On the Operational Treatment of the Green Function Formalism

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Introduction

In 1951 Schwinger[1] has given the coupled integral equations for the nucleon and meson fields in interaction, in the form of functional differential equations in terms of an arbitrary external field. On the other hand, Edwards and Peierls[2] have recently developed it in the form that, using a functional Fourier transform, the Green function of a single nucleon moving in an external field *with radiative corrections* can be related to that of a nucleon moving in an arbitrary external field *without radiative corrections*. They have also shown that in the case of neutral pseudoscalar meson theory in which the recoil of the nucleon is neglected, the Green function for one nucleon can be solved in a closed form, and mass and Green function renormalizations are easily and completely done.

Green[3] has also shown, independently of and in another way from Edwards and Peierls, that the above transformation can be obtained by the use of the functional differential operator for the components of Fourier analysis of an external field. Recently, Anderson[4] and Kinoshita-Nambu[5] have described that some functional operators as shown below (1), can be applied significantly to connect Dyson's T-Product and Wick's N-Product.

The treatment of Edwards-Peierls which used the functional Fourier transform of the Green function, is too roundabout to see through the formalism. Green's treatment in which functional operators are used for the components of Fourier analysis of the external fields, has also the same defects. Considering these facts we have proposed, in this paper, another method by which the external field is treated directly by the same functional differential operators, as introduced above by Anderson and Kisoshita-Nambu. Our method will be simpler and by which better country views will be obtained than by the method of Edwards-Peierls and Green, while our method has retained many characters above mentioned.

Already, in his non-linear theory of elementary particles Heisen-

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berg[6] has utilized the transformation from the τ-function to the φ-function which are proposed and discussed by Freese[7], and tried to reduce the order of singularity in order to make them clear. It seems that his treatment is fundamentally identical with our treatment, but we shall postpone it to the future to apply the method of functional operators to the functional non-linear equation proposed by Heisenberg.

In this paper we describe first the definition of the Green function and the commutation relation of functional differential operators (1). Then, we discuss the methods of Edwards-Peierls and Green (2). Lastly, we obtain the explicit solutions of Greens functions for one and two particles (3), comparing their results with the Bethe-Salpeter equation.

1. Preliminary notes

(A) Definition of Green functions

A generalized Green function governing processes involving n nucleons and m external meson lines may be defined as follows

$$
\begin{aligned}
& G(x_1\cdots x_n,\ x_1'\cdots;\ \xi_1\cdots\xi_m) \\
& \quad = i^n i^{[m/2]} <(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(x_1')\cdots\overline{\psi}(x_n') \\
& \qquad \times\varphi(\xi_1)\cdots\varphi(\xi_1)\cdots\varphi(\xi_m))_+\varepsilon(x_1\cdots x_n')\,. \qquad (1)
\end{aligned}
$$

For any Heisenberg operator $F(x)$, the symbol $<F(x)>$ is defined such as

$$<F(x)>\frac{<\Omega_1|F(x)|\Omega_2>}{<\Omega_1|\Omega_2>}\ ,$$

where Ω_1 and Ω_2 are true vacuum states at two space like surfaces at $t=\pm\infty$ respectively: [m/2] is the integral part of m/2. $\varepsilon(x_1\cdots x_n')$ is $+1$ for an even permutation of the order of the times $x_{01}\cdots x_{0n}'$ and -1 for an odd permutation. The suffix $+$ means the chronological ordering. The spinor and isotropic indices have been supressed. Especially, for one nucleon and one meson system the Green function can be written

$$G(x_1x_1'\xi)=i<(\psi(x_1)\overline{\psi}(x_1')\varphi(\xi))_+>\varepsilon(x-x')\,. \qquad (2)$$

For two nucleons and no meson system the Green function G_{12} can be expressed by

$$G_{12}=-<(\psi(x_1)\psi(x_2)\overline{\psi}(x_1')\overline{\psi}(x_2'))_+>\varepsilon\ , \qquad (3)$$

where

$$\varepsilon=\varepsilon(x_1-x_2)\varepsilon(x_1-x_1')\varepsilon(x_1-x_2')\varepsilon(x_1'-x_2)\varepsilon(x_1'-x_2')\varepsilon(x_2-x_2'). \qquad (4)$$

When an external meson field $\varphi(x)$ exists, the Green function is described as follows including the expression of its influences.

$$\mathrm{G}(x_1\cdots x_n x_1'\cdots x_n';\ \varphi)\,. \qquad (5)$$

(B) Operator calculus of functional differential operator

We shall introduce the following functional differential operator with respect to a function $\varphi(x)$, as introduced by Anderson and Kinoshita-Nambu

$$R_\varphi = e^{-\frac{i}{2}\int \Delta(\xi,\, \xi';\, \varphi)\frac{\delta}{\delta\varphi(\xi)}\frac{\delta}{\delta\varphi(\xi')}d\xi d\xi'} . \tag{6}$$

Then we obtain a commutation relation between the operator R_φ and the function $\varphi(x)$

$$[R_\varphi,\ \varphi] = -i\int \Delta(\xi,\, \xi';\, \varphi)\frac{\delta}{\delta\varphi(\xi)}R_\varphi d\xi\ . \tag{7}$$

From Eq. (7) we obtain

$$R_\varphi \varphi = \left\{\varphi - i\int \Delta(\xi,\ \xi';\ \varphi)\frac{\delta}{\delta\varphi(\xi)}d\xi\right\}R_\varphi . \tag{7}$$

We can define the following functional operator with respect to another function $\psi(x)$

$$R_\psi = e^{-i\int S_F'(\xi,\, \xi';\, \psi)\frac{\delta}{\delta\psi(\xi)}\frac{\delta}{\delta\bar{\psi}(\xi')}d\xi d\xi'} . \tag{9}$$

The commutation relation between the operator R_ψ and the function $\psi(x)$ is

$$[R_\psi,\ \psi] = -i\int S_F'(\xi,\ \xi;\ \psi)\frac{\delta}{\delta\bar{\psi}(\xi')}R_\psi d\xi'\ . \tag{10}$$

2. Remarks about Methods of Edwards-Peierls and Green

As delivered in the introduction Edwards-Peierls has thrown light for obtaining the explicit solution of one nucleon's Green function perturbed by meson fields. On the other hand Green had discussed that solution through the Fourier transform of meson field by applying Feynman's conception about two fields' interaction. For the fact that Edwards-Peierls had successfully obtained the above solution it seems to be the following two reasons,

(a) The introduction of Schwinger's external field

By the introduction of an external field to the nucleon by Schwinger a relation is obtained between a certain Green function for several nucleons, meson field and another Green function of the field for the same nucleon, but with one less mesons. As its example we shall take up one nucleon system, for which functional derivative by external current is expressed as follows

$$\frac{1}{i}\frac{\delta}{\delta J}\langle(\psi\bar{\psi})_+\rangle = \langle(\psi\bar{\psi}\varphi)_+\rangle_\varepsilon - \langle\varphi\rangle\langle(\psi\bar{\psi})_+\rangle_\varepsilon\ .$$

Considering the following relation

$$\frac{\delta}{\delta J}=\frac{\delta}{\delta\varphi}\frac{\delta\varphi}{\delta J}=\Delta\frac{\delta}{\delta\varphi}$$

we obtain

$$<(\psi\overline{\psi}\varphi)_+>\varepsilon=<\varphi><(\psi\overline{\psi})_+>\varepsilon-i\int d\xi\,\Delta\frac{\delta}{\delta\varphi}(\psi\overline{\psi})_+>\varepsilon$$

$$=\left\{<\varphi>-i\int d\xi\,\Delta\frac{\delta}{\delta\varphi}\right\}<(\psi\overline{\psi})_+>\varepsilon\,. \qquad (1)$$

(b) The introduction of an operator R_φ by Green, Anderson and Kinoshita-Nambu

Because of not adopting this functional derivative operator R_φ the calculation of Edwards-Peierls is too roundabout, though they used functional Fourier transform. Green made use of the same operator with ours R_φ for the Fourier transform of an external field. In our paper we shall adopt the operator R_φ used by Anderson and Kinoshita-Nambu and apply it to the same calculation with Edwards-Peierls. For adopting this operator we shall specially consider the following properties obtained from the commutation relation of these operators.

$$R_\varphi<\varphi>R_\varphi^{-1}=<\varphi>-i\int\Delta(\xi,\ \xi';\ \varphi)\frac{\delta}{\delta\varphi}d\xi\,. \qquad (2)$$

From (1) and (2) we have

$$G(x,\ x',\ y;\ \varphi)=R_\varphi<\varphi>R_\varphi^{-1}G(x,\ x';\ \varphi)\,,$$

or

$$<(\psi\overline{\psi}\varphi>_+\varepsilon=R_\varphi<\varphi>R_\varphi^{-1}<(\psi\overline{\psi})_+>\varepsilon\,.$$

3. Explicit Solutions of Green Functions for one and two Particles

(A) Green function for one particle and its explicit solution

For a nucleon moving in an external neutral pseudoscalar field $\varphi(x)$ the functional equation to be satisfied by Schwinger's Green function is given as follows, neglecting the reaction of the nucleon field to the meson field.

$$\left\{-p+m-g\gamma_5\varphi+ig\gamma_5\int d\xi\Delta(x,\ \xi;\ \varphi)\frac{\delta}{\delta\varphi(\xi)}\right\}G(x,\ x';\ \varphi)=\delta(x-x')\,, \qquad (1)$$

where

$$p=i\gamma_\mu\frac{\partial}{\partial x_\mu}\,.$$

As shown below we can simplify the treatment of Edwards-Peierls by applying the operational calculation prepared in the preceding secttion to a function G_1, which is defined by the following transformation

$$G=R_\varphi G_1 R_\varphi{}^{-1} \tag{2}$$

Then we have

$$R_\varphi \varphi G_1 R_\varphi{}^{-1}=\left\{\varphi-i\int\Delta(\xi,\ \xi';\ \varphi)\frac{\delta}{\delta\varphi(\xi)}d\xi\right\}R_\varphi G_1 R_\varphi{}^{-1}. \tag{3}$$

Further considering that the p and m are commutable with the operator R_φ we obtain

$$\left\{-p+m-g\gamma_5\varphi+ig\gamma_5\int d\xi\Delta(x,\ \xi;\ \varphi)\frac{\delta}{\delta\varphi(\xi)}\right\}G=R_\varphi\{-p+m-g\gamma_5\varphi\}G_1R_\varphi{}^{-1}. \tag{4}$$

We have from Eq. (1)

$$\{-p+m-g\gamma_5\varphi\}G_1=R_\varphi{}^{-1}\delta R_\varphi=\delta\ , \tag{5}$$

which is nothing but the Schrödinger differential equation with respect to G_1 in an external field $\varphi(x)$.

The solution of Eq. (5) is expressed usually by the following integral equations

$$G_1(x,\ x';\ \varphi)=S(x,\ x')-\int S(x,\ \xi)\varphi(\xi)G_1(\xi,\ x';\ \varphi)d\xi\ , \tag{6}$$

where

$$\{-p+m\}S(x,\ x')=\delta(x-x')\,.$$

But in a special case of an external field $\varphi(x)$ the explicit solution of Eq. (5) can be expressed formally as follows.

$$G_1(x,\ x';\ \varphi)=S(x,\ x')e^{-ig\gamma_5\int\varphi(\xi)d\xi}. \tag{7}$$

Then the explicit solution for G can be expressed as follows

$$\begin{aligned}G(x,\ x';\ \varphi)&=R_\varphi G_1(x,\ x';\ \varphi)R_\varphi{}^{-1}\\ &=S(x,\ x')R_\varphi e^{-ig\gamma_5\int\varphi(\xi)d\xi}\\ &=S(x,\ x')e^{-\frac{i}{2}\iint\Delta(\xi,\ \xi';\ \varphi)\frac{\delta}{\delta\varphi(\xi)}\frac{\delta}{\delta\varphi(\xi')}d\xi d\xi'}e^{-ig\gamma_5\int\varphi(\xi)d\xi}\\ &=S(x,\ x')e^{\frac{ig^2}{2}\iint\gamma_5\Delta(\xi,\ \xi';\ \varphi)\gamma_5 d\xi d\xi'}e^{-ig\gamma_5\int\varphi(\xi)d\xi}.\end{aligned} \tag{8}$$

Especially when $\varphi=0$ we obtain

$$G_{\varphi=0}=S(x,x')e^{\frac{ig^2}{2}\iint\gamma_5\Delta(\xi',\ \xi'')\gamma_5 d\xi' d\xi''}. \tag{9}$$

Edwards-Peierls did not express the general solution such as their Eq. (21) in the form of our result. But our result coincides exactly with their Eq. (35) which is obtained for the case when $\varphi(x)$, G and G_1 contain only time variable, as the solution can be expressed in the form of Eq. (6) for this case. If we express Eq. (9) in terms of the integral equation, we have

$$G(x,\ x')=S(x,\ x')\Big\{1-\frac{ig^2}{2}\iint S^{-1}(x,\ \xi)\gamma_5\Delta(\xi,\ \xi')\gamma_5 S(\xi',\ x')e^{\frac{ig^2}{2}\iint\gamma_5\Delta(\xi'',\xi''')\gamma_5 d\xi'' d\xi'''}$$
$$\times d\xi d\xi'\Big\}=S(x,\ x')\Big\{1-\iint S^{-1}(x\xi)\gamma_5\Delta(\xi,\ \xi')\gamma_5 G(\xi',\ x')d\xi d\xi'\Big\}\ . \tag{10}$$

(B) Green function for two particles and its explicit solution

As an example of our treatment for many nucleon system we shall take up the case of two nucleon system. The functional equation to be satisfied by the Green function for two particles such as Eq. (1.3) are, as shown by Schwinger, expressed as follows

$$\Big\{-p_1+m_1-g\gamma_5^1\varphi+ig\gamma_5^1\int d\xi'\Delta(x_1,\ \xi';\ \varphi)\frac{\delta}{\delta\varphi(\xi')}\Big\}$$
$$\times\Big\{-p_2+m_2-g\gamma_5^2\varphi+ig\gamma_5^2\int d\xi''\Delta(x_2,\ \xi'';\ \varphi)\frac{\delta}{\delta\varphi(\xi'')}\Big\}G_{12}=\mathbf{1}_{12}\,, \tag{11}$$

where suffixes 1 and 2 stand for two particles respectively, and G_{12} and $\mathbf{1}_{12}$ are defined as

$$G_{12}=G(x_1,\ x_2;\ x_1',\ x_2\ )\ ,$$
$$\mathbf{1}_{12}=(x_1,\ x_2|1|x_1'x_2')=\delta(x_1-x_1')\delta(x_2-x_2')-\delta(x_1-x_2')\delta(x_2-x_1'). \tag{12}$$

Applying the operator R_φ^{-1} to the G_{12} we transform from G_{12} to K_{12}

$$R_\varphi^{-1}G_{12}R_\varphi=K_{12}\quad\text{or}\quad G_{12}=R_\varphi K_{12}R_\varphi^{-1}\ . \tag{13}$$

Then, considering that the relation

$$R_\varphi\varphi K_{12}=\Big\{\varphi-i\int\Delta(\xi,\ \xi';\ \varphi)\frac{\delta}{\delta\varphi(\xi)}d\xi\Big\}R_\varphi K_{12} \tag{14}$$

can be applied both particles 1 and 2, we obtain

$$\{-p_1+m_1-g\gamma_5^1\varphi\}\{-p_2+m_2-g\gamma^2_5\varphi\}K_{12}=\mathbf{1}_{12}\ . \tag{15}$$

Operational solution of the above equation is

$$K_{12}=K(x_1x_2x_1'x_2';\ \varphi)=S(x_1x_1')S(x_2x_2')e^{-ig\gamma_5^1\int\varphi(\xi)d\xi-ig\gamma_5^2\int\varphi(\xi')d\xi'}\ . \tag{16}$$

Transforming from K_{12} to G_{12} we obtain

$$G_{12}=G(x_1x_2,\ x_1'x_2';\ \varphi)$$
$$=S(x_1,\ x_1')S(x_2,\ x_2')R_\varphi e^{-ig\gamma_5^1\int\varphi(\xi)d\xi-ig\gamma_5^2\int\varphi(\xi)\,d\xi}R_\varphi^{-1}$$
$$=S(x_1,\ x_1')S(x_2,\ x_2')e^{\frac{ig^2}{2}\iint\gamma_5^1\Delta(\xi,\ \xi';\ \varphi)\gamma_5^1 d\xi d\xi'+\frac{ig^2}{2}\iint\gamma_5^2\Delta(\xi,\ \xi';\ \varphi)\gamma_5^2 d\xi d\xi'}$$

$$\times e^{\frac{ig^2}{2}\iint(\gamma_5{}^1\Delta\gamma_5{}^2+\gamma_5{}^2\Delta\gamma_5{}^1)d\xi d\xi}, e^{-ig\gamma_5{}^1\int\varphi(\xi)d\xi-ig\gamma_5{}^2\int\varphi(\xi)d\xi}$$

$$=S_F'(x_1,x_1')S_F'(x_2,x_2')e^{\frac{ig^2}{2}\iint(\gamma_5{}^1\Delta\gamma_5{}^2+\gamma_5{}^2\Delta\gamma_5{}^1)d\xi d\xi'-ig\gamma_5{}^1\int\varphi(\xi)d\xi-ig\gamma_5{}^2\int\varphi(\xi)d\xi}, \tag{17}$$

where

$$S_F'(x,\ x_1')=S(x_1,\ x_1')e^{\frac{ig^2}{2}\iint\gamma_5\Delta(\xi,\ \xi';\ \varphi)d\xi d\xi'} \tag{18}$$

When $\varphi=0$ we obtain

$$G_{12}|_{\varphi=0}=S_F'(x_1,x_1')S_F'(x_2,x_2')e^{\frac{ig^2}{2}\iint(\gamma_5{}^1\Delta(\xi,\ \xi')\gamma_5{}^2+\gamma_5{}^2\Delta(\xi,\ \xi')\gamma_5{}^1)d\xi d\xi'}. \tag{19}$$

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