDelta_1\cdot\varDelta_2\geqq\varDelta_2$, from which we can conclude that to each pair of divisions $\varDelta_1$ and $\varDelta_2$ there is a division $\varDelta$ which is simultaneously a subdivision of both Δ_1 and Δ_2. Let N be the totality of natural numbers. We denote by π and π_j a finite subset of N. Then we shall write $\pi_1 \leqq \pi_2$ for $\pi_1 \subset \pi_2$. $\{x_j\}$ being a point-sequence in $\mathbf{E}$, $\sum_{\pi} x_j$ denotes the finite sum of x_j for all j belonging to π. Accordingly, when $\{A_j\}$ is a finite or countable sequence of sets in $\mathbf{E}$, $\sum_{\pi} A_j$ means the set $\{\sum_{\pi} x_j \,;\, x_j \in A_j\}$.

It seems convenient for our purpose to give here a brief mention of Moore-Smith's limit [5. p. 103] with respect to the system $\mathfrak{V}$ of neighbourhoods about θ. Given $\{x_V \,;\, V \in \mathfrak{V}\}$, suppose, for a certain $x \in \mathbf{E}$, there exists, to each V a $V_0 \in \mathfrak{V}$ such that $V_1 \subset V_0$ implies $(x_{V_1} - x) \in V$: in notation

$$\forall V, \exists V_0 \;;\quad V_1 \subset V_0 \rightarrow (x_{V_1} - x) \in V.$$

Then x is said the *limit of* $\{x_V\}$ *with respect to* $\mathfrak{V}$ *in the sense of Moore-Smith.* Also we say that $\{x_V\}$ *converges to* x. By this definition of limit, the idea of completeness may be generalized by considering, instead of Cauchy's fundamental sequence, the *fundamental* $\mathfrak{V}$*-sequence* $\{x_V\}$, that is, in notation as follows :

$$\forall V, \exists V_0 \;;\quad V_1, V_2 \subset V_0 \rightarrow (x_{V_1} - x_{V_2}) \in V.$$

We are lead to say that $\mathbf{E}$ is *complete* in the sense of Moore-Smith if and only if every fundamental $\mathfrak{V}$-sequence has a limit in $\mathbf{E}$. A set-function $F(\sigma)$, defined on $\mathbf{B}$, with range in $\mathbf{E}$, is said to be *completely additive* if and only if $\sigma = \bigcup_{j=1}^{\infty} \sigma_j$, $\sigma_i \cap \sigma_j = 0 \; (i \neq j)$, $\sigma_j \in \mathbf{B}$ imply

$$\sum_{j=1}^{\infty} F(\sigma_j) = F(\sigma),$$

which means that the left hand-side is convergent and the sum is equal to the right hand-side. For later use, we need the notion of absolute continuity of set-functions. Let $F(\sigma)$ be a completely additive set-function. If, to each V, there exists a $\delta_0 > 0$ such that

$$m(\sigma) < \delta_0 \quad \text{implies} \quad F(\sigma) \in V,$$

then $F(\sigma)$ is said to be *absolutely continuous* with respect to the measure m.

2. Basic functions.

Definition 2.1. *Let $x(s)$ be a function defined on S with range in $\mathbf{E}$. $x(s)$ is called basic on $\sigma \in \mathbf{B}$ if and only if to each neighbourhood $V \in \mathfrak{V}$ there exists a finite division $\Delta = \{\sigma_j \,;\, j = 1, \cdots, n\}$ of σ such that for any pair of points s', s'' of the same σ_j holds $x(s') - x(s'') \in V$, or more briefly $x(\sigma_j) - x(\sigma_j) \subset V (j = 1, \cdots, n)$.*

From this definition we see immediately that if $x(s)$ is basic on S, then it is basic on any measurable subset $\sigma \subset S$, and that a basic function is bounded. It is also evident that the family $B(S)$ of all basic function on S forms a linear space over R. A simple function is basic. If we introduce a topology into $B(S)$ by means of uniform convergence in the sense of Moore-Smith, then the subspace of all the simple functions is dense in $B(S)$. From this we can easily derive the following Lemma 2.1.

Lemma 2.1. *Let $x(s)$ be a function belonging to $B(S)$ in which* **E** *is assumed complete. Then there is a set-function $F(\sigma)=F(x,\sigma)$ defined on* **B** *with range in* **E**, *satisfying the following conditions (I_1) and (I_2).*

(I_1) $F(x,\sigma)\in m(\sigma)(Co\, x(\sigma))^a$ *for all* $\sigma\in \mathbf{B}$

(I_2) $F(x,\sigma)$ *is completely additive as a set-function.*

Proof. If $x(s)$ is a simple function, given by $x(s)=\sum_{j=1}^{n}\chi_{\sigma_j}(s)c_j$ $(c_j\in \mathbf{E}, \sigma_i\cap\sigma_j=0(i\neq j))$, then the set-function $F(x,\sigma)$ defined by $F(x,\sigma)=\sum_{j=1}^{n}m(\sigma\cap\sigma_j)c_j$ has the properties (I_1) and (I_2).

Now, every $x(s)\in B(S)$ is a uniform limit of some sequence $\{x_V(s)\}$ in the sense of Moore-Smith in which each x_V is a simple function. Consequently with each x_V, a set-function $F(x_V,\sigma)$ satisfying (I_1) and (I_2) can be associated. It is easy to see that $\{F(x_V,\sigma)\ ;\ V\in\mathfrak{B}\}$ is a fundamental $\mathfrak{L}$-sequence for each fixed σ and converges to a limit $\lim F(x_V,\sigma)=F(x,\sigma)$, which is easily seen to be independent of the choice of $\{x_V\}$. Thus $F(x,\sigma)$, satisfying (I_1) and (I_2), is the required one and our lemma is proved.

3. The definition of the integrals and their properties.

We have shown that there exists a set-function satisfying (I_1) and (I_2) for the family of basic functions. To define our integrals, we shall follow after the definition mentioned before of integrals of real valued functions, but take into account of their uniqueness condition.

Definition 3.1. *Given a function $x(s)$ taken values in* **E** *defined on S, let us suppose that there exists a set-function $I(x,\sigma)$ with range in* **E** *such that*

(I_1) $I(x,\sigma)\in m(\sigma)(Co\, x(\sigma))^a$ *for any measurable* $\sigma\subset S$,

(I_2) $I(x,\sigma)$ *is completely additive.*

If a set-function satisfying (I_1) and (I_2) is uniquely determined, then we shall say that $I(x,\sigma)$ is the integral over S of $x(s)$ with respect to the measure m and also that $x(s)$ is integrable over S and shall write

$$I(x,\sigma)=\int_{\sigma}x(s)\,dm$$

From the definition we see immediately that if $x(s)$ is integrable then $\alpha x(s)$ $(\alpha\in R)$ is also integrable and if further $x(s)$ is bounded, then $I(x,\sigma)$ becomes absolutely continuous as a set-function. Further we shall prove the following :

Theorem 3.1. *If $x(s)$ is integrable over S, it follows that it is also integrable over any set $\sigma\in$* **B**.

Proof. We have only to show that the uniqueness condition of the integral over σ_o is verified.

Supposing the contrary, let $F_o(\sigma)$ be any set-function satisfying (I_1) and (I_2) for any measurable $\sigma\subset\sigma_o$ (instead of S) and

$$F_o(\sigma_o)\neq\int_{\sigma_0}x(s)\ dm \qquad (3.1)$$

Let F be the extension of F_o defined by

$$F(\sigma)=F_o(\sigma\cap\sigma_o)+\int_{\sigma-\sigma_o} x(s)\ \mathrm{dm} \qquad \text{for all } \sigma\in\mathbf{B}.$$

Since $F(\sigma)$ must satisfy (I_1) and (I_2), we would have from the uniqueness condition of the integral over S

$$F(\sigma)=\int_\sigma x(s)\ dm \qquad \text{for all } \sigma\in\mathbf{B},$$

whence especially

$$\int_{\sigma_o} x(s)dm=F(\sigma_o)=F_o(\sigma_o)$$

This contradicts (3.1) and proves our theorem.

To investigate the properties of integrable functions we shall make use of the following definition which may be regarded as a generalization of the notion of basic functions.

Definition 3.2. *We shall say that a function $x(s)$ defined on S with range in* $\mathbf{E}$ *is measurable on σ_o if and only if to each $\sigma\in\mathbf{B}$, $\sigma\subset\sigma_o$ and to each $V\in\mathfrak{V}$, there exists a countable division Δ_V of σ_o such that for any subdivision $\Delta=\{\sigma_j\}$ of Δ_V there exists a π_Δ for which $\pi\geqq\pi_\Delta$ implies*

$$\sum_\pi(Co\, m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j))^a-\sum_\pi(Co\, m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j))^a\subset V,$$

which we shall denote in notation as follows :

$$\forall V,\ \exists\Delta_V\ ;\ \forall\Delta\geqq\Delta_V$$
$$\exists\pi_\Delta\ ;\ \pi\geqq\pi_\Delta\rightarrow\sum_\pi(Co\, m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j))^a$$
$$-\sum_\pi(Co\, m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j))^a\subset V.$$

We see easily that basic functions are measurable. Moreover we shall see later in section 4, that any function which is integrable in the sense of Phillips is measurable.

Let us remark here that if $F(\sigma)$ is completely additive, then to each subdivision $\Delta=\{\sigma_j\}$ of σ and to each V there exists a π_V such that

$$(\sum_\pi F(\sigma_j)-F(\sigma))\in V \qquad \text{for all } \pi\geqq\pi_V,$$

which means that the series $\sum F(\sigma_j)$ is *unconditionally summable* to $F(\sigma)$ [9. p. 501].

Theorem 3.2. *If for any function $x(s)$, measurable on σ_o, a set-function $I(x,\sigma)$ satisfies the conditions (I_1) and (I_2), it is uniquely determined on σ_o.*

Proof. Let $I_o(x,\sigma)$ be any set-function satisfying (I_1) and (I_2). The condition of measurability of $x(s)$ shows that to each σ and to each V, there corresponds a Δ_V and π_V such that

$$\sum_\pi(Co\, m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j))^a-\sum_\pi(Co\, m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j))^a\subset V \text{ for all } \pi\geqq\pi_V.$$

Since I and I_o are both completely additive, there exists a π'_V for which if $\pi\geqq\pi'_V$ then

$$(I(x,\sigma)-\sum_\pi I(x,\sigma\cap\sigma_j))\in V$$
$$(I_o(x,\sigma)-\sum_\pi I_o(x,\sigma\cap\sigma_j))\in V,$$

so that, for any $\pi\geqq\pi_V\cup\pi'_V$, we have

$$I(x,\sigma)-I_0(x,\sigma)\in\sum_\pi I(x,\sigma\cap\sigma_j)-\sum_\pi I_0(x,\sigma\cap\sigma_j)+2V$$
$$\subset 3V.$$

whence

$$I(x,\sigma)-I_0(x,\sigma)\in\bigcap_{V\in\mathfrak{V}}(3V)=\{\theta\}$$

taking into account of the arbitrariness of V. Thus the theorem is proved.

Corollary 3.1. *Basic functions are integrable over* S *provided that* **E** *be complete.*

Corollary 3.2. *If $x(s)$, a function defined on* S, *is measurable on σ_0 and permits set-functions satisfying (I_1) and (I_2), then $x(s)$ is integrable over σ_0.*

Lemma 3.1. *The sum and constant multiple of measurable functions on σ_0 are measurable on σ_0.*

Proof. Since the proof is obvious, we shall omit it.

Lemma 3.2. *Let $x(s)$ be a function defined on* S *with range in* **E** *and let $\{\sigma_j\}$ be any countable division of $\sigma\in$* **B** *into measurable sets $\sigma_j\in$* **B**. *Then to each V there exists a π_0 such that if $\pi\geqq\pi_0$ then*

$$m(\bigcup_\pi\sigma_j)Co\,x(\sigma)\subset m(\sigma)Co\,x(\sigma)+V. \tag{3.2}$$

Proof. Let us remark here that for any set A in **E** we have

$$\alpha\,Co\,A\subset\beta\,Co\,A-(\beta-\alpha)x_0 \tag{3.3}$$

provided that $0\leqq\alpha<\beta$, $x_0\in Co\,A$.

If we replace A, α and β in (3.3) by $x(\sigma)$, $m(\bigcup_\pi\sigma_j)$ and $m(\sigma)$ respectively, then we shall obtain

$$m(\bigcup_\pi\sigma_j)Co\,x(\sigma)\subset m(\sigma)Co\,x(\sigma)+(m(\sigma)-m(\bigcup_\pi\sigma_j))x_0,$$

whence taking π large enough so that $(m(\sigma)-m(\bigcup_\pi\sigma_j))x_0\in V$ may hold, we shall have the required result (3.2).

Theorem 3.3. *If measurable function $x(s)$ and $y(s)$ are integrable over σ_0, then so are the sum $(x+y)(s)$ and*

$$\int_{\sigma_0}(x+y)(s)\,dm=\int_{\sigma_0}x(s)\,dm+\int_{\sigma_0}y(s)\,dm.$$

Proof. Let us define a set-function $F(x+y,\sigma)$, for $\sigma\subset\sigma_0$, $\sigma\in$ **B** by

$$F(x+y,\sigma)=\int_\sigma x(s)\,dm+\int_\sigma y(s)\,dm.$$

We have to show that $F(x+y,\sigma)$ satisfies (I_1) and (I_2), which means by Lemma 3.1 that $x+y$ is measurable and by Corollary 3.2 that $x+y$ becomes integrable. By this that consideration, we have only to show that $F(x+y,\sigma)$ satisfies (I_1), as we find with ease from the complete additivity of $\int_\sigma x(s)\,dm$ and $\int_\sigma y(s)\,dm$ that $F(x+y,\sigma)$ satisfies (I_2).

From the measurability of $y(s)$, we may find, to each V, a Δ_V such that for every $\Delta=\{\sigma_j\}\geqq\Delta_V$, there exists a π_Δ for which

$$\pi\geqq\pi_\Delta\rightarrow\sum_\pi m(\sigma\cap\sigma_j)y(\sigma\cap\sigma_j)-\sum_\pi m(\sigma\cap\sigma_j)y(\sigma\cap\sigma_j)\subset\mathrm{V}.$$

By the complete additivity of integrals $\int_\sigma y(s)\,dm$ and $\int_\sigma x(s)\,dm$, there exists a π_0 such that for all $\pi \geqq \pi_0$ holds

$$\int_\sigma x(s)\ dm - \sum_\pi \int_{\sigma \cap \sigma_j} x(s)\ dm \in V \tag{3.5}$$

and

$$\int_\sigma y(s)\ dm - \sum_\pi \int_{\sigma \cap \sigma_i} y(s)\ dm \in V. \tag{3.6}$$

By Lemma 3.2 there exists a π_0' such that for all $\pi \geqq \pi_0'$

$$Co\ m(\sigma \cap \bigcup_\pi \sigma_j)(x+y)(\sigma) \subset Co\ m(\sigma)(x+y)(\sigma)+V. \tag{3.7}$$

Then we have, for all $\pi \geqq \pi_\Delta \cup \pi_0 \cup \pi_0'$,

$$\int_\sigma x(s)\ dm + \int_\sigma y(s)\ dm$$

$$\in \sum_\pi \int_{\sigma \cap \sigma_j} x(s)\ dm + V + \sum_\pi \int_{\sigma \cap \sigma_j} y(s)\ dm + V \qquad \text{(by (3.5) and (3.6))}$$

$$\subset \sum_\pi m(\sigma \cap \sigma_j)(Co\ x(\sigma \cap \sigma_j))^a + \sum_\pi m(\sigma \cap \sigma_j)(Co\ y(\sigma \cap \sigma_i))^a + 2V$$

$$\subset \sum_\pi m(\sigma \cap \sigma_j)(Co\ x(\sigma \cap \sigma_j)+V) + \sum_\pi m(\sigma \cap \sigma_j)(Co\ y(\sigma \cap \sigma_j)+V)+2V$$

$$\subset \sum_\pi m(\sigma \cap \sigma_j) Co(x(\sigma \cap \sigma_j)+y(\sigma \cap \sigma_j))+2m(\sigma)V+2V$$

$$\subset \sum_\pi m(\sigma \cap \sigma_j) Co((x+y)(\sigma \cap \sigma_j)+y(\sigma \cap \sigma_j)-y(\sigma \cap \sigma_j))+2m(\sigma)V+2V$$

$$\subset Co \sum_\pi m(\sigma \cap \sigma_j)(x+y)(\sigma \cap \sigma_j)+V+2m(\sigma)V+2V \qquad \text{(by (3.4))}$$

$$\subset Co\ m(\sigma \cap \bigcup_\pi \sigma_j)(x+y)(\sigma)+V+2m(\sigma)V+2V$$

$$\subset Co\ m(\sigma)(x+y)(\sigma)+V+V+2m(\sigma)V+2V \qquad \text{(by (3.7))}$$

$$= Co\ m(\sigma)(x+y)(\sigma)+(4+2m(\sigma))V.$$

Since V is arbitrary, we can conclude

$$\int_\sigma x(s)\ dm + \int_\sigma y(s)\ dm \in \bigcap_{V \in \mathfrak{B}} (Co\ m(\sigma)(x+y)(\sigma)+(4+2m(\sigma))V)$$
$$= (Co\ m(\sigma)(x+y)(\sigma))^a.$$

This shows $F(x+y, \sigma)$ satisfies the condition (I_1) and our theorem is proved.

4. **Relations to other integrals.**

To investigate the relations of our integrals to other integrals we shall make use of the following theorem the essential part of which is due to C. E. Rickart and R.S. Phillips.

Theorem 4.1. *For a function $x(s)$, defined on S with range in* **E**, *to be measurable and permit set-functions satisfying (I_1) and (I_2) it is necessary and sufficient that there exists a set-function $I(x, \sigma)$ with range in* **E** *such that*

$$\forall V,\ \exists \Delta_V\ ;\ \Delta \geqq \Delta_V,\ \exists \pi_\Delta$$
$$\pi \geqq \pi_\Delta \rightarrow \sum_\pi m(\sigma \cap \sigma_j)x(\sigma \cap \sigma_j) - I(x, \sigma) \subset V \tag{4.1}$$

The condition is nothing but a condition that the set-function $F(\sigma) = m(\sigma)x(\sigma)$ should be SV-integrable in the sense of Rickart [9. p. 502]. Also we remark here that the same condition is that of the integrability of $x(\sigma)$ in the sense of Phillips [8. p. 119].

Proof. *Necessity.* Let $x(s)$ be a measurable function for which there exist set-functions $I(x,\sigma)$ satisfying (I_1) and (I_2), then $I(x,\sigma)$ is our integral by Corollary 3.2. From the measurability of $x(s)$ and the complete additivity of $I(x,\sigma)$, we may easily conclude that the condition (4.1) is necessary.

Sufficiency. The set-function, satisfying the condition (4.1), is Phillips's integral of $x(\sigma)$ as mentioned above. According to one of the results obtained by R.S. Phillips [8. p. 122], $I(x,\sigma)$ satisfies (I_2). Since the measurability of $x(s)$ is obvious by (4.1), we have only to show that $I(x,\sigma)$ satisfies (I_1). From our assumption, there exist some division $\Delta=\{\sigma_j\}$ and a set $\pi\subset N$ which satisfy

$$I(x,\sigma)\in\sum_{\pi} m(\sigma\cap\sigma_j)x(\sigma\cap\sigma_j)+V.$$

Further, from Lemma 3.2, we have for a sufficiently large π

$$Co\ m(\sigma\cap\bigcup_{\pi}\sigma_j)x(\sigma\cap\bigcup_{\pi}\sigma_j)\subset Co\ m(\sigma)x(\sigma)+V.$$

Thus we may conclude

$$I(x,\sigma)\in Co\ m(\sigma)x(\sigma)+2V$$

which shows

$$I(x,\sigma)\in\bigcap_{V\in\mathfrak{B}}(Co\ m(\sigma)x(\sigma)+2V)=m(\sigma)(Co\ x(\sigma))^a$$

and our theorem is proved.

This theorem shows that at least for measurable function $x(s)$ our integral is in the sense mentioned above identical with that of Phillips or Rickart's SV-integral.

Pettice's integral is concerning with the functions with ranges in Banach spaces. The following theorem gives some relation of our integral to that of Pettice's.

Theorem 4.2. *Let $x(s)$ be a function defined on S with range in Banach space. If $x(s)$ is integrable in the sense of Pettice, then it is also integrable in our sense.*

Proof. It is sufficient to show that Pettice's integral (P) $\int_\sigma x(s)dm$ satisfies (I_1) and the condition of uniqueness, since Pettice himself gives the complete additivity of his integral as a set-function [7. p. 283]. But the condition of uniqueness is almost immediate since $\bar{x}(x(s))$ is integrable as a real-valued function where $\bar{x}$ is an element of the set $\bar{\mathbf{E}}$ of all the linear functionals defined on $\mathbf{E}$.

To prove that (I_1) is satisfied, let us remark first that $\bar{x}(x(s))$ is integrable for every $\bar{x}\in\bar{\mathbf{E}}$, so that we have

$$(\mathrm{P})\int_\sigma x(s)\,dm\in\bigcap_{\bar{x}\in\bar{\mathbf{E}}}\{x\,;\,\bar{x}(x)\in m(\sigma)(Co\,\bar{x}(x(\sigma)))^a\}.$$

Now we shall make use of the fact that every closed convex subset A of $\mathbf{E}$ is the intersection of all the closed half-spaces $\{x\,;\,\bar{x}(x)\leqq\alpha\}$ $(\alpha\in R,\ \bar{x}\in\bar{\mathbf{E}})$ which contain A[3. p. 73].

From this fact we can conclude

$$\bigcap_{\bar{x}\in\bar{E}}\{x\,;\,\bar{x}(x)\in m(\sigma)(Co\,\bar{x}(x(\sigma)))^a\}=m(\sigma)(Co\,x(\sigma))^a$$

which shows

$$\text{(P)}\ \int_\sigma x(s)\,dm\in m(\sigma)(Co\,x(\sigma))^a.$$

Thus the theorem is proved.

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