A NOTE ON SEMIGROUPS OF LOCALLY LIPSCHITZ OPERATORS ASSOCIATED WITH SEMILINEAR EVOLUTION EQUATIONS

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Abstract. In this note we shall give a simple proof for a part of proof of T. Matsumoto and N. Tanaka [6] Theorem 2.2. This theorem is applied to the global solvability of the mixed problem for the complex Ginzburg-Landau equation by T. Matsumoto and N. Tanaka [5][6].

In this note we shall give a simple proof for a part of proof of T. Matsumoto and N. Tanaka[6] Theorem 2.2.

1. Preliminaries

Let \((X, \| \cdot \|)\) be a Banach space with norm \(\| \cdot \|\) and \(D\) be a closed subset of \(X\).

Definition 1. A one-parameter family \(\{S(t); t \geq 0\}\) of Lipschitz operators from \(D\) into itself is called a semigroup of Lipschitz operators on \(D\) if the following three conditions are satisfied:

(S1) \(S(0)x = x\) \(\text{for } x \in D\), \(S(t+s)x = S(t)S(s)x\) \(\text{for } s, t \geq 0 \text{ and } x \in D\).

(S2) For each \(x \in D\), \(S(\cdot)x: [0, \infty) \to X\) is continuous.

(S3) For each \(\tau > 0\), there exists \(L_{\tau} > 0\) such that

\[\|S(t)x - S(t)y\| \leq L_{\tau}\|x - y\| \text{ for } x, y \in D \text{ and } t \in [0, \tau].\]

For semigroups of Lipschitz operators we have the following properties.

Proposition 1. Let \(\{S(t); t \geq 0\}\) be a semigroup of Lipschitz operators on \(D\). Then there exist \(M \geq 1\), \(\omega \geq 0\) and a nonnegative functional \(\Phi\) on \(X \times X\) satisfying the following three conditions:

(i) \(\|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| \leq M(\|x_1 - x_2\| + \|y_1 - y_2\|)\) \(\text{for } (x_1, y_1), (x_2, y_2) \in X \times X\),

(ii) \(\|x - y\| \leq \Phi(x, y) \leq M\|x - y\|\) \(\text{for } (x, y) \in D \times D\),

(iii) \(\Phi(S(t)x, S(t)y) \leq e^{\omega t}\Phi(x, y)\) \(\text{for } t \geq 0\) and \((x, y) \in D \times D\).


We consider a semilinear Cauchy problem in \(X\) of the form

\[u'(t) = Au(t) + Bu(t) \quad (t > 0), \quad u(0) = u_0\]  

\((SP; u_0)\).

Here we assume :

(A) \(A\) is the infinitesimal generator of an analytic \(C_0\)-semigroup \(\{T(t); t \geq 0\}\) on \(X\) with

\[\|T(t)\| \leq Const. e^{\omega_A t}\]  \(\text{for all } t \geq 0\), where \(Const. \geq 1\) and \(\omega_A < 0\) are some constants.

Remark 1. We may assume without loss of generality that \(Const. = 1\).

We know that, for any integer \(n \in \mathbb{Z}\), the operator \(A^n\) is defined. We are then concerned with extending the definition for all real exponents \(\alpha \in \mathbb{R}\).

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**Definition 2** (Fractional powers). Let $\alpha > 0$. Define $(-A)^{-\alpha}$ by

$$(-A)^{-\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1}T(t)x\,dt \quad \text{for} \quad x \in X$$

where, $\Gamma(\alpha)$ is the Gamma function. An operator $(-A)^{\alpha}$ is defined by $(-A)^{\alpha} = ((-A)^{-\alpha})^{-1}$.

**Proposition 2.** $(-A)^{\alpha}$ satisfies the following conditions:

(i) For $x \in D((-A)^{\alpha})$

$$T(t)(-A)^{\alpha}x = (-A)^{\alpha}T(t)x \quad \text{for} \quad t > 0.$$  

(ii) For $\alpha > 0$ there exists $M_\alpha > 0$ such that

$$\|(-A)^{\alpha}T(t)\| \leq M_\alpha t^{-\alpha} \quad \text{for} \quad t > 0.$$  

(iii) For $\gamma \in (\alpha, 1)$ there exists $M_{\alpha, \gamma} > 0$ such that

$$\|(-A)^{\alpha}(T(t)x - x)\| \leq M_{\alpha, \gamma} t^{-\alpha} \|(-A)^{\gamma}x\| \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x \in D((-A)^{\gamma}).$$

(iv) If $0 \leq \alpha < \theta < \gamma \leq 1$, then there exists $M_{\alpha, \theta, \gamma} > 0$ such that

$$\|(-A)^{\theta}x\| \leq M_{\alpha, \theta, \gamma} \|(-A)^{\alpha}x\|^{\frac{\theta - \alpha}{\gamma - \alpha}} \quad \text{for} \quad x \in D((-A)^{\gamma}).$$

**Proof.** Cf. H. Tanabe[7].

2. **Assumptions and main result**

Let $\alpha \in (0, 1)$ and $Y = D((-A)^{\alpha})$. Then $Y$ is a Banach space equipped with norm

$$\|v\|_Y := \|(-A)^{\alpha}v\| \quad \text{for} \quad v \in Y \equiv D((-A)^{\alpha})$$

Obviously $Y \subset X$ and $Y$ is dense in $X$ with $X$-norm.

Let $\mathcal{C} = D \cap Y$. We assume that $\mathcal{C}$ is dense in $D$ with $X$-norm. In this case $\mathcal{C}$ is closed in $Y$.

**B** For the operator $B$ we make the following assumptions:

**B-i** The operator $B$ is continuous from $(\mathcal{C}, \| \cdot \|_Y)$ into $(X, \| \cdot \|)$.

**B-ii** There exists $M_B > 0$ such that $\|Bx\| \leq M_B (1 + \|x\|_Y)$ for $x \in \mathcal{C}$.

**Φ** Let $\Phi$ be a nonnegative functional on $X \times X$ satisfying the following two conditions:

**Φ-i** There exists $L \geq 0$ such that

$$|\Phi(x_1, y_1) - \Phi(x_2, y_2)| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|) \quad \text{for} \quad (x_1, y_1), (x_2, y_2) \in X \times X.$$

**Φ-ii** There exist $M \geq m > 0$ such that

$$m\|x - y\| \leq \Phi(x, y) \leq M\|x - y\| \quad \text{for} \quad (x, y) \in D \times D.$$

**F** Let $\{F_h; h \in (0, h_0]\}(h_0 > 0)$ be a family of nonlinear operators from $\mathcal{C}$ into $\mathcal{C}$ which satisfies the following two conditions:

**F-i** There exists $\omega \geq 0$ such that for any sequence $\{h_n\}_{n=1}^{\infty}$ with $h_n \downarrow 0$ as $n \to \infty$ and any bounded sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with respect to $Y$-norm in $\mathcal{C}$,

$$\limsup_{n \to \infty} \left\{ \frac{\Phi(F_{h_n}x_n, F_{h_n}y_n) - \Phi(x_n, y_n)}{h_n} - \omega \Phi(x_n, y_n) \right\} \leq 0.$$

**F-ii** There exists $\beta \in (0, 1)$ such that for any sequence $\{h_n\}_{n=1}^{\infty}$ with $h_n \downarrow 0$ as $n \to \infty$ and any convergence sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{C}$ with respect to $Y$-norm,

$$\lim_{n \to \infty} \frac{\|F_{h_n}x_n - J(h_n)x_n\|}{h_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\|F_{h_n}x_n - J(h_n)x_n\|_Y}{h_n^\beta} = 0.$$
where
\[ J(h)w = T(h)w + \int_0^h T(s)Bw\,ds \quad \text{for} \quad w \in \mathcal{C} \quad \text{and} \quad h > 0. \]  
(7)

**Remark 2.** We may assume that in condition (F-ii), \( \beta \in (0, 1 - \alpha] \).

**Remark 3.** It is easily seen that (F-i) is equivalent to the following condition:

(F-i)' There exists \( \omega \geq 0 \) such that for any \( Y \)-bounded set \( W \subset \mathcal{C} \),
\[
\limsup_{h \to 0} \left( \sup_{x, y \in W} \left\{ \frac{\Phi(F_hx, F_hy) - \Phi(x, y)}{h} - \omega \Phi(x, y) \right\} \right) \leq 0.
\]

The main theorem in this note is given by

**Theorem 1** ([6] Theorem 2.2.). Assume that (B), (Φ) and (F) hold. Then there exists a semigroup \( \{S(t); t \geq 0\} \) of Lipschitz operators on \( D \) such that

(i) \( BS(\cdot)x \in C([0, \infty); \mathcal{X}) \quad \text{for} \quad x \in \mathcal{C} \),
(ii) \( BS(\cdot)x \in C((0, \infty); \mathcal{X}) \cap L^1_{\text{loc}}(0, \infty; \mathcal{X}) \quad \text{for} \quad x \in D \),
(iii) \[ S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds \quad \text{for} \quad x \in D \quad \text{and} \quad t \geq 0. \]  
(8)

Moreover, the following product formula hold:

(iv) \[ S(t)x = \lim_{[0,1]} F_{[t]}^h x \quad \text{for} \quad x \in \mathcal{C} \quad \text{and} \quad t \geq 0, \]  
(9)

where the convergence of (9) is uniform on every compact subset of \([0, \infty)\). Here \([\alpha]\) is the greatest integer that is less than or equal to \( \alpha \).

For the proof of the existence of a semigroup \( \{S(t); t \geq 0\} \) of Lipschitz operators on \( D \) satisfying (i)-(iii) T. Matsumoto, and N. Tanaka used [4] Theorem 5.2. But this theorem treated more general case.

### 3. Mild solutions

We need the following notion of solutions.

**Definition 3.** Let \( u_0 \in D \) and \( \tau > 0 \). A function \( u \in C([0, \tau]; \mathcal{X}) \cap C((0, \tau]; \mathcal{Y}) \) is called a mild solution to \((SP; u_0)\) on \([0, \tau]\) if

(i) \( u(t) \in \mathcal{C} \) for \( t \in (0, \tau] \),
(ii) \( Bu \in C((0, \tau]; \mathcal{X}) \cap L^1(0, \tau; \mathcal{X}) \),
(iii) \( u \) satisfies the integral equation:
\[ u(t) = T(t)u_0 + \int_0^t T(t-s)Bu(s)\,ds \quad \text{for} \quad t \in [0, \tau]. \]  
(10)

A function \( u \in C([0, \infty); \mathcal{X}) \cap C((0, \infty); \mathcal{Y}) \) is called a global mild solution to \((SP; u_0)\) if for each \( \tau > 0 \) the restriction \( u \) to \([0, \tau]\) is a mild solution to \((SP; u_0)\) on \([0, \tau]\).

The continuous dependence of mild solutions to the Cauchy problem for \((SP)\) on their initial data is given by following Proposition.
Proposition 3. Let $\tau > 0$ and $x_1, x_2 \in D$. Let $u : [0, \tau] \rightarrow X$ be a mild solution to $(SP; x_1)$ on $[0, \tau]$ and $v : [0, \tau] \rightarrow X$ be a mild solution to $(SP; x_2)$ on $[0, \tau]$. Suppose that conditions $(\Phi)$ and $(F)$ are satisfied. Then there exist $M > 0$ and $\omega > 0$ such that

$$\|u(t) - v(t)\| \leq M e^{\omega t} \|x_1 - x_2\| \quad \text{for} \quad t \in [0, \tau].$$

Proof. Let $\omega > 0$ be a number appearing in condition $(F-i)$. From $(\Phi-i)$, we have

$$|\Phi(u(s), v(s)) - \Phi(u(t), v(t))| \leq L \left( \|u(s) - u(t)\| + \|v(s) - v(t)\| \right) \quad \text{for} \quad s, t \in [0, \tau].$$

The definition of mild solutions shows that $u, v \in C([0, \tau] ; X)$. Therefore we see that the map $t \mapsto \Phi(u(t), v(t))$ is continuous on $[0, \tau]$. Let $t \in (0, \tau)$ and let $h > 0$ be such that $t + h \leq \tau$. By the semigroup property of $\{T(t) ; t \geq 0\}$ and (10), we obtain that

$$u(t + h) = T(t + h)x + \int_0^{t+h} T(t + h - s)Bu(s)ds$$

$$= T(h)T(t)x + T(h) \int_0^t T(t - s)Bu(s)ds$$

$$- T(h) \int_0^t T(t - s)Bu(s)ds + \int_{t}^{t+h} T(t + h - s)Bu(s)ds$$

$$= T(h) \left( T(t)x + \int_0^t T(t - s)Bu(s)ds \right)$$

$$- \int_0^t T(t + h - s)Bu(s)ds + \int_{t}^{t+h} T(t + h - s)Bu(s)ds$$

$$= T(h)u(t) + \int_{t}^{t+h} T(t + h - s)Bu(s)ds$$

$$= T(h)u(t) + \int_0^{h} T(s)Bu(t + h - s)ds. \quad (11)$$

With this equation and (7) we have

$$u(t + h) = T(h)u(t) + \int_0^{h} T(s)Bu(t)ds$$

$$- \int_0^{h} T(s)Bu(t)ds + \int_{0}^{h} T(s)Bu(t + h - s)ds$$

$$= J(h)u(t) + \int_0^{h} T(s) \left( Bu(t + h - s) - Bu(t) \right)ds. \quad (12)$$

From the definition of mild solutions we get $Bu \in C(\{0, \tau\}; X)$. Then with assumption (A) it follows that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^{h} \|T(s) \left( Bu(t + h - s) - Bu(t) \right)\|ds = 0. \quad (13)$$

(12) and (13) yields that

$$\lim_{h \downarrow 0} \frac{1}{h} \|u(t + h) - J(h)u(t)\| = 0. \quad (14)$$

Similarly we have

$$\lim_{h \downarrow 0} \frac{1}{h} \|v(t + h) - J(h)v(t)\| = 0. \quad (15)$$
With condition \((\Phi-i)\), we have the following estimate:

\[
\frac{1}{h} \left( \Phi(u(t + h), v(t + h)) - \Phi(u(t), v(t)) \right) \\
\leq \frac{1}{h} \left( \Phi(J(h)u(t), J(h)v(t)) - \Phi(u(t), v(t)) \right) \\
+ L \frac{1}{h} \left( \|u(t + h) - J(h)u(t)\| + \|v(t + h) - J(h)v(t)\| \right) \\
\leq \frac{1}{h} \left( \Phi(F_h u(t), F_h v(t)) - \Phi(u(t), v(t)) \right) \\
+ L \frac{1}{h} \left\{ \|u(t + h) - J(h)u(t)\| + \|v(t + h) - J(h)v(t)\| \\
+ \|J(h)u(t) - F_h u(t)\| + \|J(h)v(t) - F_h v(t)\| \right\}. \tag{16}
\]

From (14),(15) and condition \((F)\) we obtain that

\[
\lim_{h \to 0} \sup \frac{1}{h} \left( \Phi(u(t + h), v(t + h)) - \Phi(u(t), v(t)) \right) \\
\leq \lim_{h \to 0} \sup \frac{1}{h} \left( \Phi(F_h u(t), F_h v(t)) - \Phi(u(t), v(t)) \right) \leq \omega \Phi(u(t), v(t)). \tag{17}
\]

Therefore we have

\[
D^+ \Phi(u(t), v(t)) \leq \omega \Phi(u(t), v(t)) \quad \text{for} \quad t \in (0, \tau), \tag{18}
\]

where \(D^+\) denotes the upper right Dini derivative which defined by

\[
D^+ f(a) = \limsup_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

Since \(\Phi(u(\cdot), v(\cdot))\) is continuous on \([0, \tau]\) solving the differential inequality (18) yields that

\[
\Phi(u(t), v(t)) \leq e^{\omega t} \Phi(x_1, x_2) \quad \text{for} \quad t \in [0, \tau].
\]

An application of condition \((\Phi-ii)\) shows that

\[
\|u(t) - v(t)\| \leq \frac{1}{m} e^{\omega t} \Phi(x_1, x_2) \leq \frac{M}{m} e^{\omega t} \|x_1 - x_2\| \quad \text{for} \quad t \in [0, \tau]. \tag{19}
\]

Then we obtain the desired inequality. \hfill \square

**Proposition 4.** Suppose that \((\Phi)\) and \((F)\) are satisfied. Suppose that for each \(x \in C\) there exist \(\tau > 0\) and a mild solution \(u\) to \((SP; x)\) on \([0, \tau]\). Then for every \(x \in C\) there exists a global mild solution \(u\) to \((SP; x)\).

**Proposition 5.** Suppose that \((\Phi)\) and \((F)\) are satisfied. Suppose that for each \(x \in C\) there exist a global mild solution \(u\) to \((SP; x)\). Then for every \(x \in D\) there exists a global mild solution \(u\) to \((SP; x)\).

**Proof.** From Proposition 2.5 in [4](resp Proposition 2.6 in [4]) with \(\varphi\) defined by

\[
(\varphi) \left\{ \begin{array}{ll}
\varphi(x) = 0 & x \in D \\
\varphi(x) = \infty & x \in X \setminus D
\end{array} \right.,
\]

we have Proposition 4 (resp. Proposition 5). \hfill \square
4. Key Estimate

In this section we give a key estimate to showing the convergence of approximate solutions.

**Lemma 1.** There exists $K \geq 1$ such that for any $\tau \in (0, 1]$ and for any finite sequence $\{s_k\}_{k=0}^{N}$ satisfying $0 \leq s_0 < s_1 < \cdots < s_N \leq \tau$, the following two assertions hold:

(i) Let $M_G > 0$ and let $G : [0, \tau) \to X$ be a measurable function satisfying $\|G(\xi)\| \leq M_G$ for $\xi \in [0, \tau)$. Then
\[
\int_{s_i}^{s_{i+1}} \|T(s_i - \xi)G(\xi)\| d\xi \leq K M_G (s_i - s_{i+1})^3 \quad \text{for} \quad 0 \leq i \leq N.
\]

(ii) Let $\varepsilon > 0$. Then for any finite sequence $\{\zeta_i\}_{i=1}^{N}$ in $Y$ satisfying $\|\zeta_i\| \leq \varepsilon (s_i - s_{i-1})$ and $\|\zeta_i\|_Y \leq \varepsilon (s_i - s_{i-1})^3$ for $1 \leq i \leq N$, we have
\[
\sum_{i=k+1}^{N} \|T(s_i - s_k)\zeta_i\|_Y \leq K \varepsilon (s_i - s_k)^3 \quad \text{for} \quad 0 \leq k \leq i \leq N.
\]

Here as usual we put $\sum_{i=k+1}^{N} = 0$.


In the rest of this paper the symbol $K$ stands for the constant specified in Lemma 1 and we define
\[
E_h w = F_h w - J(h) w \quad \text{for} \quad h \in (0, h_0] \quad \text{and} \quad w \in C.
\]

For $w_0 \in C$, $h > 0$, $\rho > 0$, $M > 0$ and $\varepsilon > 0$ we introduce the condition
\[
V(w_0; h, \rho, M, \varepsilon) \equiv \left\{ \begin{array}{ll}
(i) & \|Bx\| \leq M \quad \text{for} \quad x \in U_Y(w_0, \rho) \cap C, \\
(ii) & K(M + \varepsilon)h^\beta \sup_{s \in [0, \rho]} \|T(s)w_0 - w_0\|_Y \leq \rho.
\end{array} \right\}
\]

where $U_Y(w_0, \rho)$ denotes the closed ball in $Y$ with center $w_0$ and radius $\rho$ and $\beta$ is a constant appearing in condition (F-ii).

**Lemma 2.** Let $w_0 \in C$. Assume that $0 < h \leq 1$, $\rho > 0$, $M > 0$ and $\varepsilon > 0$, satisfy condition $V(w_0; h, \rho, M, \varepsilon)$. And take $\sigma > 0$ satisfy $\sigma \leq h$. Assume that there exists a sequence $\{(s_i, w_i, \zeta_i)\}_{i=1}^{N}$ in $[0, \sigma] \times C \times Y$ satisfies the following three conditions:

(i) $0 = s_0 < s_1 < \cdots < s_N \leq \sigma$,

(ii) $w_i = T(s_i - s_{i-1})w_{i-1} + \int_{s_{i-1}}^{s_i} T(s_i - \xi)Bw_{i-1}d\xi + \zeta_i \quad \text{for} \quad 1 \leq i \leq N$,

(iii) $\|\zeta_i\| \leq \varepsilon (s_i - s_{i-1})$ and $\|\zeta_i\|_Y \leq \varepsilon (s_i - s_{i-1})^3 \quad \text{for} \quad 1 \leq i \leq N$.

Then the following assertions (a) and (b) hold:

(a) We have the following estimates with $X$-norm and $Y$-norm respectivery:

\[\begin{align*}
& (a-1) \|T(s_j - s_k)w_k - w_j\| \leq (M + \varepsilon)(s_j - s_k) \quad \text{for} \quad 0 \leq k \leq j \leq N, \\
& (a-2) \|T(s_j - s_k)w_k - w_j\|_Y \leq K(M + \varepsilon)(s_j - s_k)^3 \quad \text{for} \quad 0 \leq k \leq j \leq N.
\end{align*}\]

(b) $w_j \in U_Y(w_0, \rho)$ and $\|Bw_j\| \leq M \quad \text{for} \quad 0 \leq j \leq N$.

**Proof.** To prove this lemma we use Lemma 1 inductively. □
Given \((t_0, x_0) \in [0, \infty) \times C\) we set

\[
\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^N) \equiv \left\{ \begin{array}{l}
(i) 0 = t_0 < t_1 < \ldots < t_N < \tau,
(ii) t_j - t_{j-1} \leq \varepsilon
(iii) x_j = T(t_j - t_{j-1})x_{j-1} + \int_{t_{j-1}}^{t_j} T(t - \xi)Bx_{j-1}d\xi + \zeta_j,
(iv) \|\zeta_j\| \leq \varepsilon(t_j - t_{j-1}) \quad \text{and} \quad \|\zeta_j\|_{Y} \leq \varepsilon(t_j - t_{j-1})^3
(v) \text{If } x \in C \text{ satisfies the inequality}
\quad \|x - x_j\|_{Y} \leq K(M_B + 1)(t_j - t_{j-1})^3 + \sup_{s \in [0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_j\|_{Y}
\quad \text{then} \quad \|Bx - Bx_{j-1}\| \leq \frac{\varepsilon}{4K}
(vi) (t_j - t_{j-1})(M_B + 1) + \sup_{s \in [0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_j\| \leq \varepsilon
\end{array} \right. 
\]

where \(j = 1, 2, \ldots, N\).

\[(vii) \lim_{j \to \infty} t_j = \tau.\]

**Proposition 6.** Suppose that condition (F) is satisfied. Let \(x_0 \in C\) and \(\varepsilon \in (0, 1/2]\). Assume that \(\tau \in (0, 1]\), \(\rho_0 > 0\) and \(M_B > 0\) satisfy condition \(V(x_0; \tau, \rho_0, M_B, 1)\). Then there exists a sequence \(\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty\) in \([0, \tau) \times C \times Y\) satisfying the condition \(\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^\infty)\) and condition (vii).

**Proof.** We shall construct inductively a sequence \(\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty\) in \([0, \tau) \times C \times Y\) satisfying condition \(\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^\infty)\). For this purpose, let \(i \in \mathbb{N}\) and assume that a sequence \(\{(t_j, x_j, \zeta_j)\}_{j=1}^{i-1}\) in \([0, \tau) \times C \times Y\) can be constructed so that it satisfies condition \(\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{i-1})\). For \(h > 0, t \in [0, \tau), y \in C\) and \(\varepsilon > 0\) we set

\[
\theta(h; t, y, \varepsilon) \equiv \left\{ \begin{array}{l}
h < \tau - t,
(h(M_B + 1) + \sup_{s \in [0, h]} \|T(s)y - y\| \leq \varepsilon,
\|Bx - By\| \leq \frac{\varepsilon}{4K} \quad \text{for} \quad x \in U_Y(y, \rho) \cap C,
\text{where} \quad \rho = K(M_B + 1)h^3 + \sup_{s \in [0, h]} \|T(s)y - y\|_Y
\end{array} \right. 
\]

By condition (B-i), the strong continuity of \(T(\cdot)\) and (F-ii), there exist \(h \in (0, \varepsilon]\) such that

\[
\|E_{h,x_{i-1}}\| \leq h \varepsilon \quad \text{and} \quad \|E_{h,x_{i-1}}\|_Y \leq h^3 \varepsilon
\]

and \((h; t_{i-1}, x_{i-1}, \varepsilon)\) satisfying condition \(\theta(h; t_{i-1}, x_{i-1}, \varepsilon)\). We define \(\bar{h}_i\) by supremum of such numbers \(h\). Then there exists \(h_i \in (0, \varepsilon]\) such that \(\bar{h}_i/2 < h_i\) which satisfy \(\theta(h_i; t_{i-1}, x_{i-1}, \varepsilon)\). We set \(t_i = t_{i-1} + h_i\), then condition (ii) is satisfied. From (22) we get conditions (i), (vi) and (v) in \(\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{i})\). Next we shall show that there exist \(x_i \in C\) and \(\zeta_i \in Y\) satisfying (iii) and (iv) in \(\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{i})\). Here, we define \(x_i = F_{h_i,x_{i-1}}\) and \(\zeta_i = E_{h_i,x_{i-1}}\). Obviously \(F_{h_i,x_{i-1}} \in C\) and \(E_{h_i,x_{i-1}} \in Y\) and condition (iv) is satisfied by (23). With (7) and (20), we have

\[
x_i = F_{h_i,x_{i-1}} = J(h_i)x_{i-1} + E_{h_i,x_{i-1}}
= T(h_i)x_{i-1} + \int_{0}^{h_i} T(s)Bx_{i-1}ds + E_{h_i,x_{i-1}}
= T(t_i - t_{i-1})x_i - 1 + \int_{t_i-1}^{t_i} T(t - s)Bx_{i-1}ds + \zeta_i.
\]

It remains to show that condition (vii) is satisfied. We can show it in a way similar to that of T. Matsumoto, and N. Tanaka[4,Proposition 3.7]. It is concluded that a sequence
\{(t_j, x_j, \zeta_j)\}_{j=1}^\infty in [0, \tau) \times C \times Y can be constructed so that the condition \(W(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^\infty)\) and condition (vii) are satisfied.

**Proposition 7.** Let \(x_0 \in C\), \(0 < \tilde{\tau} \leq \min\{\tau, 1\}\), \(\rho_0 > 0\), \(M_B > 0\) and \(0 < \varepsilon, \lambda, \mu \leq 1/2\) and suppose condition \(V(x_0; \tilde{\tau}, \rho_0, M_B, 1)\) satisfied. For each \(\varepsilon = \lambda or \mu\), suppose that there exists a sequence \((t_j^\varepsilon, x_j^\varepsilon, \zeta_j^\varepsilon)\) in \([0, \tilde{\tau}) \times C \times Y\) satisfying conditions in \(W(\tilde{\tau}; \varepsilon, \{(t_j^\varepsilon, x_j^\varepsilon, \zeta_j^\varepsilon)\}_{j=1}^\infty)\) and condition (vii). Set \(P = \{t_i^\varepsilon; i = 0, 1, \cdots\} \cup \{t_j^\varepsilon; j = 0, 1, \cdots\}\), and define \(s_0 = 0 and s_k = \inf(P \setminus \{s_0, s_1, \cdots, s_{k-1}\})(k \in \mathbb{N})\). Then there exists a sequence \((z_k^\varepsilon, z_k^\varepsilon)\) in \(C \times C\) satisfying the following conditions (where \(\varepsilon = \lambda or \mu\)):

(a) If \(s_k = t_i^\varepsilon\), then \(z_k^\varepsilon = x_i^\varepsilon\).

(b) If \(s_k \neq t_i^\varepsilon\), then the element \(f_k^\varepsilon\) on \(Y\) defined by

\[
f_k^\varepsilon = T(s_k - s_{k-1})z_k^\varepsilon_{k-1} + \int_{s_{k-1}}^{s_k} T(s_k - \xi)Bz_k^\varepsilon_{k-1}d\xi - z_k^\varepsilon,
\]

satisfies \(\|f_k^\varepsilon\| \leq \varepsilon(s_k - s_{k-1})\) and \(\|f_k^\varepsilon\| \leq \varepsilon(s_k - s_{k-1})\).

(c) \(\Phi(z_k^\varepsilon, z_k^\varepsilon) \leq e^{\omega X}\{L(\lambda + \mu)\tilde{\tau} + \eta(\lambda, \mu)\}\) for \(k \geq 0\), where

\[
\eta(\lambda, \mu) = 3L \left( \lambda \sum_{t_i^\varepsilon \in \{s_1, \cdots, s_k\}} (t_i^\varepsilon - t_{i-1}^\varepsilon) + \mu \sum_{t_j^\varepsilon \in \{s_1, \cdots, s_k\}} (t_j^\varepsilon - t_{j-1}^\varepsilon) \right).
\]

Here \(\omega\) is constants appearing in condition (F-i).

**Proof.** The proof is assured by Proposition 4.2 in [4] with \(\varphi\) defined by

\[
(\varphi) \begin{cases} 
\varphi(x) = 0 & x \in D \\
\varphi(x) = \infty & x \in X \setminus D
\end{cases}
\]

\(\square\)

5. **Characterization of Semigroups**

We characterize semigroups of locally Lipschitz operators associated with semilinear evolution equations of parabolic type.

**Theorem 2.** Assume that condition (B) is satisfied. Then, the following two statements are equivalent:

(i) There exists a semigroup \(\{S(t); t \geq 0\}\) of Lipschitz operators on \(D\) such that for each \(x \in D\), \(S(\cdot)x\) is a global mild solution to \((SP; x)\).

(ii) There exist a nonnegative functional \(\Phi\) on \(X \times X\) satisfying conditions (F) and a family \(\{F_h; h \in (0, h_0]\}\) of nonlinear operators from \(C\) into \(C\) satisfying conditions (F).

**Proof.** We begin by showing that (i) implies (ii). Applying Proposition 1 with \(L = M\) and \(m = 1\) there exists a nonnegative functional \(\Phi\) on \(X \times X\) satisfying condition (F). It remains to check the existence of a family \(\{F_h; h \in (0, h_0]\}\) of nonlinear operators from \(C\) into \(C\) satisfying conditions (F). Let \(h > 0\). From (iv) in Proposition 1 we have

\[
\Phi(S(h)x, S(h)y) \leq e^{\omega h}\Phi(x, y) \quad \text{for} \quad (x, y) \in D \times D.
\]

Then from the definition of mild solution we obtain that \(S(h)x\) belongs to \(C\). We define \(F_hx = S(h)x\). Now we shall show that \(\{F_h; h \in (0, h_0]\}\) satisfies condition (F).
Let $W$ be a bounded subset of $\mathcal{C}$ with respect to $Y$-norm. By (26), we have
\[
\frac{1}{h} \left( \Phi(F_h x, F_h y) - \Phi(x, y) \right) - \omega \Phi(x, y)
\]
\[
= \frac{1}{h} \left( \Phi(S(h)x, S(h)y) - \Phi(x, y) \right) - \omega \Phi(x, y)
\]
\[
\leq \left( \frac{1}{h} (e^{\omega h} - 1) - \omega \right) \Phi(x, y)
\]
for $h \in (0, h_0]$ and $(x, y) \in W \times W$.

Since $W$ is bounded in $Y$, we have $\sup \{\Phi(x, y); (x, y) \in W \times W\} < \infty$. This and (27) imply that condition (F-i)' is satisfied. That is to say, condition (F-i) is valid. Next we shall show condition (F-ii). Let take any sequence $\{h_n\}_{n=1}^{\infty}$ such that $h_n \downarrow 0$ as $n \to \infty$ and any convergence sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{C}$.

Note that $S(\cdot)x$ is a mild solution in $(SP; x)$. From (7) and (10) we obtain that
\[
F_h x - J(h)x = S(h)x - J(h)x
\]
\[
= \left( T(h)x + \int_0^h T(h - s)BS(s)xdx \right) - \left( T(h)x + \int_0^h T(s)Bxdx \right)
\]
\[
= \int_0^h T(h - s)(BS(s)x - Bx)dx.
\]

From (28) we have
\[
\|F_{h_n} x_n - J(h_n)x_n\| \leq \frac{1}{h_n} \int_0^{h_n} \|T(h_n - s)(BS(s)x_n - Bx_n)\|ds
\]
\[
\leq \max_{0 \leq s \leq h_n} \|BS(s)x_n - Bx_n\|.
\]

With the strong continuity of $S(\cdot)$ and condition (B-i), from (29) it follows that
\[
\lim_{n \to \infty} \frac{\|F_{h_n} x_n - J(h_n)x_n\|}{h_n} = 0.
\]

By (3) and (6), it follows that
\[
\|F_{h_n} x_n - J(h_n)x_n\| \leq \int_0^{h_n} \|T(h_n - s)(BS(s)x_n - Bx_n)\|ds
\]
\[
= \int_0^{h_n} \|(-A)^{\alpha}T(h_n - s)(BS(s)x_n - Bx_n)\|ds
\]
\[
\leq \int_0^{h_n} \|A^{1-\alpha}BS(s)x_n - Bx_n\|ds
\]
\[
\leq M_\alpha \frac{1}{1 - \alpha} h_n^{1-\alpha} \max_{0 \leq s \leq h_n} \|BS(s)x_n - Bx_n\|.
\]

With the strong continuity of $S(\cdot)$ and condition (B-i), from (31) we have that
\[
\lim_{n \to \infty} \frac{\|F_{h_n} x_n - J(h_n)x_n\|}{h_n^{1-\alpha}} = 0.
\]

If $\beta = 1 - \alpha$, then (32) is the desired estimate. Therefore condition (F-ii) is showed.

To prove the converse implication, let $x_0 \in \mathcal{C}$. Then, condition (B-i) ensures the existence of $\rho_0 > 0$ and $M_B > 0$ satisfying condition $V(x_0; \tau, \rho_0, M_B, 1)$. Therefore, Proposition 6 asserts that for each $\varepsilon \in (0, 1/2]$ there exists a sequence $\{(t_j^\varepsilon, x_j^\varepsilon, \zeta_j^\varepsilon)\}_{j=1}^{\infty}$ in $[0, \tau) \times \mathcal{C} \times Y$.
satisfying \( W(\tau; \varepsilon, \{(t^\varepsilon_i, x^\varepsilon_i, \zeta^\varepsilon_i)\}_{i=1}^{\infty}) \) and condition (vii). For each \( \varepsilon \in (0, 1/2] \), we define a family \( \{u^\varepsilon\} \) of step functions by
\[
 u^\varepsilon(t) = x^\varepsilon_i \quad \text{for} \quad t \in [t^\varepsilon_i, t^\varepsilon_{i+1}) \quad \text{and} \quad i \in \mathbb{N}.
\]
The purpose is to demonstrate that the family \( \{u^\varepsilon\} \) converges in the space \( C([0, \tau]; X) \cap C((0, \tau]; Y) \). For this purpose, let \( \lambda, \mu \in (0, 1/2] \), and let \( \{s_k\}_{k=0}^{\infty} \) be a sequence constructed as in Proposition 7. Then, applying Proposition 7 we find a sequence \( \{(z^\lambda_k, z^\mu_k)\}_{k=0}^{\infty} \in C \times C \) satisfying (a),(b) and (c) in Proposition 7, which plays an important role in accomplishing the above-mentioned purpose. In the following, \( \omega \) stands for the constants in (c), which are specified by condition (F-i) in Proposition 7.

The first step: We shall show that \( \lim_{\varepsilon \to 0} u^\varepsilon(t) = u(t) \) in \( X \). Let \( t \in [0, \tau) \). We begin by estimating the difference \( |u^\varepsilon(t) - u^\mu(t)| \). Take \( i, j, k \in \mathbb{N} \) such that:
\[
 t \in [s_{k-1}, s_k), \quad t^\lambda_{i-1} \leq s_{k-1} < s_k \leq t^\lambda_i \quad \text{and} \quad t^\mu_{j-1} \leq s_{k-1} < s_k \leq t^\mu_j.
\]
Then, from the definition of \( u^\varepsilon \) we have \( u^\lambda(t) = x^\lambda_{i-1} \) and \( u^\mu(t) = x^\mu_{j-1} \). Take \( p \in \mathbb{Z} \) such that \( t^\lambda_{i-1} = s_p \). By (a) in Lemma 7, we have \( z^\lambda_p = x^\lambda_{i-1} \). From Lemma 1 it follows that \( \|B_x \| \leq M_B \). This inequality and condition (v) together imply that,
\[
 \|Bx\| \leq M_B + \frac{\lambda}{4K} \quad \text{for} \quad x \in U_Y(x^\lambda_{i-1}, \rho_i \lambda) \cap C.
\]
It follows (b) in Lemma 7 that
\[
 z^\varepsilon_k = T(s_k - s_{k-1})z^\varepsilon_{k-1} + \int_{s_{k-1}}^{s_k} T(s_k - \xi)Bz^\varepsilon_{k-1}d\xi - f^\varepsilon_k,
\]
satisfies \( \|f^\varepsilon_k\| \leq \varepsilon(s_k - s_{k-1}) \) and \( \|f^\varepsilon_k\|_Y \leq \varepsilon(s_k - s_{k-1})^2 \). Since \( 0 = s_p - t^\lambda_{i-1} < s_{p+1} - t^\lambda_{i-1} < \cdots < s_k - t^\lambda_{i-1} < \cdots < t^\lambda_i - t^\lambda_{i-1} \). We apply the sequence \( \{(z^\lambda_{p+k} - t^\lambda_{i-1}, x^\mu_{p+k} - f^\lambda_{p+k})\}_{k=1}^{\infty} \) in \( [0, t^\lambda_i - t^\lambda_{i-1}] \times C \times Y \) for (a-1) in Lemma 2, it follows that
\[
 \|z^\lambda_{k-1} - T(s_{k-1} - t^\lambda_{i-1})x^\lambda_{i-1}\| \leq (M_B + \frac{\lambda}{4K} + \lambda)(s_{k-1} - t^\lambda_{i-1}).
\]
This inequality and (vi) in Lemma 6 together imply that \( \|z^\lambda_{k-1} - x^\lambda_{i-1}\| \leq \lambda \). Similarly we have \( \|z^\mu_{k-1} - x^\mu_{j-1}\| \leq \mu \). Since it follows from (F-i) that
\[
 |\Phi(x^\lambda_{i-1}, x^\mu_{j-1}) - \Phi(z^\lambda_{k-1}, z^\mu_{k-1})| \leq L\left(\|x^\lambda_{i-1} - z^\lambda_{k-1}\| + \|x^\mu_{j-1} - z^\mu_{k-1}\|\right) \leq L(\lambda + \mu).
\]
With inequality (33), (F-ii) and (c) in proposition 7, we obtain that
\[
 m\|u^\lambda(t) - u^\mu(t)\| = m\|x^\lambda_{i-1} - x^\mu_{j-1}\| \leq \Phi(x^\lambda_{i-1}, x^\mu_{j-1}) \leq \Phi(z^\lambda_{k-1}, z^\mu_{k-1}) + L(\lambda + \mu) \leq e^{\omega \tau}\left\{L(\lambda + \mu)\tau + \eta_{k-1}(\lambda, \mu)\right\} + L(\lambda + \mu) \leq 4Le^{\omega \tau}(\lambda + \mu)\tau + L(\lambda + \mu).
\]
This implies the existence of a measurable function \( u : [0, \tau) \to X \) such that \( \lim_{\varepsilon \to 0} u^\varepsilon(t) = u(t) \) uniformly for \( t \in [0, \tau) \).

The second step: We shall show that for any \( t \in (0, \tau) \), \( \lim_{\varepsilon \to 0} u^\varepsilon(t) = u(t) \) in \( Y \).

The third step: We shall prove that \( u \in C(0, \tau); X) \cap C((0, \tau); Y) \).

The proof of The second step and The third step is obtained in a way similar to that of T.
Matsumoto, and N. Tanaka [4, Theorem 5.2]. Therefore we have proved that to each \( x \in \mathcal{C} \) there corresponds \( \tau_x > 0 \) such that the \( (SP; x) \) has a mild solution \( u \) on \([0, \tau_x]\). Proposition 4 and Proposition 5 therefore assert that for any \( x \in D \) and \( t \geq 0 \), the \( (SP; x) \) has a global mild solution \( u(t; x) \). Next we shall show that the family \( \{S(t)x; t \geq 0\} \), defined by \( S(t)x = u(t; x) \) for \( x \in D \) and \( t \geq 0 \), is a semigroup of locally Lipschitz operators on \( D \). From the semigroup property of \( T(\cdot) \) it follows that

\[
S(0)x = u(0; x) = x,
\]

\[
S(t+s)x = u(t+s; x) = T(t+s)x + \int_0^{t+s} T(t+s-\xi)Bu(\xi)d\xi
\]

\[
= T(t)T(s)x + \int_0^{t+s} T(t)T(s-\xi)Bu(\xi)d\xi
\]

\[
= T(t)\left(T(s)x + \int_0^s T(s-\xi)Bu(\xi)d\xi\right) + \int_s^{t+s} T(t+s-\xi)Bu(\xi)d\xi
\]

\[
= T(t)u(s; x) + \int_0^t T(t-\xi)Bu(\xi+s)d\xi
\]

\[
= u(t; u(s)) = S(t)u(s; x) = S(t)x.
\]

Therefore we obtain the semigroup property of \( \{S(t); t \geq 0\} \). Note that \( u(t; x) \) is a global mild solution. For each \( \tau > 0 \) we have that \( S(\cdot)x = u(\cdot) \in C([0, \tau]; X) \). It proved that \( S(\cdot)x : [0, \infty) \to X \) is continuous. Finally, we check condition (S3). With Lemma 3 we have

\[
\|S(t)x - S(t)y\| = \|u(t; x) - u(t; y)\| \leq Me^{\omega t}\|x - y\|.
\]

If we take \( L = Me^{\omega t} \), we obtain the estimate in (S3).

The above argument proves that there exists semigroup \( \{S(t)x; t \geq 0\} \) of locally Lipschitz operators on \( D \), which is a global mild solutions to \( (SP; x) \).

\[ \square \]

6. Proof of the theorem 1


(ii) and (iii) in Theorem 1 is assured by Theorem 2. (i) follows from Theorem 2 and condition (B-i) too. The proof of (iv) follows the one given in T. Matsumoto, and N. Tanaka [6, Chapter 4]. Then the proof is complete.

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References

[6. Chapter 4]


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