On Generalized Logarithmic Spirals in Riemann Spaces[1]

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1. Consider a Riemann space V_n with a line element $ds^2=g_{ij}\,dx^i\,dx^j$ $(i, j, =1, 2, \cdots, n)$ and a curve with a parametric representation $x^i=x^i(s)$. We denote tangent, 1st normal, $\cdots$, $(n-1)$-th normal by $\xi^i_{(1)}, \xi^i_{(2)}, \cdots, \xi^i_{(n)}$ respectively and i-th curvature by $\kappa_{(i)}$. Then the Frenet's formulas are given by the following equations:

$$(1)\qquad \frac{\delta}{\delta s}\,\xi^i_{(a)}=-\kappa_{(a-1)}\,\xi^i_{(a-1)}+\kappa_{(a)}\,\xi^i_{(a+1)}\qquad (a=1, \cdots, n),$$

$$\kappa_{(0)}=\kappa_{(n)}=0,$$

where $\delta/\delta s$ denotes covariant differentiation along the curve.

Now we shall define a curve called generalized logarithmic spiral of order p by the following two conditions:

(i) $\kappa_{(p+1)}=0,\ p<n$, (ii) a point field $M(s)+\sum_{a=1}^{p+1}\alpha_a\,\xi^i_{(a)}\,e_i$ along the curve is invariant by the development along the curve provided that the following conditions are satisfied:

$$(2)\qquad \alpha_b=\alpha\,c_{b-1},\quad c_{b-1}=\text{const.}\neq 0,\ c_0=1,\ \alpha\neq 0,\ (b=1, \cdots, p+1),$$

where $M(s)$ is an arbitrary point on the curve and e's denote natural repères at M.

In this note we shall obtain the explicit forms of $\kappa_{(i)}$ of the curve and differential equations of the one of order 1.

2. In the 1st place we shall consider the curve of order p. From the condition (ii),

$$\frac{d}{ds}\left[M+\sum_{a=1}^{p+1}\alpha_a\,\xi^i_{(a)}\,e_i\right]=0$$

holds good along the curve. Using (1), we get from the last equation,

$$\xi^i_{(1)}+\sum_{a=1}^{p+1}\left[\alpha'_a\,\xi^i_{(a)}+\alpha_a\left(-\kappa_{(a-1)}\,\xi^i_{(a-1)}+\kappa_{(a)}\,\xi^i_{(a+1)}\right)\right]=0,$$

where dash denotes the derivative with respect to s. If we substitute (2) into the last equation, then

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$$(3)_1 \qquad 1+\alpha'-\kappa_{(1)}c_1\alpha=0,$$

$$(3)_b \qquad c_{b-1}\alpha'+c_{b-2}\kappa_{(b)}\alpha=c_b\kappa_{(b)}\alpha, \qquad (b=2,\cdots,p+1),$$

hold good. From $(3)_b$ for $b=p+1$, we have

$$(3)_{p+1} \qquad \alpha'=-\frac{c_{p-1}}{c_p}\kappa_{(p)}\alpha.$$

Substituting $(3)_{p+1}$ into $(3)_1$, $(3)_b$, we get

$$(4)_1 \qquad c_p-c_{p-1}\kappa_{(p)}\alpha-c_1c_p\kappa_{(1)}\alpha=0,$$

$$(4)_b \qquad -c_{b-1}c_{p-1}\kappa_{(p)}+c_pc_{b-2}\kappa_{(b-1)}=c_bc_p\kappa_{(b)}.$$

From $(4)_b$ for $b=p$, it follows

$$\kappa_{(p-1)}=\frac{c_p^2+c_{p-1}^2}{c_pc_{p-2}}\kappa_{(p)}.$$

Substituting it into $(4)_{p-1}$, we get

$$\kappa_{(p-2)}=\frac{c_{p-1}(c_p^2+c_{p-1}^2+c_{p-2}^2)}{c_pc_{p-2}c_{p-3}}\kappa_{(p)}.$$

If we continue the same process step by step, then we have

$$(5) \qquad \kappa_{(p-a)}=\frac{c_{p-1}\sum_{c=0}^{a}c_{p-c}^2}{c_pc_{p-a}c_{p-(a+1)}}\kappa_{(p)}, \qquad (a=1,2,\cdots,p-1),$$

especially for $a=p-1$, the relation

$$(6) \qquad \kappa_{(1)}=\frac{c_{p-1}\sum_{a=1}^{p}c_a^2}{c_pc_1}\kappa_{(p)}$$

holds good. On the other hand, from $(4)_1$ it follows

$$(7) \qquad \kappa_{(p)}=\frac{c_p(1-c_1\kappa_{(1)}\alpha)}{c_{p-1}\alpha}.$$

Substituting (7) into (6), we get

$$(8) \qquad c_1\kappa_{(1)}\alpha\left(1+\sum_{a=1}^{p}c_a^2\right)=\sum_{a=1}^{p}c_a^2.$$

Differentiating (8) with respect to s, we get

$$\alpha'\kappa_{(1)}+\kappa'_{(1)}\alpha=0.$$

From $(3)_{p+1}$ and the last equation, it follows

$$(9) \qquad c_p\kappa'_{(1)}=c_{p-1}\kappa_{(1)}\kappa_{(p)}.$$

Hence we have from (7), (8) and (9)

$$(10) \qquad \kappa'_{(1)}=\frac{c_1}{\sum_{a=1}^{p}c_a^2}\kappa_{(1)}^2.$$

By integration we get

$$(11)_1 \qquad \frac{1}{\kappa_{(1)}} = -\frac{c_1}{\sum_{a=1}^{p} c_a^2}\, s + c, \qquad (c = \text{const.}).$$

From (5), (6) and (11) we get

$$(11)_b \qquad \frac{1}{\kappa_{(b)}} = \frac{c_{b-1}\, c_b}{\sum_{a=b}^{p} c_a^2}\left(-s + \frac{c}{c_1}\sum_{a=1}^{p} c_a^2\right), \qquad (b = 2, \cdots, p).$$

3. In the next place we shall investigate the curve of order 1. In this case by virtue of $(11)_1$ and (i) we have

$$(12) \qquad \kappa_{(1)} = \frac{c_1}{c\, c_1 - s}, \qquad \kappa_{(2)} = 0,$$

and the Frenet's formulas are given by

$$\frac{\delta}{\delta s}\, \xi^i_{(1)} = \kappa_{(1)}\, \xi^i_{(2)}, \qquad \frac{\delta}{\delta s}\, \xi^i_{(2)} = -\kappa_{(1)}\, \xi^i_{(1)}.$$

Differentiate the 1st equations covariantly, then we get

$$\frac{\delta^2}{\delta s^2}\, \xi^i_{(1)} = \kappa'_{(1)}\, \xi^i_{(2)} + \kappa_{(1)} \frac{\delta}{\delta s}\, \xi^i_{(2)} = \frac{\kappa'_{(1)}}{\kappa_{(1)}} \frac{\delta}{\delta s}\, \xi^i_{(1)} - \kappa^2_{(1)}\, \xi^i_{(1)}.$$

On the other hand it follows from (10) that $\kappa'_{(1)} = \kappa^2_{(1)}/c_1$. Hence we get

$$\frac{\delta^2}{\delta s^2}\, \xi^i_{(1)} - \frac{\kappa_{(1)}}{c_1} \frac{\delta}{\delta s}\, \xi^i_{(1)} + \kappa^2_{(1)}\, \xi^i_{(1)} = 0.$$

Substituting (12) in the last equation, we get

$$(13) \qquad \frac{\delta^3 x^i}{\delta s^3} + \frac{1}{s - c c_1} \frac{\delta^2 x^i}{\delta s^2} + \left(\frac{c_1}{s - c c_1}\right)^2 \frac{dx^i}{ds} = 0,$$

because $\xi^i_{(1)} = dx^i/ds$. (13) is the differential equations of the generalized logarithmic spirals of order 1.