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On a General Divisor Problem in Algebraic Number Fields [1)]

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Introduction. Let $U(n)$ be the number of integral solutions of $u^2+v^2=n$, where n is a given positive integer.[2)] Hardy [1] (see the list of references at the end) proved that, for $x>0$,

$$\sum_{n\leqq x}{}' U(n)=\pi x-1+\sqrt{x}\sum_{n=1}^{\infty}\frac{U(n)}{\sqrt{n}}J_1(2\pi\sqrt{nx}) .^{3)}$$

Subsequently Hardy-Landau [2] discovered the identity

$$\sqrt{x}\sum_{n\leqq y}\frac{U(n)}{\sqrt{n}}J_1(2\pi\sqrt{nx})=\sum_{n\leqq x}{}' U(n)-\pi x+1-J_0(2\pi\sqrt{yx})$$
$$+\sqrt{\frac{x}{y}}P(y)J_1(2\pi\sqrt{yx})+\pi\frac{x}{y}Q(y)J_2(2\pi\sqrt{yx})$$
$$-x\sum_{n=1}^{\infty}\frac{U(n)}{n}\int_{2\pi\sqrt{yx}}^{\infty}J_2\left(z\sqrt{\frac{n}{x}}\right)J_3(z)dz ,^{3)}$$

where $x>0$, $y>0$ and

$$P(y)=\sum_{n\leqq y}U(n)-\pi y+1 , \qquad Q(y)=\int_0^y P(z)dz ;$$

and they deduced Hardy's identity from this, letting $y\to\infty$.

In the present paper we shall investigate the extension of these two formulae to a general divisor problem generated by a finite product $Z(s)$ of Dedekind zeta-functions of algebraic number fields.[4)] (The above circle-problem of Gauss itself belongs properly to the Gaussian number field, as is well known.)

We preassume on the part of the reader the knowledge of Landau's book [9] (chiefly of Chapter 5: *Die Anzahl der Ideale mit Norm* $\leqq x$).

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1) This research was made by means of the *Grant in Aid for Miscellaneous Scientific Research* (1952).

2) The letters m, n denote positive integers, and k, l non-negative integers, throughout this paper.

3) The dash attached to $\sum$ means that, if x is an integer, the last term $U(x)$ is to be halved; and J_ν represents the Bessel function usually so denoted, ν being a positive integer.

4) This divisor problem was first considered by Hasse-Suetuna [4]. It should be noted, however, that the generating function for their *Allgemeines Teilerproblem* was not $Z(s)$ itself, but $Z(s)$ multiplied by a certain factor; they considered $Z(s)$ as auxiliary means.

§ 1. Preliminaries

Let k_j be for each $j=1, 2, \ldots, \tau$ (τ a positive integer) an algebraic number field of (absolute) degree n_j, where we assume once for all that[5)]

$$N=n_1+n_2+\cdots+n_\tau>2\ .$$

Denoting by $\zeta_j(s)$ the Dedekind zeta-function of k_j we find, by Hecke's functional equation (Satz 156 of Landau [9]),

$$\begin{aligned} g_j(s)&=\zeta_j(s)/\zeta_j(1-s)\\ &=\left(\frac{(2\pi)^s}{\pi}\right)^{n_j} d_j^{\frac{1}{2}-s}\left(\sin\frac{\pi s}{2}\right)^{r_1(j)+r_2(j)}\left(\cos\frac{\pi s}{2}\right)^{r_2(j)}\Gamma^{n_j}(1-s)\ , \end{aligned}$$

where d_j denotes the absolute value of the discriminant of k_j, and $r_1^{(j)}$ and $2r_2^{(j)}$ denote the numbers of the real and imaginary conjugate fields of k_j respectively (so that $n_j=r_1^{(j)}+2r_2^{(j)}$). By Satz 166 of Landau [9] the asymptotic behaviour of $g_j(s)$ is expressed for large t by

$$g_j(s)=A_j t^{n_j(\frac{1}{2}-\sigma)} e^{-ti(n_j\log t-\Xi_j)}\left(1+O\left(\frac{1}{t}\right)\right)\ ,^{6)}$$

uniformly in the strip $\sigma_1\leqq\sigma\leqq\sigma_2$ ($\sigma_1<\sigma_2$), where $\sigma=\Re s$ and $t=\Im s$. The number Ξ_j is real and depends on k_j alone; and $A_j\neq 0$ is complex and depends on k_j and σ only, but bounded for $\sigma_1\leqq\sigma\leqq\sigma_2$, so that we have uniformly in this strip

$$g_j(s)=O(t^{n_j(\frac{1}{2}-\sigma)})\ .$$

Hence, writing

$$\begin{gathered} Z(s)=\zeta_1(s)\cdots\zeta_\tau(s)\ ,\\ d=d_1\cdots d_\tau\ ,\\ r=(r_1^{(1)}+r_2^{(1)})+\cdots+(r_1^{(\tau)}+r_2^{(\tau)})-1\ , \end{gathered}$$

we find that

$$Z(s)=g(s)Z(1-s)\ ,$$

where

$$g(s)=\left(\frac{(2\pi)^s}{\pi}\right)^N d^{\frac{1}{2}-s}\left(\sin\frac{\pi s}{2}\right)^{r+1}\left(\cos\frac{\pi s}{2}\right)^{N-r-1}\Gamma^N(1-s)\ , \tag{1}$$

and this is, uniformly for $\sigma_1\leqq\sigma\leqq\sigma_2$,

$$=At^{N(\frac{1}{2}-\sigma)}e^{-ti(N\log t-\Xi)}\left(1+O\left(\frac{1}{t}\right)\right) \tag{2}$$

$$=O\left(t^{N(\frac{1}{2}-\sigma)}\right)\ , \tag{3}$$

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5) As for the case $N\leqq 2$ of our problem, see Walfisz [10].

6) $O(W)$ means a quantity which is absolutely less than a constant multiple of $W(>0)$ under the stated circumstance (namely for large t in the above case).

where $A=A_1\cdots A_\tau \neq 0$ and $\Xi=\Xi_1+\cdots+\Xi_\tau$. Also, by Satz 171 of Landau [9],

$$Z(s)=O\left(t^{N(\frac{1}{2}-c)}\right) \tag{4}$$

uniformly for $\sigma \geqq c$, when $c<0$.

It should be marked that each of $Z(s)$ and $g(s)$ takes conjugate values for conjugate values of s.

Lemma 1. *Writing*

$$Z(s)=\sum_{n=1}^{\infty} F(n)n^{-s} \quad (\sigma>1)\ ,$$

we have, when $x>2$,

$$H(x)=\sum_{n\leqq x} F(n)=O(x\log^{\tau-1}x)\ .$$

Remark. In the following proof we exceptionally suppose that N may be any positive integer, for the sake of convenience.

Proof (induction on τ). 1) If $\tau=1$, the result coincides with Satz 202 of Landau [9].

2) Suppose that the lemma holds for τ and consider, in place of $Z(s)$,

$$Z_0(s)=\zeta_0(s)Z(s)=\sum_{n=1}^{\infty} F_0(n)n^{-s} \quad (\sigma>1),$$

where $\zeta_0(s)$ is the Dedekind zeta-function of an algebraic number field. Writing

$$\zeta_0(s)=\sum_{n=1}^{\infty} f(n)n^{-s} \quad (\sigma>1)\ ,$$

we find (a, b denote positive integers)

$$Z_0(s)=\sum_{n=1}^{\infty}\frac{F(n)}{n^s}\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{ab=n}F(a)f(b)\ ,$$

so that

$$F_0(n)=\sum_{ab=n}F(a)f(b)\ .$$

We therefore have for $x>2$, by the case 1),

$$\sum_{n\leqq x}F_0(n)=\sum_{ab\leqq x}F(a)f(b)=\sum_{a\leqq x}F(a)\sum_{b\leqq \frac{x}{a}}f(b)$$

$$=O\left(\sum_{n\leqq x}\frac{x}{n}F(n)\right).$$

But, by hypothesis ($H(0)=0$),

$$\sum_{n\leqq x}\frac{F(n)}{n}=\sum_{n\leqq x}\frac{H(n)-H(n-1)}{n}=\sum_{n\leqq x}H(n)\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{H(x)}{[x]+1}$$

$$=O\left(\sum_{n\leqq x}\frac{\log^{\tau-1}x}{n}\right)+O(\log^{\tau-1}x)=O(\log^{\tau}x)\ , \tag{5}$$

as $x\to\infty$. This proves the lemma.

Lemma 2. *We have, for given $\vartheta>0$ and all $x>2$,*

$$\sum_{n>x} F(n)n^{-\vartheta}=O(x^{1-\vartheta}\log^{\tau-1}x) \qquad (\vartheta>1)\ , \tag{6}$$

$$\sum_{n\leqq x} F(n)n^{-\vartheta}=O(x^{1-\vartheta}\log^{\tau-1}x) \qquad (0<\vartheta<1), \tag{7}$$

$$\sum_{n\leqq x} F(n)n^{-\vartheta}=O(\log^{\tau}x) \qquad (\vartheta=1)\ . \tag{8}$$

Proof. (8) follows from (5) ; and (6), (7) follow from Lemma 1 in almost the same way as for Satz 203 of Landau [9].

Lemma 3. *Let $w>0$ and consider the following $N+3$ integrals :*

$$\frac{1}{2\pi i}\int_{(\frac{7}{4})} g(s)\frac{w^{s+N}}{s(s+1)\cdots(s+N)}ds=L_0(w)=L(w)\ ,$$

$$\frac{1}{2\pi i}\int_{(\frac{7}{4})} g(s)\frac{w^{s+N-1}}{s(s+1)\cdots(s+N-1)}ds=L_1(w)\ ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{2\pi i}\int_{(\frac{7}{4})} g(s)\frac{w^{s}}{s}ds=L_N(w)\ ,$$

$$\frac{1}{2\pi i}\int_{(\frac{7}{4})} g(s)w^{s-1}ds=L_{N+1}(w)\ ,$$

$$\frac{1}{2\pi i}\int_{(\frac{7}{4})} g(s)(s-1)w^{s-2}ds=L_{N+2}(w)\ ,$$

where each integral is obtained from the preceding one by formal differentiation with respect to w.

Then these integrals converge absolutely and uniformly for $w_1\leqq w\leqq w_2$ $(0<w_1<w_2)$, so that the $L_k(w)$ $(0\leqq k\leqq N+2)$ are continuous in w and

$$L_k(w)=L^{(k)}(w)\ .$$

Proof is similar to that of Satz 205, Landau [9].

Lemma 4. *We have*

$$L(w)=R(w)+K(w)\ ,$$

where

$$K(w)=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})} g(s)\frac{w^{s+N}}{s\cdots(s+N)}ds \tag{9}$$

converges absolutely and

$$R(w)=w^{N+1}(A_0\log^{r}w+\cdots+A_r)$$

denotes the residue of the integrand at the pole $s=1$ of order $r+1$, the A's being certain constants.[7]

[7] Similarly for the B's, C's, D's, E's in the sequel.

Proof is the same as for Satz 206, Landau [9].

Lemma 5. *We have, for* $0 \leqq k \leqq N+2$ *and all* $w>1$,

$$K^{(k)}(w)=O\left(w^{(N+\frac{1}{2}-k)(1-\frac{1}{N})}\right).$$

Proof is similar to that of Satz 207, Landau [9].

Lemma 6. *Suppose that* $w>1$, $0<v<w$ *and denote by* $\Delta_v K(w)$ *the difference of* $K(w)$ *for the increase of* w *by* v:

$$\Delta_v K(w)=K(w+v)-K(w) ;$$

so that we have, repeating this operation k *times* $(k>0)$,

$$\Delta_v^k K(w)=\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}K(w+lv) .$$

We put further $\Delta_v^0 K(w)=K(w)$.

Then, for $0 \leqq k \leqq N$,

$$\Delta_v^k K(w)=\begin{cases} O\left(v^k w^{(N+\frac{1}{2}-k)(1-\frac{1}{N})}\right) , \\ O\left(w^{N-\frac{1}{2}-\frac{1}{2N}}\right) . \end{cases}$$

Proof is the same as for Satz 208, Landau [9].

Lemma 7. *Write for* $y>0$

$$S(y)=\operatorname*{Res}_{s=1}\left(Z(s)\frac{y^{s+N}}{s\cdots(s+N)}\right)$$

$$=y^{N+1}(B_0\log^{\tau-1}y+\cdots+B_{\tau-1}) ,$$

$$H_0(y)=H(y)=\sum_{n\leq y}F(n) \quad (\textit{see Lemma } 1),$$

$$H_1(y)=\int_0^y H(y_1)dy_1=\sum_{n\leqq y}F(n)(y-n) ,$$

$$H_2(y)=\int_0^y dy_1\int_0^{y_1}H(y_2)dy_2=\sum_{n\leqq y}F(n)\frac{(y-n)^2}{2!} ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$H_N(y)=\int_0^y dy_1\int_0^{y_1}dy_2\cdots\int_0^{y_{N-1}}H(y_N)dy_N=\sum_{n\leqq y}F(n)\frac{(y-n)^N}{N!} ,$$

$$Q_k(y)=H_k(y)-S^{(N-k)}(y)-\frac{y^k}{k!}Z(0) \quad (0\leqq k\leqq N).$$

Then

(10) $$H_N(y)=\frac{1}{2\pi i}\int_{(\gamma)}Z(s)\frac{y^{s+N}}{s\cdots(s+N)}ds \quad (\gamma>1) ,$$

(11) $$Q_N(y)=\sum_{n=1}^{\infty}\frac{F(n)}{n^{N+1}}K(ny) .$$

Proof is the same as for Satz 210, Landau [9].

Remark. We easily find that $(0 \leqq k \leqq N)$

$$S^{(N-k)}(y) = \operatorname*{Res}_{s=1}\left(Z(s)\frac{y^{s+k}}{s\cdots(s+k)}\right) \tag{12}$$

$$= y^{k+1}(C_0^{(k)}\log^{\tau-1}y + \cdots + C_{\tau-1}^{(k)})\ , \tag{13}$$

$$S^{(N+1)}(y) = \operatorname*{Res}_{s=1}(Z(s)y^{s-1})$$

$$= D_0\log^{\tau-1}y + \cdots + D_{\tau-1}\ . \tag{14}$$

Also we get immediately, in virtue of (13),

$$\lim_{y\to 0} S^{(l)}(y) = 0 \quad (0 \leqq l \leqq N)\ , \tag{15}$$

and hence

$$Q_k(y) = \int_0^y dy_1 \int_0^{y_1} dy_2 \cdots \int_0^{y_{k-1}} Q_0(y_k)dy_k \quad (1 \leqq k \leqq N). \tag{16}$$

§ 2. Extension of Hardy-Landau's identity

Lemma 8. *We have, for* $y > 2$,

$$Q_l(y) = O\left(y^{(l+\frac{1}{2})(1-\frac{1}{N})}\right) \quad \left(\frac{N-1}{2} < l \leqq N\right)\ , \tag{17}$$

$$Q_l(y) = O\left(y^{l+1-\frac{2}{N+1-2l}}\log^\tau y\right) \quad \left(0 \leqq l \leqq \frac{N-1}{2}\right)\ . \tag{18}$$

Proof (cf. the proof of Satz 210, Landau [9]).

1) We differentiate (11) $N-l$ times formally with respect to y and obtain, in virtue of (16),

$$Q_l(y) = \sum_{n=1}^{\infty} \frac{F(n)}{n^{l+1}} K^{(N-l)}(ny)\ . \tag{19}$$

The formal process is justified as follows. We have by Lemma 5, for $y > 1$ and all n,

$$K^{(N-l)}(ny) = O\left((ny)^{(l+\frac{1}{2})(1-\frac{1}{N})}\right)\ ,$$

and so

$$\frac{F(n)}{n^{l+1}} K^{(N-l)}(ny) = O\left(y^{(l+\frac{1}{2})(1-\frac{1}{N})} \frac{F(n)}{n^{\frac{1}{2}+\frac{1}{2N}+\frac{l}{N}}}\right)\ . \tag{20}$$

But

$$\sum_{n=1}^{\infty} \frac{F(n)}{n^{\frac{1}{2}+\frac{1}{2N}+\frac{l}{N}}} = Z\left(\frac{1}{2}+\frac{1}{2N}+\frac{l}{N}\right)$$

is convergent, since

$$\frac{1}{2}+\frac{1}{2N}+\frac{l}{N} > \frac{1}{2}+\frac{1}{2N}+\frac{N-1}{2N} = 1\ .$$

Hence the series on the right of (19) converges absolutely and uniformly for $y_1 \leqq y \leqq y_2$ $(1 < y_1 < y_2)$, so that (19) is true.

The result now follows from (19) and (20).

2) (18) is equivalent to

$$Q_{N-k}(y) = O(y^{N-k+\theta}\log^{\tau} y) \quad \left(\frac{N+1}{2} \leqq k \leqq N\right), \tag{21}$$

where $$\theta = 1 - \frac{2}{2k-N+1} \quad (\text{so that } 0 \leqq \theta < 1).$$

We write, for $y > 2$,

$$w = ny, \quad v = n\xi, \quad \xi = y^{\theta}, \quad \eta = y^{\psi},$$

where $$\psi = \frac{3N-2k-1}{2k-N+1} \quad (>0),$$

and obtain, by (11) and Lemma 6,

$$\begin{aligned} \Delta_{\xi}^{k} Q_N(y) &= \sum_{n=1}^{\infty} \frac{F(n)}{n^{N+1}} \Delta_v^k K(w) \\ &= O \sum_{n=1}^{\infty} \frac{F(n)}{n^{N+1}} \mathrm{Min}\left((n\xi)^k (ny)^{(N+\frac{1}{2}-k)(1-\frac{1}{N})}, \quad (ny)^{N-\frac{1}{2}-\frac{1}{2N}}\right) \\ &= O\left(y^{(N+\frac{1}{2}-k)(1-\frac{1}{N})+k\theta} \sum_{n \leqq \eta} \frac{F(n)}{n^{\frac{3}{2}+\frac{1}{2N}-\frac{k}{N}}}\right) \\ &\quad + O\left(y^{N-\frac{1}{2}-\frac{1}{2N}} \sum_{n > \eta} \frac{F(n)}{n^{\frac{3}{2}+\frac{1}{2N}}}\right). \end{aligned}$$

Marking that

$$\frac{3}{2} + \frac{1}{2N} - \frac{k}{N} = 1 - \frac{1}{N}\left(k - \frac{N+1}{2}\right) \leqq 1, \tag{22}$$

we apply Lemma 2 and get

$$\begin{aligned} \Delta_{\xi}^{k} Q_N(y) &= O(y^{(N+\frac{1}{2}-k)(1-\frac{1}{N})+k\theta} \eta^{\frac{k}{N}-\frac{1}{2}-\frac{1}{2N}} \log^{\tau} \eta) \\ &\quad + O(y^{N-\frac{1}{2}-\frac{1}{2N}} \eta^{-\frac{1}{2}-\frac{1}{2N}} \log^{\tau-1} \eta) \\ &= O(y^{N-\frac{1}{2}-\frac{1}{2N}-k(1-\frac{1}{N}-\theta)+\psi(\frac{k}{N}-\frac{1}{2}-\frac{1}{2N})} \log^{\tau} y) \\ &\quad + O(y^{N-\frac{1}{2}-\frac{1}{2N}-\psi(\frac{1}{2}+\frac{1}{2N})} \log^{\tau-1} y). \end{aligned}$$

But $$\frac{\psi+1}{N} = \frac{2}{2k-N+1} = 1-\theta, \quad \text{hence} \quad 1 - \frac{1}{N} - \theta = \frac{\psi}{N},$$

and so the exponents of y in the last two O-terms coincide; therefore

$$\begin{aligned} \Delta_{\xi}^{k} Q_N(y) &= O(y^{N-(\psi+1)(\frac{1}{2}+\frac{1}{2N})} \log^{\tau} y) \\ &= O(y^{N-(1-\theta)(\frac{N}{2}+\frac{1}{2})} \log^{\tau} y). \end{aligned} \tag{23}$$

On the other hand

$$\Delta_\xi^k Q_N(y)=\Delta_\xi^k H_N(y)-\Delta_\xi^k S(y)-\Delta_\xi^k \frac{y^N}{N!}Z(0) . \tag{24}$$

Now, by (15),

$$S(y)=\int_0^y dy_1\int_0^{y_1} dy_2\cdots\int_0^{y_{k-1}} S^{(k)}(y_k)dy_k ,$$

hence

$$\Delta_\xi^k S(y)=\Delta_\xi^k\int_0^y dy_1\int_0^{y_1} dy_2\cdots\int_0^{y_{k-1}} S^{(k)}(y_k)dy_k$$
$$=\int_y^{y+\xi} dy_1\int_{y_1}^{y_1+\xi} dy_2\cdots\int_{y_{k-1}}^{y_{k-1}+\xi} S^{(k)}(y_k)dy_k .$$

Here we find for $y\leqq u\leqq y+k\xi$, by (13) and (14),

$$S^{(k)}(u)-S^{(k)}(y)=\int_y^u S^{(k+1)}(t)dt=O(\xi y^{N-k}\log^{\tau-1}y) . \tag{25}$$

Hence $\quad \Delta_\xi^k S(y)=\xi^k S^{(k)}(y)+O(\xi^{k+1}y^{N-k}\log^{\tau-1}y)$.

Further $\quad \Delta_\xi^k y^N=N(N-1)\cdots(N-k+1)y^{N-k}\xi^k+\cdots$.

It thus follows from (24) and (23) that

$$\Delta_\xi^k H_N(y)=\xi^k S^{(k)}(y)+O(y^{N+1-(k+1)(1-\theta)}\log^{\tau-1}y)$$
$$+O(y^{N-(1-\theta)(\frac{N}{2}+\frac{1}{2})}\log^\tau y) .$$

Here both exponents of y coincide on account of $1-\theta=\dfrac{2}{2k-N+1}$, and we finally obtain

$$\Delta_\xi^k H_N(y)=\xi^k S^{(k)}(y)+O(\xi^{k+1}y^{N-k}\log^\tau y) . \tag{26}$$

Now (see Lemma 7)

$$\Delta_\xi^k H_N(y)=\Delta_\xi^k\int_0^y dy_1\int_0^{y_1} dy_2\cdots\int_0^{y_{k-1}} H_{N-k}(y_k)dy_k$$
$$=\int_y^{y+\xi} dy_1\int_{y_1}^{y_1+\xi} dy_2\cdots\int_{y_{k-1}}^{y_{k-1}+\xi} H_{N-k}(y_k)dy_k ,$$

where $H_{N-k}(u)$ is clearly a non-decreasing function of u. Hence

$$\xi^k H_{N-k}(y)\leqq\Delta_\xi^k H_N(y)\leqq\xi^k H_{N-k}(y+k\xi) .$$

Comparing this with (26) we deduce that

$$H_{N-k}(y)\leqq S^{(k)}(y)+O(y^{N-k+\theta}\log^\tau y) , \tag{27}$$
$$H_{N-k}(y+k\xi)\geqq S^{(k)}(y)+O(y^{N-k+\theta}\log^\tau y) .$$

Here we write $z=y+k\xi$. Then

$$y=z-ky^\theta=z+O(z^\theta) ,$$
$$H_{N-k}(z)\geqq S^{(k)}(y)+O(z^{N-k+\theta}\log^\tau z) .$$

Hence, by (25), $\quad H_{N-k}(z)\geqq S^{(k)}(z)+O(z^{N-k+\theta}\log^\tau z)$.

On combining this with (27), it follows directly that

$$H_{N-k}(y)=S^{(k)}(y)+O(y^{N-k+\theta}\log^{\tau}y)\,,$$

whence we ultimately get

$$Q_{N-k}(y)=H_{N-k}(y)-S^{(k)}(y)-\frac{y^{N-k}}{(N-k)!}Z(0)=O(y^{N-k+\theta}\log^{\tau}y)\,,$$

which proves (21).

Remark. If we take $k=N$ in 2), we have in place of (22)

$$\frac{3}{2}+\frac{1}{2N}-\frac{k}{N}=\frac{1}{2}+\frac{1}{2N}<1\,,$$

so that the case (8) of Lemma 2 does not occur, and we find that, instead of (23),

$$\Delta_{\xi}^{k}Q_{N}(y)=O(y^{N-(1-\theta)(\frac{N}{2}+\frac{1}{2})}\log^{\tau-1}y)\,.$$

Hence our final result becomes

$$Q_0(y)=O\left(y^{\frac{N-1}{N+1}}\log^{\tau-1}y\right)\,,$$

which coincides with the result of Hasse-Suetuna [4].

Lemma 9. *We have, for $x>0$, $y>0$ and $0\leqq k\leqq\frac{N}{2}+1$,*

$$\frac{\partial^k}{\partial x^k}\int_y^{\infty}S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz=\int_y^{\infty}S^{(N+1)}(z)\frac{K^{(k)}(xz)}{z^{N+1-k}}dz\,. \tag{28}$$

Proof. We find, by (9),

$$S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})}S^{(N+1)}(z)g(s)\frac{x^{s+N}z^{s-1}}{s\cdots(s+N)}ds\quad(z>0)\,.$$

Integrating this with respect to z from y to $\xi>y$, we get

$$\begin{aligned}\int_y^{\xi}S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz&=\frac{1}{2\pi i}\int_y^{\xi}S^{(N+1)}(z)dz\int_{(-\frac{1}{4N})}g(s)\frac{x^{s+N}z^{s-1}}{s\cdots(s+N)}ds\\&=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})}g(s)\frac{x^{s+N}}{s\cdots(s+N)}ds\int_y^{\xi}S^{(N+1)}(z)z^{s-1}dz\,;\end{aligned}$$

the reversion of the repeated integral being legitimate, since by (3)

$$\frac{g(s)}{s\cdots(s+N)}=O\left(|t|^{N(\frac{1}{2}+\frac{1}{4N})-N-1}\right)=O(|t|^{-2})\,.$$

Now noting (14) we easily get, by partial integration, the indefinite integral

$$\int S^{(N+1)}(z)z^{s-1}dz=\sum_{n=1}^{\tau}\frac{z^s}{s^n}\phi_n(z)\,,$$

where

$$\phi_n(z)=E_0^{(n)}\log^{\tau-n}z+\cdots+E_{\tau-n}^{(n)}\,. \tag{29}$$

Thus

$$\int_y^\xi S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})}g(s)\frac{x^{s+n}}{s\cdots(s+N)}\sum_{n=1}^{\tau}\frac{\xi^s\phi_n(\xi)-y^s\phi_n(y)}{s^n}ds\,.$$

Differentiating this with respect to x formally k times, we get

$$(30)\quad \left\{\begin{aligned}&\int_y^\xi S^{(N+1)}(z)\frac{K^{(k)}(xz)}{z^{N+1-k}}dz\\ &=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})}g(s)\frac{x^{s+N-k}}{s\cdots(s+N-k)}\sum_{n=1}^{\tau}\frac{\xi^s\phi_n(\xi)-y^s\phi_n(y)}{s^n}ds\,.\end{aligned}\right.$$

To justify the formal process we have only to ascertain that the last integral converges uniformly for $x_1\leqq x\leqq x_2$ $(0<x_1<x_2)$, y and ξ being fixed.

For this purpose we consider $(1\leqq n\leqq\tau)$

$$(31)\qquad I_n(x\xi)=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})}g(s)\frac{(x\xi)^s}{s^{n+1}(s+1)\cdots(s+N-k)}ds\,,$$

which transforms formally into $\left(s=-\frac{1}{4N}+ti\right)$

$$(32)\qquad =\frac{1}{2\pi}(x\xi)^{-\frac{1}{4N}}\int_{-\infty}^{\infty}g(s)\frac{(x\xi)^{ti}}{s^{n+1}(s+1)\cdots(s+N-k)}dt\,.$$

Here the integrand is by (3), for large $|t|$,

$$O\left(|t|^{N(\frac{1}{2}+\frac{1}{4N})-(N-k+n+1)}\right)=O(|t|^{k-\frac{N}{2}-\frac{3}{4}-n})=O(|t|^{\frac{1}{4}-n})\,,$$

uniformly in $x>0$ and $\xi>0$. Hence the integral of (31) converges, when $n\geqq2$ and ξ is fixed, absolutely and uniformly for $x_1\leqq x\leqq x_2$ $(0<x_1<x_2)$; and we have by (32), for all $x>0$ and all $\xi>0$,

$$(33)\qquad I_n(x\xi)=O\left((x\xi)^{-\frac{1}{4N}}\right)\quad(2\leqq n\leqq\tau)\,.$$

Next let $n=1$. Then formally $\left(s=-\frac{1}{4N}+ti\right)$

$$(34)\qquad I_1(x\xi)=\frac{1}{2\pi}(x\xi)^{-\frac{1}{4N}}\int_{-\infty}^{\infty}g(s)\frac{(x\xi)^{ti}}{s^2(s+1)\cdots(s+N-k)}dt\,.$$

Now we have by (2), when $\sigma=-\frac{1}{4N}$ and t is large,

$$g(s)=At^{N(\frac{1}{2}+\frac{1}{4N})}e^{-ti(N\log t-\Xi)}\left(1+O\left(\frac{1}{t}\right)\right);$$

further
$$\frac{1}{s^2(s+1)\cdots(s+N-k)}=\frac{1}{(it)^{N-k+2}}\left(1+O\left(\frac{1}{t}\right)\right).$$

Hence

$$\begin{aligned}\frac{g(s)}{s^2(s+1)\cdots(s+N-k)}&=A^*t^{k-\frac{N}{2}-\frac{7}{4}}e^{-ti(N\log t-\Xi)}\left(1+O\left(\frac{1}{t}\right)\right)\\ &=A^*t^{k-\frac{N}{2}-\frac{7}{4}}e^{-ti(N\log t-\Xi)}+q(t),\end{aligned}$$

where $q(t)$, which is independent of x and ξ, is continuous for $t>0$ and

$O(t^{-\frac{7}{4}})$ for large t, on account of

$$k-\frac{N}{2}-\frac{7}{4}\leqq\left(\frac{N}{2}+1\right)-\frac{N}{2}-\frac{7}{4}=-\frac{3}{4}\ ; \tag{35}$$

so that
$$\int_1^\infty (x\xi)^{ti}q(t)dt$$
converges absolutely and uniformly for $x>0$ and $\xi>0$, and

$$\left|\int_1^\infty (x\xi)^{ti}q(t)\,dt\right|\leqq\int_1^\infty |q(t)|dt=\text{const}\,.$$

Now, as is shown on p. 133 of Landau [9],

$$\left|\int_0^T t^{-\frac{1}{2}}(x\xi)^{ti}e^{-ti(N\log t-\Xi)}dt\right|<\frac{26}{\sqrt{N}}\qquad (T>0)\,.$$

Hence, for $U>T>1$,

$$\left|\int_T^U t^{-\frac{1}{2}}(x\xi)^{ti}e^{-ti(N\log t-\Xi)}dt\right|<\frac{52}{\sqrt{N}}\ ,$$

and so noting (35) we get, by the second mean-value theorem,

$$\left|\int_T^U t^{k-\frac{N}{2}-\frac{7}{4}}(x\xi)^{ti}e^{-ti(N\log t-\Xi)}dt\right|<104N^{-\frac{1}{2}}T^{-\frac{1}{4}}\,.$$

Therefore
$$\int_1^\infty t^{k-\frac{N}{2}-\frac{7}{4}}(x\xi)^{ti}e^{-ti(N\log t-\Xi)}dt$$
converges uniformly for $x>0$ and $\xi>0$, and is absolutely $\leqq 104N^{-\frac{1}{2}}$.

Consequently it follows that

$$\int_0^\infty g(s)\frac{(x\xi)^{ti}}{s^2(s+1)\cdots(s+N-k)}dt\quad\left(s=-\frac{1}{4N}+ti\right)$$

converges absolutely for $x>0$ and $\xi>0$, and that its value is uniformly bounded in x and ξ. And the same is true of

$$\int_{-\infty}^0 g(s)\frac{(x\xi)^{ti}}{s^2(s+1)\cdots(s+N-k)}dt\ ,$$

by symmetry. Thus the integral of (34), and hence of (31) for $n=1$, converges uniformly for $x_1\leqq x\leqq x_2$ when ξ is fixed; and we find by (34), for all $x>0$ and all $\xi>0$,

$$I_1(x\xi)=O\left((x\xi)^{-\frac{1}{4N}}\right)\,. \tag{36}$$

Thus the second integral of (30) converges uniformly for $x_1\leqq x\leqq x_2$ when ξ and y is fixed. Hence the equality (30) is true, and we get

$$\int_y^\xi S^{(N+1)}(z)\frac{K^{(k)}(xz)}{z^{N+1-k}}dz=x^{N-k}\sum_{n=1}^{T}\{I_n(x\xi)\phi_n(\xi)-I_n(xy)\phi_n(y)\}\,.$$

Noting (33), (36) and (29) we find moreover that, if $x_1\leqq x\leqq x_2$ $(0<x_1<x_2)$ and y is large,

$$\int_y^\xi S^{(N+1)}(z)\frac{K^{(k)}(xz)}{z^{N+1-k}}dz=O(y^{-\frac{1}{5N}})\ ,$$

uniformly in x. This shows that the integral on the right of (28) converges uniformly for $x_1 \leqq x \leqq x_2$, and thus the proof of Lemma 8 is complete.

Theorem 1 (extension of Hardy-Landau's identity).

Let $$\nu=\left[\frac{N}{2}\right]+1\,,\qquad \lambda=\left[\frac{N+1}{2}\right]=N+1-\nu\,.$$

Then, for $x>0$ *and* $y>0$,

$$\sum_{n\leqq y}\frac{F(n)}{n^{\lambda}}K^{(\nu)}(nx)=Q_{\lambda-1}(x)-\int_y^{\infty}S^{(N+1)}(z)\frac{K^{(\nu)}(xz)}{z^{\lambda}}dz$$
$$+\sum_{l=0}^{\lambda}(-1)^l Q_l(y)\frac{\partial^l}{\partial y^l}\frac{K^{(\nu)}(xy)}{y^{\lambda}}+(-1)^{\lambda}\int_y^{\infty}Q_{\lambda}(z)\frac{\partial^{\lambda+1}}{\partial z^{\lambda+1}}\frac{K^{(\nu)}(xz)}{z^{\lambda}}dz\,.$$

Proof. Noting that $H(z)=0$ for $0<z<1$, we have

$$\int_0^y H(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz\quad(=I(y),\ \text{say})$$
$$=\int_0^y\sum_{n\leqq z}F(n)\frac{\partial}{\partial z}\frac{K(xy)}{z^{N+1}}dz=\sum_{n\leqq y}F(n)\int_n^y\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz$$
$$=\sum_{n\leqq y}F(n)\left(\frac{K(xy)}{y^{N+1}}-\frac{K(nx)}{n^{N+1}}\right)=H(y)\frac{K(xy)}{y^{N+1}}-\sum_{n\leqq y}F(n)\frac{K(nx)}{n^{N+1}}\,.$$

Here we find for fixed x, by Lemma 5,

$$\frac{K(xy)}{y^{N+1}}=O(y^{-\frac{3}{2}-\frac{1}{2N}})=O(y^{-\frac{3}{2}})\quad(y\to\infty)\,.\tag{37}$$

Hence noting Lemma 1 we get, by Lemma 7,

$$I(\infty)=-\sum_{n=1}^{\infty}F(n)\frac{K(nx)}{n^{N+1}}=-Q_N(x)\,,$$

and so

$$\left\{\begin{aligned}&\sum_{n\leqq y}F(n)\frac{K(nx)}{n^{N+1}}-H(y)\frac{K(xy)}{y^{N+1}}-Q_N(x)\\&=I(\infty)-I(y)=\int_y^{\infty}H(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz\,.\end{aligned}\right.\tag{38}$$

On the other hand we have for $\xi>y$, by partial integration,

$$\int_y^{\xi}H(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz=\int_y^{\xi}(Q_0(z)+S^{(N)}(z)+Z(0))\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz$$
$$=\left\{(S^{(N)}(z)+Z(0))\frac{K(xz)}{z^{N+1}}\right\}_{z=y}^{z=\xi}-\int_y^{\xi}S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz+\int_y^{\xi}Q_0(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz\,,$$

where by (13) and (14), as $z\to\infty$,

$$S^{(N)}(z)=O(z\log^{\tau-1}z),\quad S^{(N+1)}(z)=O(\log^{\tau-1}z)\,.$$

Consequently, noting (37) and making $\xi\to\infty$, we obtain

$$\int_y^\infty H(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz=-(S^{(N)}(y)+Z(0))\frac{K(xy)}{y^{N+1}}-\int_y^\infty S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz$$
$$+\int_y^\infty Q_0(z)\frac{\partial}{\partial z}\frac{K(xy)}{z^{N+1}}dz\ ;$$

and this combined with (38) yields

$$\sum_{n\leqq y}F(n)\frac{K(nx)}{n^{N+1}}=Q_N(x)+Q_0(y)\frac{K(xy)}{y^{N+1}}-\int_y^\infty S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz \tag{39}$$
$$+\int_y^\infty Q_0(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz\ .$$

Now by (18), $Q_0(z)=O(z^{1-\frac{1}{N}})$, and so ($1\leqq l\leqq N$), by (16),

$$Q_l(z)=\int_0^z dy_1\int_0^{y_1}dy_2\cdots\int_0^{y_{l-1}}Q_0(y_l)dy_l=O(z^{l+1-\frac{1}{N}})\ .$$

Furthermore by Lemma 5, for $1\leqq l\leqq\lambda$ and fixed x,

$$\frac{\partial^l}{\partial z^l}\frac{K(xz)}{z^{N+1}}=\frac{x^lK^{(l)}(xz)}{z^{N+1}}-(N+1)\frac{x^{l-1}K^{(l-1)}(xz)}{z^{N+2}}+\cdots$$
$$=O\left(z^{(N+\frac{1}{2}-l)(1-\frac{1}{N})-(N+1)}\right)=O\left(z^{-\frac{3}{2}-\frac{1}{2N}-l(1-\frac{1}{N})}\right),$$

so that $\left(\text{note that } \lambda\leqq\dfrac{N+1}{2}\right)$

$$Q_l(z)\frac{\partial^l}{\partial z^l}\frac{K(xz)}{z^{N+1}}=O(z^{\frac{l}{N}-\frac{1}{2}-\frac{1}{2N}-\frac{1}{N}})=O(z^{-\frac{1}{N}})\ .$$

Therefore, by partial integration,

$$\int_y^\infty Q_0(z)\frac{\partial}{\partial z}\frac{K(xz)}{z^{N+1}}dz=-Q_1(y)\frac{\partial}{\partial y}\frac{K(xy)}{y^{N+1}}-\int_y^\infty Q_1(z)\frac{\partial^2}{\partial z^2}\frac{K(xz)}{z^{N+1}}dz$$
$$=\cdots=\sum_{l=1}^{\lambda}(-1)^lQ_l(y)\frac{\partial^l}{\partial y^l}\frac{K(xy)}{\cdot\, y^{N+1}}+(-1)^\lambda\int_y^\infty Q_\lambda(z)\frac{\partial^{\lambda+1}}{\partial z^{\lambda+1}}\frac{K(xz)}{z^{N+1}}dz\ ,$$

which combined with (39) gives us

$$\sum_{n\leqq y}\frac{F(n)}{n^{N+1}}K(nx)=Q_N(x)-\int_y^\infty S^{(N+1)}(z)\frac{K(xz)}{z^{N+1}}dz$$
$$+\sum_{l=0}^{\lambda}(-1)^lQ_l(y)\frac{\partial^l}{\partial y^l}\frac{K(xy)}{y^{N+1}}+(-1)^\lambda\int_y^\infty Q_\lambda(z)\frac{\partial^{\lambda+1}}{\partial z^{\lambda+1}}\frac{K(xz)}{z^{N+1}}dz\ .$$

We now differentiate the last result ν ($<N$) times with respect to x and obtain, by Lemma 9,

$$\sum_{n\leqq y}\frac{F(n)}{n^\lambda}K^{(\nu)}(nx)=Q_{\lambda-1}(x)-\int_y^\infty S^{(N+1)}(z)\frac{K^{(\nu)}(xz)}{z^\lambda}dz$$
$$+\sum_{l=0}^{\lambda}(-1)^lQ_l(y)\frac{\partial^l}{\partial y^l}\frac{K^{(\nu)}(xy)}{y^\lambda}+(-1)^\lambda\frac{\partial^\nu}{\partial x^\nu}\int_y^\infty Q_\lambda(z)\frac{\partial^{\lambda+1}}{\partial z^{\lambda+1}}\frac{K(xz)}{z^{N+1}}dz\ .$$

On comparing this with the statement of our theorem, it is found that what remains to be proved is only the uniform convergence of

$$\int_y^\infty Q_\lambda(z)\frac{\partial^{\lambda+1}}{\partial z^{\lambda+1}}\frac{K^{(k)}(xz)}{z^{N+1-k}}dz \quad (1\leqq k\leqq \nu)$$

for $x_1\leqq x\leqq x_2$ $(0<x_1<x_2)$. Noting that $\lambda+1+\nu=N+2$ we find by Lemma 5 and (17), uniformly for such x,

$$Q_\lambda(z)\frac{\partial^{\lambda+1}}{\partial z^{\lambda+1}}\frac{K^{(k)}(xz)}{z^{N+1-k}}=O\left(z^{(\lambda+\frac{1}{2})(1-\frac{1}{N})+(N+\frac{1}{2}-\lambda-1-k)(1-\frac{1}{N})-(N+1-k)}\right)$$

$$=O(z^{\frac{k}{N}-2})=O\left(z^{\frac{1}{N}(\frac{N}{2}+1)-2}\right)=O(z^{-\frac{7}{6}}) \quad (z\to\infty).$$

This concludes the proof of Theorem 1.

§ 3. Extension of Hardy's identity

The object of this section is to establish the following extension of Hardy's identity:

Theorem 2. *We have, for $x>0$,*

$$Q_{\lambda-1}(x)=\sum_{n=1}^\infty \frac{F(n)}{n^\lambda}K^{(\nu)}(nx),$$

or, written out in full,

$$\sum_{n\leqq x}F(n)(x-n)^M=\operatorname*{Res}_{s=1}\left(Z(s)\frac{M!\,x^{s+M}}{s\cdots(s+M)}\right)+Z(0)x^M$$
$$+\sum_{n=1}^\infty\frac{F(n)}{n^{M+1}}\frac{M!}{2\pi i}\int_{(\frac{1}{2})}\frac{Z(s)}{Z(1-s)}\frac{(nx)^{s+M}}{s\cdots(s+M)}ds,$$

where
$$M=\left[\frac{N-1}{2}\right].$$

Remarks. 1) The equivalence of the two forms may be seen as follows. We apply Cauchy's theorem to replace $\left(-\frac{1}{4N}\right)$ in

$$K(w)=\frac{1}{2\pi i}\int_{(-\frac{1}{4N})}g(s)\frac{w^{s+N}}{s\cdots(s+N)}ds \quad (w>0)$$

by $(\frac{1}{2})$. This is possible since by (3), uniformly in the strip $-\frac{1}{4N}\leqq\sigma\leqq\frac{1}{2}$, for fixed w and large $|t|$,

$$g(s)\frac{w^{s+N}}{s\cdots(s+N)}=O\left(|t|^{N(\frac{1}{2}+\frac{1}{4N})-N-1}\right)=O(|t|^{-2}).$$

Thus
$$K(w)=\frac{1}{2\pi i}\int_{(\frac{1}{2})}g(s)\frac{w^{s+N}}{s\cdots(s+N)}ds.$$

On differentiating this formally k times $(1\leqq k\leqq\nu)$ with respect to w it is found that

$$K^{(k)}(w)=\frac{1}{2\pi i}\int_{(\frac{1}{2})}g(s)\frac{w^{s+N-k}}{s\cdots(s+N-k)}ds.$$

To justify the process we show that the last integral converges uniformly for $0<w\leqq w_0$ $(w_0>0)$. This is clear since we have by (3), on the line $\sigma=\frac{1}{2}$, uniformly for such w,

$$g(s)\frac{w^{s+N-k}}{s\cdots(s+N-k)}=O(|t|^{k-N-1})=O(|t|^{-\frac{N}{2}})=O(|t|^{-\frac{3}{2}})\ .$$

From the above result and (12) the equivalence follows at once.

2) The interest of this theorem consists chiefly in the fact that the convergence of the series can not be established easily. In fact Lemma 5 gives merely, for fixed x,

$$\frac{K^{(\nu)}(nx)}{n^{\lambda}}=O\left(n^{(N+\frac{1}{2}-\nu)(1-\frac{1}{N})-\lambda}\right)=O\left(n^{\frac{\nu}{N}-\frac{3}{2}-\frac{1}{2N}}\right)\ ;$$

and since $$\frac{\nu}{N}-\frac{3}{2}-\frac{1}{2N}\geqq\frac{1}{N}\frac{N+1}{2}-\frac{3}{2}-\frac{1}{2N}=-1\ ,$$

we deduce no conclusion as to the convergence.

3) We prove Theorem 2 first for the case $3\leqq N\leqq 11$, using Lemma 8 (which was established by Landau's method). It will not be until we have developed some further lemmas that we shall be able to prove it in the general case, using a method similar to that for Theorem F of Hardy-Littlewood [3]. It thus appears that Hardy-Littlewood's method is more powerful than Landau's (as far as we are concerned with the present problem).

Proof for $3\leqq N\leqq 11$. In view of Theorem 1 we have only to show that, when $0\leqq l\leqq\lambda$,

$$Q_l(y)\frac{\partial^l}{\partial y^l}\frac{K^{(\nu)}(xy)}{y^{\lambda}}\rightarrow 0\quad(y\rightarrow\infty). \tag{40}$$

This expression is by (17) and Lemma 5, for $l=\lambda$,

$$O\left(y^{(\lambda+\frac{1}{2})(1-\frac{1}{N})+(N+\frac{1}{2}-\lambda-\nu)(1-\frac{1}{N})-\lambda}\right)=O\left(y^{-\frac{\lambda}{N}}\right)\ , \tag{41}$$

and by (18) and Lemma 5, for $0\leqq l\leqq\lambda-1$ $\left(\leqq\frac{N-1}{2}\right)$,

$$O\left(y^{l+1-\frac{2}{N+1-2l}+(N+\frac{1}{2}-l-\nu)(1-\frac{1}{N})-\lambda}\log^{\tau}y\right)$$
$$=O\left(y^{\frac{l+\nu}{N}-\frac{2}{N+1-2l}-\frac{1}{2}-\frac{1}{2N}}\log^{\tau}y\right)=O\left(y^{\frac{\nu}{N}-\left(\frac{2}{N+1-2l}+\frac{N+1-2l}{2N}\right)}\log^{\tau}y\right)\ .$$

Now suppose that $N+1-2\xi>0$. Then, since $\dfrac{2}{N+1-2\xi}\dfrac{N+1-2\xi}{2N}=\dfrac{1}{N}$, the function

$$G(\xi)=\frac{\nu}{N}-\left(\frac{2}{N+1-2\xi}+\frac{N+1-2\xi}{2N}\right) \tag{42}$$

attains its maximum when

$$\frac{2}{N+1-2\xi}=\frac{N+1-2\xi}{2N}=\frac{1}{\sqrt{N}}\quad\left(\text{or }\ \xi=\tfrac{1}{2}(\sqrt{N}-1)^2\right)\ .$$

Hence $$G(l)\leqq\frac{1}{N}\left(\frac{N}{2}+1\right)-\frac{2}{\sqrt{N}}=\frac{(\sqrt{N}-2)^2-2}{2N},$$

and the last expression is negative if

$$\sqrt{N}<2+\sqrt{2}, \quad \text{or} \quad N<6+4\sqrt{2}=11.6\cdots.$$

This proves the stated result.

Example. Let us take $N=3$ (so that $M=1$) and let k_j be for each $j=1, 2, 3$ the rational number field. Then

$$Z(s)=\zeta^3(s)=\sum_{n=1}^{\infty}d_3(n)n^{-s} \quad (\sigma>1),$$

where $\zeta(s)$ is the Riemann zeta-function. Further

$$\zeta(0)=-\tfrac{1}{2}, \quad Z(0)=-\tfrac{1}{8},$$

and by (1)

$$\frac{Z(s)}{Z(1-s)}=g(s)=\left(\frac{(2\pi)^s}{\pi}\sin\frac{\pi s}{2}\Gamma(1-s)\right)^3.$$

We thus obtain $(x>0)$

$$\int_0^x\sum_{n\leqq y}d_3(n)dy=\sum_{n\leqq x}d_3(n)(x-n)$$
$$=\operatorname*{Res}_{s=1}\left(\frac{x^{s+1}\zeta^3(s)}{s(s+1)}\right)-\frac{1}{8}x$$
$$+\sum_{n=1}^{\infty}\frac{d_3(n)}{n^2}\frac{1}{2\pi i}\int_{(\frac{1}{2})}\left(\frac{(2\pi)^s}{\pi}\sin\frac{\pi s}{2}\Gamma(1-s)\right)^3\frac{(nx)^{s+1}}{s(s+1)}ds.$$

Lemma 10. *Let γ, T, y be positive, $y\neq 1$, and k a natural number. Then*

$$\left|\frac{1}{2\pi i}\left(\int_{\gamma-iT}^{\gamma+iT}-\int_{\gamma-i\infty}^{\gamma+i\infty}\right)\frac{y^s}{s\cdots(s+k)}ds\right|<\frac{1}{T^{k+1}}\frac{y^\gamma}{|\log y|}.$$

(The convergence of the infinite integral is obvious.)

Proof. The left-hand side equals

$$\left|\frac{1}{2\pi i}\left(\int_{\gamma+iT}^{\gamma+i\infty}+\int_{\gamma-i\infty}^{\gamma-iT}\right)\frac{y^s}{s\cdots(s+k)}ds\right|. \tag{43}$$

Now, by partial integration,

$$\left|\int_{\gamma+iT}^{\gamma+i\infty}\frac{y^s}{s\cdots(s+k)}ds\right|=\left|\left\{\frac{y^s}{\log y}\frac{1}{s\cdots(s+k)}\right\}_{\gamma+iT}^{\gamma+i\infty}\right.$$
$$\left.+\int_{\gamma+iT}^{\gamma+i\infty}\frac{y^s}{\log y}\frac{1}{s\cdots(s+k)}\left(\frac{1}{s}+\cdots+\frac{1}{s+k}\right)ds\right|$$
$$\leqq\frac{y^\gamma}{|\log y|}\left(\frac{1}{T^{k+1}}+\int_T^\infty\frac{k+1}{t^{k+2}}dt\right)=\frac{2}{T^{k+2}}\frac{y^\gamma}{|\log y|};$$

and, by symmetry,

$$\left|\int_{\gamma-i\infty}^{\gamma-iT}\frac{y^s}{s\cdots(s+k)}ds\right|\leqq\frac{2}{T^{k+1}}\frac{y^\gamma}{|\log y|}.$$

Hence (43) does not exceed

$$\frac{1}{2\pi}\frac{4}{T^{k+1}}\frac{y^{\gamma}}{|\log y|}<\frac{1}{T^{k+1}}\frac{y^{\gamma}}{|\log y|}\,.$$

Lemma 11. *We have, for any fixed* $\varepsilon>0$,

$$F(n)=O(n^{\varepsilon})\,.$$

Proof. Let $\zeta_0(s)$ be the Dedekind zeta-function of an algebraic number field of degree m and write

$$\zeta_0(s)=\sum_{n=1}^{\infty}f(n)n^{-s}\quad(\sigma>1)\,.$$

Then we know, by Hilfssatz 1 of Landau [7], that

$$f(n)\leqq d_m(n)\,,$$

where $d_m(n)$ is defined by

$$\zeta^m(s)=\sum_{n=1}^{\infty}d_m(n)n^{-s}\quad(\sigma>1)\,,$$

$\zeta(s)$ being agein the Riemann zeta-function. Hence, writing $(\sigma>1)$

$$\zeta^N(s)=\sum_{n=1}^{\infty}d_N(n)n^{-s}=\zeta^{n_1}(s)\cdots\zeta^{n_\tau}(s)$$

and comparing this with

$$Z(s)=\sum_{n=1}^{\infty}F(n)n^{-s}=\zeta_1(s)\cdots\zeta_\tau(s)\,,$$

we at once see that

$$F(n)\leqq d_N(n)\,.$$

But it is known (Hilfssatz 12 of Landau [6]) that

$$d_N(n)=O(n^{\varepsilon})\,.$$

Hence the result.

Lemma 12. *Let* $\gamma>1$, $T>0$, $y=m+\frac{1}{2}$, *m a positive integer*, $1\leqq k\leqq N$. *Then, for all T and y,*

$$H_k(y)=\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT}Z(s)\frac{y^{s+k}}{s\cdots(s+k)}ds+O\left(\frac{y^{k+\gamma}}{T^{k+1}}\right)\,.$$

Proof. Differentiating (10) with respect to y formally $N-k$ times, we get

$$H_k(y)=\frac{1}{2\pi i}\int_{(\gamma)}Z(s)\frac{y^{s+k}}{s\cdots(s+k)}ds\,,$$

and the formal process is legitimate, since the last integral clearly converges uniformly for bounded y. Hence

$$(44)\quad\left\{\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT}Z(s)\frac{y^{s+k}}{s\cdots(s+k)}ds-H_k(y)\\&=\frac{1}{2\pi i}\left(\int_{\gamma-iT}^{\gamma+iT}-\int_{\gamma-i\infty}^{\gamma+i\infty}\right)Z(s)\frac{y^{s+k}}{s\cdots(s+k)}ds\,.\end{aligned}\right.$$

Now we easily see that

$$(45) \qquad \left|\frac{1}{2\pi i}\left(\int_{\gamma-iT}^{\gamma+iT}-\int_{\gamma-i\omega}^{\gamma+i\omega}\right)Z(s)\frac{y^{s+k}}{s\cdots(s+k)}ds\right| \quad (\omega>T)$$

$$=\left|\sum_{n=1}^{\infty}F(n)y^k\frac{1}{2\pi i}\left(\int_{\gamma-iT}^{\gamma+iT}-\int_{\gamma-i\omega}^{\gamma+i\omega}\right)\frac{\left(\dfrac{y}{n}\right)^s}{s\cdots(s+k)}ds\right| ,$$

and this is, by Lemma 10,

$$\leqq y^k\sum_{n=1}^{\infty}F(n)\frac{2}{T^{k+1}}\left(\frac{y}{n}\right)^{\gamma}\frac{1}{\left|\log\dfrac{y}{n}\right|}$$

$$=\frac{y^{k+\gamma}}{T^{k+1}}\sum_{n=1}^{\infty}\frac{F(n)}{n^{\varepsilon}}\frac{2}{n^{1+\varepsilon}\left|\log\dfrac{y}{n}\right|} \qquad (\gamma=1+2\varepsilon) ,$$

which is independent of ω.

Here $F(n)n^{-\varepsilon}$ is bounded by Lemma 11, and we may prove without difficulty that, for all $y=m+\frac{1}{2}$,

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}\left|\log\dfrac{y}{n}\right|}=O(1) ;$$

the proof being similar to that for Hilfssatz 3 on p. 841 of Landau [5]. Hence (45) is $O(y^{k+\gamma}T^{-k-1})$ uniformly in ω.

On making $\omega\to\infty$, it follows that

$$\frac{1}{2\pi i}\left(\int_{\gamma-iT}^{\gamma+iT}-\int_{\gamma-i\infty}^{\gamma+i\infty}\right)Z(s)\frac{y^{s+k}}{s\cdots(s+k)}ds=O\left(\frac{y^{k+\gamma}}{T^{k+1}}\right) ,$$

and comparing this with (44) we get the desired result.

Lemma 13. *For any given $\varepsilon>0$ we have, uniformly for $0\leqq\sigma\leqq1+\varepsilon$,*

$$Z(s)=O\left(|t|^{N\left(\frac{1-\sigma}{2}+\varepsilon\right)}\right) .$$

Proof. Let $\delta>0$. Then, by (4),

$$Z(s)=O\left(|t|^{N(\frac{1}{2}+\delta)}\right)$$

uniformly for $\sigma\geqq-\delta$. But clearly $Z(1+\varepsilon+ti)=O(1)$. Hence by the Phragmén-Lindelöf theorem (Satz 405 of Landau [8]) we have, uniformly for $-\delta\leqq\sigma\leqq1+\varepsilon$,

$$Z(s)=O\left(|t|^{N(\frac{1}{2}+\delta)\frac{1+\varepsilon-\sigma}{1+\varepsilon+\delta}}\right) .$$

The lemma now follows if we take δ sufficiently small.

Lemma 14. *Let $\delta>0$ be fixed, $1\leqq l\leqq N$, $y=m+\frac{1}{2}$, m a variable positive integer. Then*

(46) $$Q_l(y)=O(y^{l+\frac{1}{2}}) \qquad \left(l>\frac{N}{4}\right),$$

(47) $$Q_l(y)=O(y^{l+1-\frac{2(l+1)}{N+4}+\delta}) \quad \left(l\leqq\frac{N}{4}\right).$$

Proof. Let $\varepsilon>0$ and write

$$Z_l(s)=Z(s)\frac{y^{s+l}}{s\cdots(s+l)}.$$

Noting (12) we find for $T>1$, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} Z_l(s)ds=S^{(N-l)}(y)+I_1+I_2+I_3,$$

where we write for brevity

$$I_1=\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{1+\varepsilon+iT} Z_l(s)ds, \quad I_2=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{\frac{1}{2}-iT} Z_l(s)ds, \quad I_3=\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} Z_l(s)ds,$$

so that, by symmetry,

(48) $$|I_1|=|I_2|.$$

Hence, by Lemma 12,

$$Q_l(y)-\frac{y^l}{l!}Z(0)=H_l(y)-S^{(N-l)}(y)$$

(49) $$=I_1+I_2+I_3+O\left(\frac{y^{l+1+\varepsilon}}{T^{l+1}}\right).$$

As for the order of $Z_l(s)$ we have by Lemma 13, uniformly for $0\leqq\sigma\leqq1+\varepsilon$,

(50) $$Z_l(s)=O\left(y^{\sigma+l}t^{N(\frac{1-\sigma}{2}+\varepsilon)-l-1}\right) \quad (t\to\infty).$$

1) Let $l>\frac{N}{4}$. We take $T=y$ and obtain, by (50),

$$I_1=O\left(y^{l+1+\varepsilon}T^{N(\frac{1}{4}+\varepsilon)-l-1}\right)=O(y^{l1(N++)\varepsilon});$$

further, by symmetry,

$$I_3=O\int_1^T y^{l+\frac{1}{2}}t^{N(\frac{1}{4}+\varepsilon)-l-1}dt+O(y^{l+\frac{1}{2}})=O(y^{l+\frac{1}{2}}),$$

if $\varepsilon<\frac{l}{N}-\frac{1}{4}$. Hence, noting (48) we get, by (49),

$$Q_l(y)=O(y^{l+(N+1)\varepsilon})+O(y^{l+\frac{1}{2}})+O(y^{\varepsilon})+\frac{y^l}{l!}Z(0),$$

and this is $O(y^{l+\frac{1}{2}})$ if ε is small enough.

2) Let $l\leqq\frac{N}{4}$. We take $T=y^{\frac{2}{N+4}}$, so that $y=T^{\alpha}$, $\alpha=\frac{N}{2}+2$. Then by (50), uniformly for $\frac{1}{2}\leqq\sigma\leqq1+\varepsilon$,

$$Z_l(s)=O\left(T^{\alpha(\sigma+l)+N(\frac{1-\sigma}{2}+\varepsilon)-l-1}\right) \quad (t=T).$$

Now the linear function of σ

$$f(\sigma)=\alpha(\sigma+l)+N\Big(\frac{1-\sigma}{2}+\varepsilon\Big)-l-1$$

attains its maximum for $\frac{1}{2}\leqq\sigma\leqq 1+\varepsilon$ at one of the end-points of the interval, where

$$(51)\qquad f(\tfrac{1}{2})=\alpha(l+\tfrac{1}{2})+N(\tfrac{1}{4}+\varepsilon)-l-1=\alpha(l+1)-l-2+N\varepsilon\,,$$
$$f(1+\varepsilon)=\alpha(l+1+\varepsilon)+\tfrac{1}{2}N\varepsilon-l-1=\alpha(l+1)-l-1+(N+2)\varepsilon\,,$$

so that we get, in view of (48),

$$(52)\qquad |I_1|=|I_2|=O(T^{(\alpha-1)(l+1)+(N+2)\varepsilon})\,.$$

Further, by (50) and (51),

$$I_3=O\int_1^T y^{l+\frac{1}{2}}t^{N(\frac{1}{4}+\varepsilon)-l-1}dt+O(y^{l+\frac{1}{2}})$$
$$=O(y^{l+\frac{1}{2}}T^{N(\frac{1}{4}+\varepsilon)-l})=O(T^{\alpha(l+\frac{1}{2})+N(\frac{1}{4}+\varepsilon)-l})$$
$$=O(T^{(\alpha-1)(l+1)+N\varepsilon})\,.$$

From (49), (52) and this it follows that

$$Q_l(y)=O(T^{(\alpha-1)(l+1)+(N+2)\varepsilon})+O(T^{(\alpha-1)(l+1)+\alpha\varepsilon})+O(T^{l\alpha})\,,$$

where $\qquad (\alpha-1)(l+1)-l\alpha=\alpha-(l+1)\geqq(\frac{N}{2}+2)-(\frac{N}{4}+1)>0\,.$

Thus $\qquad Q_l(y)=O(T^{(\alpha-1)(l+1)+(N+2)\varepsilon})=O\Big(y^{l+1-\frac{l+1}{\alpha}+2\varepsilon}\Big)\,,$

whence the result follows if ε is sufficiently small.

Proof of Theorem 2 (in the general case). It suffices to show the validity of (40), where $y=m+\frac{1}{2}$. It holds for $l=\lambda$ by (41) ; and we have for $l=0$ and $l=1$, in (42),

$$G(0)=\frac{\nu}{N}-\frac{2}{N+1}-\frac{N+1}{2N}\leqq\frac{1}{N}\Big(\frac{N}{2}+1-\frac{N+1}{2}\Big)-\frac{2}{N+1}<0\,,$$
$$G(1)=\frac{\nu}{N}-\frac{2}{N-1}-\frac{N-1}{2N}\leqq\frac{1}{N}\Big(\frac{N}{2}+1-\frac{N-1}{2}\Big)-\frac{2}{N-1}<0\,,$$

so that (40) holds. (It should be noted that $\lambda=\Big[\dfrac{N+1}{2}\Big]\geqq 2$, where the equality holds if and only if $N=3$ or 4.)

Thus we may restrict ourselves to the case $N>4$, $2\leqq l\leqq\lambda-1$ in the following. We distinguish two cases according to Lemma 14.

1) $l>\frac{N}{4}$. Then, by (46) and Lemma 5,

$$Q_l(y)\frac{\partial^l}{\partial y^l}\frac{K^{(\nu)}(xy)}{y^\lambda}=O\Big(y^{l+\frac{1}{2}+(N+\frac{1}{2}-l-\nu)(1-\frac{1}{N})-\lambda}\Big)$$
$$=O\Big(y^{\frac{l+\nu}{N}-1-\frac{1}{2N}}\Big)=O\Big(y^{-\frac{1}{2N}}\Big)\to 0$$

as $y\to\infty$, since $l+\nu\leqq\lambda-1+\nu=N$.

2) $l\leqq\frac{N}{4}$. Then by (47) and Lemma 5, for fixed $\delta>0$,

$$Q_l(y)\frac{\partial^l}{\partial y^l}\frac{K^{(\nu)}(xy)}{y^\lambda}=O\left(y^{l+1-\frac{2(l+1)}{N+4}+\delta+(N+\frac{1}{2}-l-\nu)(1-\frac{1}{N})-\lambda}\right).$$

Here the exponent is

$$\begin{aligned}&\frac{\nu}{N}-\frac{1}{2}-\frac{1}{2N}+\frac{l}{N}-\frac{2(l+1)}{N+4}+\delta\\ &=\frac{1}{N}\left(\nu-\frac{N}{2}-\frac{1}{2}\right)+\frac{l}{N}-\frac{2l+2}{N+4}+\delta\\ &\leqq\frac{2l+1}{2N}-\frac{2l+2}{N+4}+\delta=\frac{(2l+1)(4-N)-2N}{2N(N+4)}+\delta\ ;\end{aligned}$$

and the last expression is negative if δ is small enough. Hence (40) holds.

Thus Theorem 2 is completely proved for $N\geqq 3$.

References

[1] G. H. Hardy: On the expression of a number as the sum of two squares; Quarterly Journal of Mathematics, vol. **46** (1915), pp. 263-283.

[2] G. H. Hardy and E. Landau: The Lattice Points of a Circle; Proceedings of the Royal Society, A **105** (1924), pp. 244-258.

[3] G. H. Hardy and J. E. Littlewood: The approximate functional equation in the theory of the zeta-function, with applications to the divisor problems of Dirichlet and Piltz; Proceedings of the London Mathematical Society (Second Series), vol. **21** (1922), pp. 39-74.

[4] H. Hasse and Z. Suetuna: Ein Allgemeines Teilerproblem der Idealtheorie; Journal of the Faculty of Science, Imperial University of Tokyo, Section I, vol. **2** (Part 5, 1931), pp. 133-154.

[5] E. Landau: *Handbuch der Lehre von der Verteilung der Primzahlen* (in 2 vols.); Leipzig and Berlin, 1909.

[6] E. Landau: Über die Anzahl der Gitterpunkte in gewissen Bereichen; Göttinger Nachrichten (1912), pp. 687-771.

[7] E. Landau: Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen; Göttinger Nachrichten (1918), pp. 79-97.

[8] E. Landau: *Vorlesungen über Zahlentheorie* (in 3 vols.); Leipzig, 1927.

[9] E. Landau: *Einführung in die Elementare und Analytische Theorie der Algebraischen Zahlen und der Ideale* (2nd ed.); Leipzig and Berlin, 1927.

[10] A. Walfisz: Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern; Mathematische Zeitschrift, vol. **22** (1925), pp. 153-188.

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