On Cohomology Groups in a Field, which is Complete with respect to a Discrete Valuation

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Let K be a field, which is complete with respect to a discrete valuation, and $\mathfrak{K}$ be its residue class field. In the case, $\mathfrak{K}$ is perfect, Witt considered central division algebras over K and obtained a theorem on the structure of Brauer group of K. This result was generalized by Nakayama to the case, where $\mathfrak{K}$ is not necessarily perfect. If we consider from the viewpoint of cohomology theory, Witt's theorem can be replaced by a theorem on two-dimensional cohomology group in a Galois extension of K, but Nakayama's condition on division algebras can hardly be interpreted by cohomology theory. In the present paper we aim to investigate the structure of higher dimensional cohomology groups, under the assumption that $\mathfrak{K}$ is perfect. For this purpose it seems more natural to consider cohomology groups in a maximal separable algebraic extension of K than to consider those in a finite Galois extension of K. We find that the structure of cohomology groups depends solely on properties of the residue class field. As a typical example we treat the case, where generalized local class field theory holds. The author is much indebted to Mr. Kawada for his collaboration in preparing Theorem 2.

I. Let G be the Galois group of a maximal separable algebraic extension L of K, and assign to G the usual topology. We denote by L^* the multiplicative group of all non-zero elements in L, and assign to L^* the discrete topology. Then, according to Artin, a continuous cochain in L^* is no other than the lifting to L of a cochain, which is defined on the Galois group of some finite Galois extension of K. In the following by cochains we understand always continuous cochains and by a coboundary a coboundary of a continuous cochain. We consider the unique prolongation in L of the valuation in K. A subfield Ω of L over K is called unramified over K, when the value group of Ω coincides with the value group of K. Obviously there exists the maximal unramified subfield Σ of L over K. Since $\mathfrak{K}$ is perfect, every algebraic extension of $\mathfrak{K}$ is separable over $\mathfrak{K}$. The residue class field of Σ is then algebraically closed over $\mathfrak{K}$ and is identified with the residue class field $\mathfrak{L}$ of L. By a theorem of Hasse there exists no proper central division algebra over any finite extension of Σ. It follows that in every subfield W of L, which is finite normal over Σ with Galois group G_W, the n-

dimensional cohomology group $H^n(G_W, W^*)$ is trivial for every positive dimension n (See for instance (4)). If we denote by T the subgroup of G, which corresponds to Σ, we have $H^n(T, L^*)=1$. Now, by Hochschild-Serre's theorem, we have the following exact sequence

$$1 \rightarrow H^n(G/T, \Sigma^*) \rightarrow H^n(G, L^*) \rightarrow H^n(T, L)=1.$$

Therefore $H^n(G, L^*)$ is isomorphic to $H^n(G/T, \Sigma^*)$ by lifting, where the topology of G induces naturally the topology of G/T. Since every automorphism of Σ over K induces an automorphism of $\mathfrak{L}$ over $\mathfrak{K}$, we can verify that G/T is isomorphic to the Galois group $\mathfrak{G}$ of $\mathfrak{L}$ over $\mathfrak{K}$. Then we can write $H^n(\mathfrak{G}, \Sigma^*)$ in place of $H^n(G/T, \Sigma^*)$. Now every element of Σ^* can be put in the form $\pi^\nu E$, where π is a fixed prime element in K and E is an unit in Σ, ν being a rational integer. Consequently $H^n(\mathfrak{G}, \Sigma^*)$ is a direct product of $H^n(\mathfrak{G}, (\pi))$ and $H^n(\mathfrak{G}, E)$, where we denote by (π) the cyclic group with the generator π and by E the group of all units in Σ. Let Z be the additive group of all rational integers, on which $\mathfrak{G}$ operates trivially. Then the first factor $H^n(\mathfrak{G}, (\pi))$ is clearly isomorphic to the additive cohomology group $H^n(\mathfrak{G}, Z)$.

Next we shall show that the second factor $H^n(\mathfrak{G}, E)$ is naturally isomorphic to $H^n(\mathfrak{G}, \mathfrak{L}^*)$. For this purpose we consider the multiplicative group S of all units a in Σ for which $a \equiv 1$ mod. P holds, P being the prime ideal of the valuation in Σ. First we prove that $H^n(\mathfrak{G}, S)$ is trivial. Let Ω be any subfield of Σ, which is finite normal over K. It suffices to prove that every n-dimensional cocycle c with $c \in S \frown \Omega$ is a coboundary of an n-1-dimensional cochain b with $b \in S \frown \Omega$. Since Ω is complete, we have only to show the existence of an infinite sequence of n-1-dimensional cochains b_i, $i=1, 2, \cdots$, in Ω, such that

$$c \equiv \delta b_i \text{ mod. } P^i, \qquad b_i \equiv 1 \text{ mod. } P,$$
$$b_i \equiv b_{i-1} \text{ mod. } P^{i-1},$$

where δ signifies the coboundary operator. If $i=1$, we have only to put $b_1=1$. By induction hypothesis we put

$$c\delta b_i^{-1} = 1 + d_i \pi^i,$$

where d_i belongs to the valuation ring I of Ω. Since $c\delta b_i^{-1}$ is an n-dimensional (multiplicative) cocycle, d_i is an n-dimensional additive cocycle mod P. Now the n-dimensional additive cohomology group in the residue class field $\mathfrak{K}_\Omega$ of Ω is trivial. In fact, there exists an element $\theta \in \mathfrak{K}_\Omega$ such that the trace $Tr(\theta)$ of θ with respect to $\mathfrak{K}$ is not zero. We put for the additive cocycle $f(x_1, \cdots, x_n)$ in $\mathfrak{K}_\Omega$

$$g(x_1, \cdots, x_{n-1}) = \frac{(-1)^n}{Tr(\theta)} \sum_y f(x_1, \cdots, x_{n-1}, y)\, x_1 \cdots x_{n-1} y(\theta)\,.$$

Then f is an additive coboundary of g. Therefore we can find an n-1-dimensional additive cochain h_i in I, such that d_i is the additive coboundary of h_i mod. P, and we have

$$\delta(1+h_i\pi^i)\equiv 1+d_i\pi^i \text{ mod. } P^{i+1}$$

Putting $b_{i+1}=b_i\ (1+h_i\pi^i)$, we get

$$b_{i+1}\equiv b_i \text{ mod. } P^i, \qquad c\equiv\delta b_{i+1} \text{ mod. } P^{i+1}.$$

We have further $b_{i+1}\equiv 1$ mod. P from the induction hypothesis $b_i\equiv 1$ mod. P, which was to be proved. Since $H^n(\mathfrak{G}, S)$ is trivial for every dimension n, we have the following exact sequence

$$1=H^n(\mathfrak{G}, S)\to H^n(\mathfrak{G}, E)\to H^n(\mathfrak{G}, E/S)\to H^{n+1}(\mathfrak{G}, S)=1,$$

and therefore $H^n(\mathfrak{G}, E)$ is naturally isomorphic to $H^n(\mathfrak{G}, E/S)$. Since the quotient group E/S is naturally isomorphic to $\mathfrak{L}^*$, $H^n(\mathfrak{G}, E)$ is isomorphic to $H^n(\mathfrak{G}, \mathfrak{L}^*)$.

Theorem I. *Let K be a field, which is complete with respect to a discrete valuation, and its residue class field $\mathfrak{K}$ be perfect. Further let G be the Galois group of a maximal separable algebraic extension L of K, and $\mathfrak{G}$ be the Galois group of the residue class field $\mathfrak{L}$ of L over $\mathfrak{K}$. For $n\geq 1$ the n-dimensional cohomology group $H^n(G, L^*)$ is canonically isomorphic to the direct product of $H^n(\mathfrak{G}, Z)$ and $H^n(\mathfrak{G}, \mathfrak{L}^*)$, where Z is the additive group of rational integers, on which $\mathfrak{G}$ operates trivially.*

We can observe that the two-dimensional case of Theorem I is the interpretation of Witt's theorem by cohomology theory.

2. As an example of complete fields, for which the structure of our cohomology groups can be determined precisely, we shall consider the case, where generalized local class field theory, which is due to Moriya, holds. In this case, $\mathfrak{K}$ is perfect and there exists no proper central division algebra over any finite extension of $\mathfrak{K}$. Then $H^n(\mathfrak{G}, \mathfrak{L}^*)$ is trivial for every dimension n, and we have

$$H^n(G, L^*)\cong H^n(\mathfrak{G}, Z).$$

Further the following conditions are satisfied:

I. Every finite extension of $\mathfrak{K}$ is cyclic.

II. For every positive integer m there exists one and only one cyclic extension of $\mathfrak{K}$ with degree m.

By these conditions we can determine $H^n(\mathfrak{G}, Z)$ as follows. For each prime number p let $\mathfrak{U}_p$ be the subgroup of $\mathfrak{G}$, which corresponds to the extension of $\mathfrak{K}$ with degree p. For each p we choose a coset $x_p\mathfrak{U}_p$, which is different from $\mathfrak{U}_p$. The family of all cosets $x_p\mathfrak{U}_p$ has the finite intersection property. Since $\mathfrak{G}$ is compact, the intersection of all $x_p\mathfrak{U}_p$ is non-empty. We choose an element x of $\mathfrak{G}$, which is common to all $x_p\mathfrak{U}_p$, and denote by $\mathfrak{F}$ the cyclic group generated by x. One sees easily that

$\mathfrak{F}$ is everywhere dense in $\mathfrak{G}$. Then $H^n(\mathfrak{G}, Z)$ is isomorphically mapped onto $H^n(\mathfrak{F}, Z)$ by the restriction mapping: $\mathfrak{G} \to \mathfrak{F}$. When $n=1$, $H^1(\mathfrak{F}, Z)$ is obviously trivial. When $n=2$, let R be the additive group of all rational numbers. Since R is uniquely divisible, we have the following exact sequence

$$O = H^1(\mathfrak{F}, R) \to H^1(\mathfrak{F}, R/Z) \to H^2(\mathfrak{F}, Z) \to H^2(\mathfrak{F}, R) = O,$$

and $H^2(\mathfrak{F}, Z)$ is isomorphic to $H^1(\mathfrak{F}, R/Z)$. Since one-dimensional cocycle φ on $\mathfrak{F}$ in R/Z is a character of $\mathfrak{F}$, we see that $H^1(\mathfrak{F}, R/Z)$ is isomorphic onto R/Z by the mapping $\varphi \to \varphi(x)$. Therefore we have the well-known fact that $H^2(G, L^*)$ is isomorphic to the additive group of all rational numbers mod. Z. We shall show further that, for $n \geqq 3$, $H^n(G, L^*)$ is trivial. For this, it suffices to show that every cocycle in $C^n(\mathfrak{F}/\mathfrak{Z}, Z)$ becomes a coboundary by lifting, where $\mathfrak{Z}$ is a subgroup of $\mathfrak{F}$ with any finite index m. Let c be the normalized cocycle in $C^2(\mathfrak{F}/\mathfrak{Z}, \mathfrak{Z})$, which is determined by the group extension

$$\mathfrak{Z} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{F}/\mathfrak{Z},$$

where for $0 \leqq i, j < m$

$$\begin{aligned} c &= x^{ma}, \\ a(x^i\mathfrak{Z}, x^j\mathfrak{Z}) &= 1, \quad \text{if} \quad i+j \geqq m \\ &= 0, \quad \text{if} \quad i+j < m. \end{aligned}$$

Then a is a cocycle in $C^2(\mathfrak{F}/\mathfrak{Z}, Z)$. A homomorphism g of $\mathfrak{Z}$ into Z is determined uniquely by the value $g(x^m)$ and therefore $H^{n-2}(\mathfrak{F}/\mathfrak{Z}$, Hom $(\mathfrak{Z}, Z))$ is canonically isomorphic to $H^{n-2}(\mathfrak{F}/\mathfrak{Z}, Z)$. Further we have $g(c) = ag(x^m)$. Then the cup product reduction theorem asserts that $H^{n-2}(\mathfrak{F}/\mathfrak{Z}, Z)$ is isomorphically mapped onto $H^n(\mathfrak{F}/\mathfrak{Z}, Z)$ by the mapping $f \to a \smile f = af$. Further we consider the subgroup $\mathfrak{W}$ of $\mathfrak{F}$ with index m^2. Let b be a cocycle in $C^2(\mathfrak{F}/\mathfrak{W}, Z)$ such that for $0 \leqq i, j < m^2$

$$\begin{aligned} b(x^i\mathfrak{W}, x^j\mathfrak{W}) &= 1, \quad \text{if} \quad i+j \geqq m^2, \\ &= 0, \quad \text{if} \quad i+j < m^2. \end{aligned}$$

Since mb is cohomologous to the lifting of a to $\mathfrak{F}/\mathfrak{W}$, the lifting of $a \smile f$ to $\mathfrak{F}/\mathfrak{W}$ is cohomologous to $mb \smile f = b \smile mf$. Now, mf being a coboundary, the lifting of $a \smile f$ to $\mathfrak{F}/\mathfrak{W}$ is a coboundary in $C^n(\mathfrak{F}/\mathfrak{W}, Z)$. Thus we have the following:

Theorem 2. *Suppose that generalized local class field theory holds over K. Let G be the Galois group of a maximal separable algebraic extension L of K. The n-dimensional cohomology group* $H^n(G, L^*)$ *is trivial, except when* $n=2$. *If* $n=2$, $H^2(G, L^*)$ *is isomorphic to the additive group of all rational numbers* mod. Z.

This theorem is partly well known, and Kawada remarked the author that this follows immediately from Tate's theorem.

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