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Pell Equation. V. Systematic relation between the Pythagorean triples and Pell equations

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Abstract Systematic relations between the algebra of the Pell equations, $x^2 - D\,y^2 = 1$ (called Pell-1) and $x^2 - D\,y^2 = -1$ (called Llep-1), and the geometry of Pythagorean triangles or Pythagorean triples (PTs) are discussed. Although Llep-1 is solvable only for a limited number (though extending to infinity) of D values, such an algorithm is obtained that can construct a series of PTs corresponding to each D and involving rational number approximation of the square root of D. In the case of Pell-1, which is solvable for all square-free D, a simple algorithm is found for odd D, whereas some modification is necessary for even D. For each series of PTs thus obtained interesting properties regarding their recursive relations are found.

1. Introduction

Let us call Eqns. (1.1) and (1.2), respectively, Pell-N and Llep-N,[1-5]

$$x^2 - D\,y^2 = N, \qquad \text{(Pell-}N\text{)} \tag{1.1}$$

$$x^2 - D\,y^2 = -N, \qquad \text{(Llep-}N\text{)} \tag{1.2}$$

where natural number solutions (x, y) are to be sought for square-free natural number D.[1-8] The Llep-1 is sometimes called "negative Pell equation." [9] When the union of Pell-N and Llep-N is discussed, it is called Pellep-N.

A trio of positive integers (X, Y, Z) is called Pythagorean triple (PT), if they satisfy

$$X^2 + Y^2 = Z^2 \tag{1.3}$$

for forming a right triangle. If all the edge lengths have no common factor, the Pythagorean triangle or triple is called primitive[10] and abbreviated as pPT. In this paper there may appear a trio of (X, Y, Z) satisfying (1.3) but containing zero or a negative integer in a formal discussion of algebra. These trios are also included into the family of PTs.

It has long been known that a pPT (X, Y, Z) can be expressed by a pair of integers $(m>n)$ which belong to different parities and are prime with each other as follows:

$$X = m^2 - n^2$$

$$Y = 2\,m\,n \qquad (1.4)$$

$$Z = m^2 + n^2 .$$

Then X and Z are always odd, whereas Y is even.

Beiler pointed out that the PTs with consecutive legs, *i.e.*, with $|X-Y|=1$, can be obtained from the solutions of Pellep-1.[6)] Recently, Grytczuk, Luca, and Wojtowicz found an interesting theorem that the solvability of Llep-1 depends on the existence of a pPT which are derived from the constant D in (1.2) with square N.[9)] Quite recently the present author has shown[11)] that many series of PTs can be obtained from the solutions of several Pellep-N^2 and graph-theoretical interpretation for these results are derived by using the topological index Z which was proposed and developed by the present author[12-14)] In this paper more systematic relations between the algebra of Pell-1 and Llep-1 with the geometry of pPTs will be introduced. Further, as a byproduct of this study quite an effective algorithm for finding a series of rational number approximation of a certain kind of quadratic surds will be introduced.

In Section 2. the well-known case of (1, 1, v2) PTs, or "PTs with consecutive legs", will be explained with newly obtained results.

2. pPT with consecutive legs

First consider the series of pPTs with "consecutive legs", whose elements (X_k, Y_k, Z_k) are expressed by (a_k, a_k±1, c_k). Then we have

$$a^2 + (a\pm1)^2 = c^2, \qquad (2.1)$$

which can be transformed into

$$4a^2 \pm 4a + 1 = 2c^2 - 1. \qquad (2.2)$$

By putting

$$2a\pm1 = x \qquad (2.3)$$

and $$c = y, \qquad (2.4)$$

the following Llep-1 can be obtained,

$$x^2 - 2y^2 = -1. \qquad (2.5)$$

According to the standard recipe the lower members of the solutions x_k and y_k (= Z_k) of Llep-1 (2.5) can be obtained as in Table 1. Then by using (2.3) the value of a_k (= X_k) and either of the ± signs in (2.3) and (2.6) are obtained. At last the remaining Y_k can be obtained as

$$Y_k = a_k \pm 1, \qquad (2.6)$$

with the same ± sign as obtained above. Note that the ± sign in (2.3) and (2.6), which should be equal to that in (2.1), alternates, and all the Y_k members become even by this choice. By following the above procedure the set of edges (X_k, Y_k, Z_k) for the pPTs with the consecutive legs are obtained as given in Table 1, where the set of (m,n) values corresponding to (1.4) are also given. Since these pPTs converge to the isosceles right triangle, the values of $(X_k + Y_k)/Z_k$ should approach the square root of two as shown in Table 1. This series of numbers form a rather efficient rational number approximation to the square root of two, but not yet the best one.

Table 1. The edges (X_k, Y_k, Z_k) of the pPTs with consecutive legs as the solution of Llep-1 with D=2.

k	0	1	2	3	4	5
$x_k = 2a_k \pm 1$	1	7	41	239	1393	8119
$y_k = Z_k$	1	5	29	169	985	5741
±	−	+	−	+	−	+
$a_k = X_k$	1	3	21	119	697	4059
$Y_k = a_k \pm 1$	0	4	20	120	696	4060
m	1	2	5	12	29	70
n	0	1	2	5	12	29
$(X_k+Y_k)/Z_k$	1/1	7/5	41/29	239/169	1393/985	8119/5741
$= x_k/y_k$	<u>1</u>	<u>1.4</u>	<u>1.41</u>37	<u>1.4142</u>01	<u>1.414213</u>19	<u>1.41421355</u>2

The results in Table 1 have been known for a long time[6)] far back to Theodoros, a Greek philosopher in fourth century B.C.[15)] The Z_k series are found to be every other members of the Pell numbers P_k which are defined by

$$P_k = 2P_{k-1} + P_{k-2} \tag{2.7}$$

with $$P_0 = 1 \quad \text{and} \quad P_1 = 2. \tag{2.8}$$

This series of numbers were first documented by Theon of Smyrna, a Greek philosopher in second century.[16)] Note, however, that the initial condition for P_k is conventionally chosen as[16-18)]

$$P_1 = 1 \quad \text{and} \quad P_2 = 2. \tag{2.9}$$

Further, X_k and Y_k in Table 1 are found to be closely related also to the Pell numbers. Namely, all the pairs of (m,n) are composed of a pair of consecutive members of the Pell numbers, as

$$\begin{aligned} X_k &= P_k^2 - P_{k-1}^2 \\ Y_k &= 2\, P_k\, P_{k-1} \\ Z_k &= P_k^2 + P_{k-1}^2. \end{aligned} \tag{2.10}$$

Every other Pell numbers are known to obey the following recursive relation,

$$Z_k : f_k = 6\, f_{k-1} - f_{k-2}\,. \tag{2.11}$$

Further, X_k and Y_k are found to obey the following recursive relation,

$$X_k, Y_k : f_k = 5\, f_{k-1} + 5\, f_{k-2} - f_{k-3}\,. \tag{2.12}$$

Hall[19)] and Roberts[20)] have shown that all the family members of pPTs can be generated from the fundamental pPT (3, 4, 5) by multiplying three 3×3 matrices, **U**, **A**, and **D**, and especially the series of pPTs with consecutive legs given in Table 1 can be generated by using **A** as

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}. \tag{2.13}$$

For example, we have

(3 4 5)**A** = (21 20 29), (21 20 29)**A** = (119 120 169), (119 120 169)**A** = (697 696 985), *etc*..

Note that (3 4 5) can be generated from non pPT (1 0 1) as

(1 0 1)**A** = (3 4 5).

The present author has developed the operator technique for effective manipulation of complicated recursive equations.[21,22) By using this technique the close connection between the Hall's matrix and the recursive formulas (2.11) and (2.12) can be shown as follows. First define the step-up operator $\hat{\mathbf{O}}$ as

$$\hat{\mathbf{O}} f_k = f_{k+1}, \tag{2.14}$$

where f may be any series of numbers, functions, or matrices. By operating matrix **A** for the column matrix $(X_k, Y_k, Z_k)^T$ one gets

$$\mathbf{A}\,(X_k, Y_k, Z_k)^T = (X_{k+1}, Y_{k+1}, Z_{k+1})^T. \tag{2.15}$$

Similar matrix relation can be obtained by the combined use of the unit matrix **E** and the step-up operator (2.14) as

$$\hat{\mathbf{O}}\mathbf{E}(X_k\ Y_k\ Z_k)^T = (X_{k+1}\ Y_{k+1}\ Z_{k+1})^T. \tag{2.16}$$

By equating (2.15) with (2.16) one gets

$$(\mathbf{A} - \hat{\mathbf{O}}\mathbf{E})(X_k\ Y_k\ Z_k)^T = 0, \tag{2.17}$$

which can be deemed as a set of simultaneous equations of the first order with three unknown variables, X_k, Y_k, and Z_k. Then in order to have non-trivial solutions for X_k, Y_k, and Z_k, the coefficient determinant of (2.17) should be zero. Namely, we have

$$\det(\mathbf{A} - \hat{\mathbf{O}}\mathbf{E}) = \begin{vmatrix} 1-\hat{\mathbf{O}} & 2 & 2 \\ 2 & 1-\hat{\mathbf{O}} & 2 \\ 2 & 2 & 3-\hat{\mathbf{O}} \end{vmatrix}$$
$$= -(\hat{\mathbf{O}}^3 - 5\hat{\mathbf{O}}^2 - 5\hat{\mathbf{O}} + 1)$$
$$= -(\hat{\mathbf{O}} + 1)(\hat{\mathbf{O}}^2 - 6\hat{\mathbf{O}} + 1) = 0. \tag{2.18}$$

Operation of the operator polynomials $(\hat{\mathbf{O}}^2 - 6\hat{\mathbf{O}} + 1)$ and $(\hat{\mathbf{O}}^3 - 5\hat{\mathbf{O}}^2 - 5\hat{\mathbf{O}} + 1)$ on f_k gives, respectively,

$$f_k - 6f_{k-1} + f_{k-2} = 0 \tag{2.19}$$

and $$f_k - 5f_{k-1} - 5f_{k-2} + f_{k-3} = 0, \tag{2.20}$$

leading to (2.11) and (2.12). Thus mathematical relation among them is clarified by this operator technique.

Note that the operator technique does not necessarily give rigorous recursive relations, but some candidates to the desired results. However, it is a powerful tool in a variety of problems where recursive relations are involved.

The present author has proposed to define the topological index Z_G[12-14] for characterizing a graph composed of vertices and edges.[18] The interested readers can also consult one of the paper in the present series for its application to the Pell equation.[3] In Fig. 1 three series of graphs are given whose topological indices, respectively, correspond to the values of X_k, Y_k, and Z_k. Interestingly enough, as seen in the lower part of Fig. 1, the trio of graphs whose Z_G values correspond to (X_k, Y_k, Z_k) for a given k can be constructed by jointing k–1 pairs of L-shaped branches consecutively to the three kinds of the central graphs whose Z_G values correspond to the fundamental pPT (3 4 5). Note that all these graphs are the so-called caterpillar graphs. A caterpillar is a tree graph that is composed of a path graph and branches of a unit length.[18,23] The present author has proved that the Euler's continuant[4] for calculating the continued fractions is nothing else but the Z_G of the caterpillar graph constructed directly from the set of parameters forming the continuant. Calculation of the Z_G value for a given graph is easily performed by the help of the efficient recursive relations.[12,13]

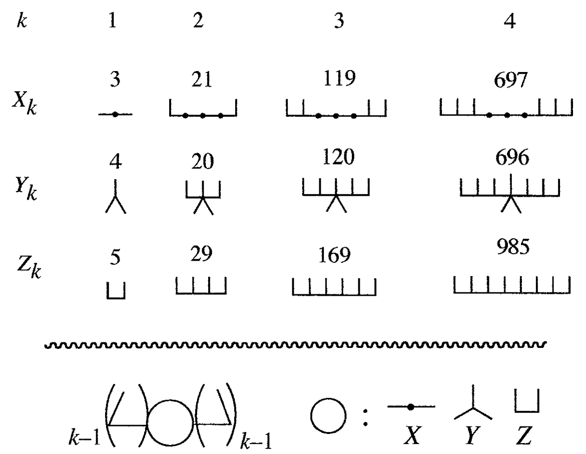

Fig. 1 Three series of graphs whose topological indices give the pPTs with consecutive legs.

Search for PTs with consecutive legs was rewarded by rational number approximation of the square root of two, which is closely related to the Llep-1 with D=2. Then let us try to obtain other quadratic surds by analyzing the cases with D larger than 2.

3. Llep-1 and pPT

Similar to the case of the right triangles approaching the isosceles triangle a number of Pell-1 and Llep-1 equations are also found to have the corresponding pPTs. At least for Llep-1, if it is solvable, one can always construct corresponding pPTs, and it is also the case with the Pell-1 with odd D. For Pell-1 with even D it depends on the value of D. The results hitherto obtained will be given as Theorems and Corollaries.

[Theorem 1]

All the solutions of Llep-1 have the corresponding Pythagorean triangles.

(Proof)

Consider Llep-1 with a square-free integer D as

$$x^2 - D y^2 = -1. \quad (3.1)$$

It is known that the necessary but not sufficient condition for the solvable Lep-1 is that D is expressed by the sum of a pair of square numbers as

$$D = a^2 + b^2. \quad (3.2)$$

where a=b=1, or a<b and prime with each other. Among the lower members of (a, b), the cases with (3,4), (3, 5), (3, 11), (5, 11), *etc*. are found to be unsolvable.

In order to obtain the PTs with the desired property as mentioned in the end of the last section, consider a right triangle (See Fig. 2) with a pair of legs,

$$X=aL\pm\alpha$$

$$Y=bL\pm\beta, \quad (3.3)$$

with all non-negative integer parameters. The hypotenuse Z which is determined by

$$X^2 + Y^2 = Z^2 \quad (3.4)$$

will approach

$$Z \rightarrow \sqrt{D}L. \quad (3.5)$$

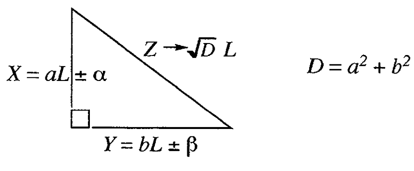

Fig. 2 Design of PTs whose hypotenuse approaches the square root of D.

From (3.3) and (3.4) we have

$$(aL\pm\alpha)^2 + (bL\pm\beta)^2 = D\,L^2 \pm 2L\,(a\alpha + b\beta) + (\alpha^2 + \beta^2) = Z^2 \qquad (3.6)$$

Multiply these equations by D, and one gets

$$[DL \pm (a\alpha + b\beta)]^2 + (b\alpha - a\beta)^2 = D\,Z^2. \qquad (3.7)$$

Under the restricted condition for a and b as described below (3.2) one can choose an appropriate pair of (α, β) to fulfill the condition,

$$|b\alpha - a\beta| = 1, \qquad (3.8)$$

and one gets the Llep-1 (3.1) with

$$x = DL \pm (a\alpha + b\beta) \quad \text{and} \quad y = Z. \quad \square \qquad (3.9)$$

By using the series of solutions of (3.1), one gets a series of L values and the PTs of the desired property.

[Corollary 1]

A series of rational number approximation to the square root of D can be obtained from the edges (X, Y, Z) of the PTs obtained in Theorem 1 by

$$\sqrt{D} = \frac{aX + bY}{Z}. \qquad (3.10)$$

(Proof) By using (3.2), (3.3), (3.7), and (3.8) one gets

$$\left[\frac{aX + bY}{Z}\right]^2 = \frac{D[(a^2L \pm a\alpha) + (b^2L \pm b\beta)]^2}{[DL \pm (a\alpha + b\beta)]^2 + (b\alpha - a\beta)^2} = \frac{D[DL \pm (a\alpha + b\beta)]^2}{[DL \pm (a\alpha + b\beta)]^2 + 1} \to D \quad . \ \square \qquad (3.11)$$

In the case of the pPTs with consecutive legs introduced in section 2, a=b=1 and α=0, β=1. Examples of Theorem 1 and Corollary 1 will be explained for the case with D=5 (See Fig. 2). Set a=1 and b=2 for D=5. The set of parameters α=β=1 satisfy (3.8). The solutions of x and y for Llep-1 (3.1) can be solved according to the standard recipe.[1-7] In Table 2 the smaller solutions of x and y are given together with the corresponding values of L and sign ± in (3.3) and (3.9). The edge lengths of the desired PTs can be obtained from (3.3) and y (=Z). The numerical values obtained from Corollary 1 are also given in the last column of Table 2. Here it is observed that the number of digits of the correct figures (underlined) increases approximately as the double of the number of digits of the edge lengths. The speed of convergence of rational number approximation to the quadratic surd is thus rather high.

Further, it was found that Z_k and the pair of X_k and Y_k obey different recursive relations as given in Table 2. Though not given in Table 2 these recursive relations were found to hold also for zero and negative k's corresponding to formal PTs carying negative integers.

Table 2. The solutions of Llep-1 with D=5 and the corresponding PTs involving $\sqrt{5}$.

k	$x=5L\pm3^{a,b}$	$y=Z^{b}$	$\pm$	L^{c}	$X=L\pm1^{a,c}$	$Y=2L\pm1^{a,c}$	$(X+2Y)/Z$
1	2	1	–	1	0	1	2.00
2	38	17	+	7	8	15	2.23529
3	682	305	–	137	136	273	2.2360655
4	12,238	5,473	+	2,447	2,448	4,895	2.2360679700

[a] a=1, b=2, α=1, and β=1, which fulfill (3.8).

[b] For $Z_k : f_k = 18 f_{k-1} - f_{k-2}$. [c] For X_k, $Y_k : f_k = 17\,(f_{k-1} + f_{k-2}) - f_{k-3}$.

Note, however, that the parities of X and Y thus obtained do not necessarily obey (1.4). If further discussion is to be extended, due caution to this point will be necessary.

4. Pell-1 and pPT

[Theorem 2]

All the solutions of Pell-1 with odd D have the corresponding Pythagorean triangles.

(Proof)

Consider Pell-1 with a square-free integer D as

$$x^2 - D\,y^2 = 1. \tag{4.1}$$

Any odd D=2n+1 can be expressed as the difference of the squares of a pair of consecutive integers as

$$D = 2n + 1 = (n+1)^2 - n^2. \qquad (4.2)$$

Then put a and b as follows:

$$a = n \quad \text{and} \quad b = n+1. \qquad (4.3)$$

Next consider a right triangle whose X (one of the legs) and Z (hypotenuse) are given as in Fig. 3.

$$X = aL \pm \alpha$$

$$Z = bL \pm \beta, \qquad (4.4)$$

Then another leg, Y, is determined by

$$Y^2 = Z^2 - X^2, \qquad (4.5)$$

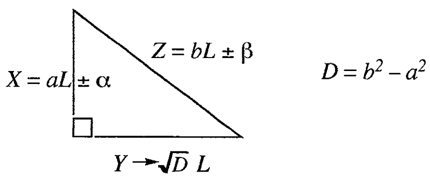

Fig. 3

which is expected to approach

$$Y \to \sqrt{D}L. \qquad (4.6)$$

From (4.4) and (4.5) we have

$$(bL\pm\beta)^2 - (aL\pm\alpha)^2 = D\,L^2 \pm 2L\,(b\beta - a\alpha) + (\beta^2 - \alpha^2) = Y^2 \qquad (4.7)$$

Multiply these equations by D, and one gets

$$[DL \pm (b\beta - a\alpha)]^2 - (b\alpha - a\beta)^2 = D\,Y^2. \qquad (4.8)$$

If a and b are chosen as (4.3), the condition (3.8) can be fulfilled by $\alpha=\beta=1$, and one gets the Pell-1 (4.1) with

$$x = DL \pm 1 \quad \text{and} \quad y = Y. \quad \square \qquad (4.9)$$

[Corollary 2]

A series of rational number approximation to the square root of odd D can be obtained from the edges (X, Y, Z) of the PTs obtained in Theorem 2 by

$$\sqrt{D} = \frac{bZ - aX}{Y} \quad . \tag{4.10}$$

(Proof) By using (3.8), (4.2), (4.3), and (4.8) one gets

$$[\frac{bZ-aX}{Y}]^2 = \frac{D[(b^2L \pm b\beta) - (a^2L \pm a\alpha)]^2}{[DL \pm (b\beta - a\alpha)]^2 - (b\alpha - a\beta)^2}$$
$$= \frac{D[DL \pm (b\beta - a\alpha)]^2}{[DL \pm (b\beta - a\alpha)]^2 - 1} \to D \quad . \quad \square \tag{4.11}$$

An example of Theorem 2 and Corollary 2 is given in Table 3 for the case with D=5. It was found that Y_k and the pair of X_k and Z_k obey different recursive relations as given in Table 3. Notice that the pair of these recursive relations are formally the same as those in Table 2.

Table 3. The solutions of Pell-1 with D=5 and the corresponding PTs involving $\sqrt{5}$.

k	$x = 5L \pm 1^{a,c}$	$y = Y^b$	$\pm$	L^c	$X = 2L \pm 1^{a,c}$	$Z = 3L \pm 1^{a,c}$	$(3Z-2X)/Y$
1	9	4	−	2	3	5	<u>2.25</u>
2	161	72	+	32	65	97	<u>2.236</u>11
3	2,889	1,292	−	578	1,155	1,733	<u>2.2360681</u>1
4	51,841	23,184	+	10,368	20,737	31,105	<u>2.23606797779</u>1

[a] a=2, b=3, α=1, and β=1, which fulfill (3.8).

[b] For $Y_k : f_k = 18 f_{k-1} - f_{k-2}$. [c] For $X_k, Z_k : f_k = 17(f_{k-1} + f_{k-2}) - f_{k-3}$.

It was found that different choice of the pair of α and β yields the same results as those in Tables 2 and 3 as long as they obey (3.8).

In order to ensure the effectiveness of the Theorems and Corollaries proved here, another pair of examples are given in Tables 4 and 5 with D=13, where the all the values corresponding to k=0 are also listed. Also in this case the pairs of recursive relations in Tables 4 and 5 are formally the same.

Table 4. The solutions of Llep-1 with D=13 and the corresponding PTs involving $\sqrt{13}$.

k	$x = 13L \pm 5^{a,b}$	$y = Z^b$	$\pm$	L^c	$X = 2L \pm 1^{a,c}$	$Y = 3L \pm 1^{a,c}$	$(2X+3Y)/Z$
0	–18	5	–	–1	–3	–4	–3.600
1	18	5	+	1	3	4	3.600
2	23,382	6,485	–	1,799	3,597	5,396	3.6055512721
3	30,349818	8,417,525	+	2,334,601	4,669,203	7,003,804	3.6055512754639873

[a] a=2, b=3, α=1, and β=1, which fulfill (3.8).

[b] For $Z_k : f_k = 1298 f_{k-1} - f_{k-2}$. [c] For X_k, $Y_k : f_k = 1297 (f_{k-1} + f_{k-2}) - f_{k-3}$.

Table 5. The solutions of Pell-1 with D=13 and the corresponding PTs involving $\sqrt{13}$.

k	$x = 13L \pm 1^{a,b}$	$y = Y^b$	$\pm$	L^c	$X = 6L \pm 1^{a,c}$	$Z = 7L \pm 1^{a,c}$	$(7Z-6X)/Y$
0	1	0	+	0	1	1	---
1	649	180	–	50	299	349	3.6055555
2	842,401	233,640	+	64,800	388,801	453,601	3.6055512754665
3	1,093,435,849	303,264,540	–	84,110,450	504,662,699	588,773,149	3.6055512754639892946

[a] a=6, b=7, α=1, and β=1, which fulfill (3.8).

[b] For $Y_k : f_k = 1298 f_{k-1} - f_{k-2}$. [c] For X_k, $Z_k : f_k = 1297 (f_{k-1} + f_{k-2}) - f_{k-3}$.

Notice that the choice of (4.2) and (4.3) in the Proof of Theorem 2 is a sufficient condition for (3.8). As the difference between the squares of a pair of odd integers is always a multiple of eight, the following theorem can be obtained.

[Theorem 3]

All the solutions of Pell-1 with D=8n excepting the square numbers have the corresponding Pythagorean triangles.

(Proof)

Set

$$a = 2n - 1 \quad \text{and} \quad b = 2n + 1. \tag{4.12}$$

Then consider Pell-1 (4.1) with D as

$$D = b^2 - a^2 = 8n, \tag{4.13}$$

but exclude square Ds.

By selecting

$$\alpha = n - 1 \quad \text{and} \quad \beta = n, \tag{4.14}$$

one gets

$$b\alpha - a\beta = -1 \tag{4.15}$$

and

$$b\beta - a\alpha = 4n - 1. \tag{4.16}$$

Then (4.8) becomes

$$[8nL \pm (4n - 1)]^2 - 8nY^2 = 1, \tag{4.17}$$

which leads to the Pell-1

$$x^2 - 2ny^2 = 1 \tag{4.18}$$

with

$$x = 8nL \pm (4n - 1) \quad \text{and} \quad y = 2Y. \quad \square \tag{4.19}$$

An example of Theorem 3 is given in Table 6 with D=8. In this case integer solutions are obtained only for odd k. However, for even k pPTs can be obtained by multiplying the (X, Y, Z) in Table 6 by 2. These two groups of edges were found to obey respectively the same recursive relations as shown in Table 6.

Table 6. The solutions of Pell-1 with D=8 and the corresponding PTs involving $\sqrt{8}=2\sqrt{2}$.

k	$x=8L\pm5$ [a]	$y=2Y$ [b]	±	L	$X=L\pm1$ [a,c]	Z=3L±2 [a,c]	Y=y/2 [b]	(3Z–X)/Y
1	3	2	–	1	0	1	1	3.00
2	17	12	+	3/2	5/2	13/2	12/2	2.833
3	99	70	–	13	12	37	35	2.82857
4	577	408	+	143/2	145/2	433/2	408/2	2.828431
5	3,363	2,378	–	421	420	1,261	1,189	2.82842724
6	19,601	13,860	+	4,899/2	4,901/2	14,701/2	13,860/2	2.8284271284
7	114,243	80,782	–	14,281	14,280	42,841	40,391	2.82842712485

[a] a=1, b=3, α=1, and β=2, which fulfill (3.8).

[b] For Y_k : $f_{2k+1}=34f_{2k-1}-f_{2k-3}$ and $f_{2k}=34f_{2k-2}-f_{2k-4}$.

[c] For X_k, Z_k : $f_{2k+1}=35\,(f_{2k-1}-f_{2k-3})+f_{2k-5}$ and $f_{2k}=35\,(f_{2k-2}-f_{2k-4})+f_{2k-6}$.

Note that Table 6 can be applied to the case of the Pell-1 with D=2. Similarly all the cases with D=2n are associated with D=8n, which is solved as in Theorem 3. This seems to mean that all the cases of Pell-1 with even D can be solved in this way. However, it has been observed that for certain D's ($\neq$8n) only the solutions of rational number can be obtained as in the case of D=24. Since we have not yet fully analyzed this problem, some preliminary result will be given in Appendix.

Anyway the advantage of our result over the former studies is that in almost all the cases of Pell-1 and Llep-1 simple algorithms were found to reach a series of the corresponding Pythagorean triangles giving an interesting geometrical meaning of the Pell equations as a whole.

Appendix Generation of pPTs involving $\sqrt{6}$

By using the same algorithm as adopted for D=8 in the last part of this paper we have tried to obtain the PTs involving the square root of 24 (n=3 in Theorem 3), where a=5, b=7, α=2, and β=3 are chosen. Then we have

$$x^2 - 6y^2 = 1 \tag{A.1}$$

with

$$x = 24L \pm 11 \quad \text{and} \quad y = 2Y, \tag{A.2}$$

and

$$X = 5L \pm 2 \quad \text{and} \quad Z = 7L \pm 3. \tag{A.3}$$

However, by using the smallest solutions of (A.1) as (5, 2), (49, 20), (485, 198), (4801, 1960), *etc.* no integer L could be obtained. Then all the PT edges, (X, Y, Z), formally obtained are fractional numbers. For example, the second smallest solution is (21/2, 20/2, 29/2), and all the PTs thus obtained are found to form two groups as in the case of D=8.

The X and Z of one group with even k are found to be half integers, while the Y an integer. Further, after multiplying them with 2 the pairs of (X, Z) are found to be expressed by (5L+1, 7L+1). Then the strategy was change to find the series of pPTs with

$$X = 5L+1 \quad \text{and} \quad Z = 7L+1. \tag{A.4}$$

The remaining Y can be given by

$$Y^2 = (7L+1)^2 - (5L+1)^2 = 24L^2 + 4L. \tag{A.5}$$

After multiplying these equations by 6 one gets

$$6Y^2 = (12L+1)^2 - 1, \tag{A.6}$$

which is equivalent to the original Pell-1 (A.1) with

$$x = 12L + 1 \quad \text{and} \quad y = Y. \tag{A.7}$$

Again the solutions of (A.1) are found to be composed of two groups. The second smallest solution and its group are found to form a family of pPTs as given in Table A. Note that in this case the parameters α and β have been changed into α=β=1 yielding $b\alpha - a\beta = 2$ instead of (3.8).

Table A. The solutions of Pell-1 with D=6 and the corresponding PTs involving $\sqrt{6}$.

k	$x = 12L+1$[a,b]	$y = Y$[b]	L[c]	$X = 5L+1$[a,c]	$Z = 7L+1$[a,c]	$(7Z-5X)/2Y$
0	1	0	0	1	1	---
2	49	20	4	21	29	2.45
4	4,801	1,960	400	2,001	2,801	2.449489795
6	470,449	192,060	39,204	196,021	274,429	2.4494897427887

[a] a=5, b=7, α=1, and β=1, which do not fulfill (3.8).

[b] For $Y_k : f_{2k} = 98 f_{2k-2} - f_{2k-4}$. [c] For $X_k, Z_k : f_{2k} = 99 (f_{2k-2} - f_{2k-4}) + f_{2k-6}$.

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