Continuant, caterpillar, and topological index *Z*. II. Novel identities involving Fibonacci, Lucas, and generalized Fibonacci numbers.

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Abstract The three series of numbers, Fibonacci (F_n), Lucas, (L_n), and generalized Fibonacci (G_n), are defined to have the same recursive relation, $u_n = u_{n-1}+u_{n-2}$. By imposing the following set of initial conditions, f_0=f_1=1, L_1=1 and L_2=3, and G_1=a>0 and G_2=b>0 with b>2a, a number of novel identities were found which systematically relate f_n, L_n, and G_n with each other. Further, graph-theoretical interpretation for these relations was obtained by the aid of the continuant, caterpillar graph, and topological index Z which was proposed and developed by the present author.

1. Introduction

In the first paper (referred as I) of this series[1] it was shown that the continuant, which was proposed by Euler[2-4] for discussing continued fractions, is identical to the topological index (Z) or Z-index of the caterpillar graph constructed from the terms comprising the given continued fraction, where the Z-index is an integer proposed by the present author for characterizing the topological property of a graph.[5,6] Then the Fibonacci (f_n) and Lucas (L_n) numbers are shown to be expressed either by the continuant or caterpillar graph.[7,8] Further, through these close relationships among different mathematical objects and concepts a number of interesting applications and graph-theoretical interpretations have been developed.

For example, in I the following identity was proved involving generalized Fibonacci numbers (G_n) as

$$G_{m+n} = G_m f_n + G_{m-1} f_{n-1}. \tag{1.1}$$

The validity of (1.1) can clearly be demonstrated by the caterpillar trees corresponding to all the entries of (1.1). It is to be noticed that the initial condition for the Fibonacci numbers f_n is chosen here as

$$f_0 = f_1 = 1, \tag{1.2}$$

different from the widely accepted one[8-11] as

$$F_1 = F_2 = 1. \tag{1.3}$$

The advantage of the choice of (1.1) over (1.2) will repeatedly be shown in this paper through a number of novel identities among the Fibonacci, Lucas, and generalized Fibonacci numbers. Graph-theoretical interpretation for these identities will also be demonstrated.

2. Definitions

2.1. Fibonacci, Lucas, and generalized Fibonacci numbers

All the series of numbers appearing in this paper commonly obey the following recursive relation,

$$u_n = u_{n-1} + u_{n-2} \qquad u_n : f_n \text{ (Fibonacci)}, L_n \text{ (Lucas)}, G_n \text{ (generalized Fibonacci)} \tag{2.1}$$

The initial conditions for f_n and L_n are chosen as

$$f_0 = f_1 = 1 \tag{2.2}$$

$$L_0 = 2, L_1 = 1. \tag{2.3}$$

Although the latter (2.3) for L_n seems to be unanimously chosen, very few authors adopt (2.2) for f_n.[12,13)]

By taking

$$G_1 = a > 0, G_2 = b > 0 \tag{2.4}$$

a series of numbers,

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, \cdots \tag{2.5}$$

with the property of (2.1) are conventionally called generalized Fibonacci numbers or generalized Fibonacci sequence.[10)] Although there seems to exist no strict restriction for the initial condition (2.4) for G_n, we dare propose to put

$$b > 2a. \tag{2.6}$$

By doing this all the three series of numbers have the same property that u_1 is not greater than u_0 as

$$u_0 \geq u_1 < u_2. \tag{2.7}$$

Actually we have $G_0 = b - a > G_1 = a$ and $L_0 > L_1$.

Further, we will mainly be concerned with the primitive G_n's with $(a, b) = 1$. For convenience of the later discussion the lower members of f_n and L_n are shown in Table 1, together with two kinds of primitive G_n's.

Table 1. Fibonacci, Lucas, and generalized Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	3	5	8	13	21	34	55	89
L_n	2	1	3	4	7	11	18	29	47	76	123
G^1_n	3	1	4	5	9	14	23	37	60	97	157
G^2_n	48	13	61	74	135	209	344	553	897	1450	2347

2.2. Continuant and the related determinant

[Def. 1] Continuant.

The continuant polynomial, or simply continuant, which was extensively discussed by Euler, can be defined recurrently, as follows:[2-4)]

$$K_0() = 1;$$

$$K_1(x_1) = x_1;$$

$$K_2(x_1, x_2) = x_1 x_2 + 1;$$

$$K_n(x_1, x_2, \cdots, x_n) = x_n K_{n-1}(x_1, x_2, \cdots, x_{n-1}) + K_{n-2}(x_1, x_2, \cdots, x_{n-2}) . \tag{2.8}$$

The continuant has been known to be reversible as shown in the following three theorems whose proofs may not be necessary here.[1)]

[Theorem 1] Reversible character.

$$K_n(x_1, x_2, \cdots, x_n) = K_n(x_n, \cdots, x_2, x_1) . \tag{2.9}$$

[Theorem 2] Reversible recursive relation (cf. the last equation in (2.8)).

$$K_n(x_1, x_2, \cdots, x_n) = x_1 K_{n-1}(x_2, x_3, \cdots, x_n) + K_{n-2}(x_3, x_4, \cdots, x_n) . \tag{2.10}$$

[Theorem 3] Tridiagonal determinantal expression.

$$K_n(x_1, x_2, \cdots, x_n) = \begin{vmatrix} x_1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & x_2 & 1 & 0 & \cdots & 0 \\ 0 & -1 & x_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x_{n-1} & 1 \\ 0 & 0 & \cdots & 0 & -1 & x_n \end{vmatrix} . \tag{2.11}$$

It is to be remembered that a tridiagonal determinant is easy to be degraded, and Theorem 3 actually plays a very important role in the calculation of the continuants and Z-indices.[1,14)]

Either from Def. 1 or Theorem 3, the Fibonacci (f_n) and Lucas (L_n) numbers can directly be obtained, respectively, to be

$$f_n = K_n(1, 1, \cdots, 1) \qquad (n \text{ 1's}), \tag{2.12}$$

$$f_n = K_{n-1}(2, 1, \cdots, 1) \qquad (n-2 \text{ 1's}) \quad (n \geq 2), \quad f_1 = K_0( ), \tag{2.13}$$

and

$$L_n = K_n(1, 2, 1, \cdots, 1) \qquad (\text{one and } n-2 \text{ 1's}) \quad (n \geq 2), \quad L_1 = K_1(1), \tag{2.14}$$

$$L_n = K_{n-1}(3, 1, \cdots, 1) \qquad (n-2 \text{ 1's}) \quad (n \geq 2), \quad L_1 = K_0( ) . \tag{2.15}$$

For the pair of G_n's in Table 1 we have

$$G^1{}_n = K_{n-1}(4, 1, \cdots, 1) \qquad (n-2 \text{ 1's}) \quad (n \geq 2), \quad G^1{}_1 = K_0( ), \tag{2.16}$$

and

$$G^2{}_n = K_{n+2}(4, 2, 1, 4, 1, \cdots, 1) \qquad (\text{one and } n-2 \text{ 1's}) \quad (n \geq 2), \quad G^2{}_1 = K_3(4, 2, 1). \tag{2.17}$$

Parallelism among the expressions (2.13), (2.15), and (2.16) for f_n, L_n, and $G^1{}_n$ is to be remarked with reference to Table 1. Derivation of (2.16) and (2.17) will be explained later.

2.3. Topological index *Z*

For characterizing a graph G the Z-index was proposed to be defined by the present author as follows.[5,6)] First define the non-adjacent number $p(G,k)$ as the number of ways for choosing k disjoint edges from G. Here $p(G,0)$ is defined to be unity for all the graphs including the vacant graph, and $p(G,1)$ is equal to the number of edges in G. By using the set of p(G,k)'s the Z-index Z(G) is defined as

$$Z(G)=\sum_{k=0}^{m} p(G,k) \quad , \qquad (2,18)$$

where m is the maximum number of k, or $m=\lfloor N/2 \rfloor$ with N being the number of vertices of G. In other words, Z(G) is the total sum of perfect and imperfect matchings. See Refs. 1, 5, and 6 for further discussion about the Z-index.

2.4. Caterpillar graphs

The path graph P_n is composed of n vertices which are consecutively connected by n–1 edges. The star graph S_n is also composed of n vertices and n–1 edges, but constructed in such a way that the central vertex is connected to all other n–1 vertices. The Z-indices of P_n and S_n are, respectively, f_n and n. A caterpillar graph is constructed in the following way. Suppose a path P_n and prepare a set of n stars $X_n=\{x_1, x_2,\cdots, x_n\}$ where natural number x_n denotes Sx_n, i.e., a star composed of x_n vertices. Then mount each element of X_n onto each vertex of S_n one by one either from left or right to another end. The resultant graph is the caterpillar graph $C_n(x_1, x_2,\cdots, x_n)$ composed of $|V|=\Sigma x_n$ vertices. Refer to Refs. 1 and 14 for further discussion about the caterpillar.

2.5. Continued fraction

In this paper the following type of simple finite continued fraction Q_N is treated, which is expressed and denoted by a finite set of elements a_n with all positive integers as

$$Q_N=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots+\cfrac{1}{a_{N-1}+\cfrac{1}{a_N}}}}}=[a_1,a_2,\cdots,a_{N-1},a_N]<1. \qquad (2.19)$$

3. Main Theorems

[Theorem 4] Fundamental set of additivity relations for f_n, L_n, and G_n.

$$f_{m+n}=f_m f_n+f_{m-1}f_{n-1} \qquad (3.1)$$

$$L_{m+n}=f_m L_n+f_{m-1}L_{n-1} \qquad (3.2)$$

$$G_{m+n}=f_m G_n+f_{m-1}G_{n-1}. \qquad (3.3)$$

(Proof) By combining (2.1) and (2.2) we have

$$\begin{aligned} u_n &= f_1\, u_{n-1}+f_0\, u_{n-2} \\ &= f_1(u_{n-2}+u_{n-3})+f_0\, u_{n-2} \\ &= f_2\, u_{n-2}+f_1\, u_{n-3}. \end{aligned} \qquad (3.4)$$

By doing similar processes we get

$$u_n=f_k\, u_{n-k}+f_{k-1}\, u_{n-k-1} \qquad (u_n : f_n,\ L_n, \text{ and } G_n). \qquad (3.5)$$

Then by putting n=n+m and k=m

$$u_{m+n} = f_m u_n + f_{m-1} u_{n-1} \quad (u_n : f_n,\ L_n, \text{ and } G_n) \tag{3.6}$$

is obtained. □

[Corollary 1]

$$L_{m+n} + L_{m+n-2} = L_m L_n + L_{m-1} L_{n-1} \tag{3.7}$$

and

$$G_{m+n} + G_{m+n-2} = L_m G_n + L_{m-1} G_{n-1}. \tag{3.8}$$

(Proof) By using the relation,

$$L_n = f_n + f_{n-2}, \tag{3.9}$$

Eq. (3.2) can be converted to

$$\begin{aligned} L_{m+n} &= (L_m - f_{m-2}) L_n + (L_{m-1} - f_{m-3}) L_{n-1} \\ &= L_m L_n + L_{m-1} L_{n-1} - (f_{m-2} L_n + f_{m-3} L_{n-1}) \\ &= L_m L_n + L_{m-1} L_{n-1} - L_{m+n-2}, \end{aligned} \tag{3.10}$$

which gives (3.7). □

Quite similarly Eq. (3.3) can be converted to

$$\begin{aligned} G_{m+n} &= (L_m - f_{m-2}) G_n + (L_{m-1} - f_{m-3}) G_{n-1} \\ &= L_m G_n + L_{m-1} G_{n-1} - G_{m+n-2}, \end{aligned} \tag{3.11}$$

which gives (3.8). □

[Theorem 5] Another set of additivity relations for f_n, L_n, and G_n.

$$f_{m+n} = f_m f_{n+1} - f_{m-2} f_{n-1} \tag{3.12}$$

$$L_{m+n} = f_m L_{n+1} - f_{m-2} L_{n-1} \tag{3.13}$$

$$G_{m+n} = f_m G_{n+1} - f_{m-2} G_{n-1}. \tag{3.14}$$

(Proof) Equation (3.6) can be converted to

$$\begin{aligned} u_{m+n} &= f_m(u_{n+1} - u_{n-1}) + (f_m - f_{m-2}) u_{n-1} \\ &= f_m u_{n+1} - f_{m-2} u_{n-1} \quad (u_n : f_n,\ L_n, \text{ and } G_n). \end{aligned} \tag{3.15}$$ □

[Theorem 6]

$$f_{n+1} L_n - f_n L_{n+1} = (-1)^n, \tag{3.16}$$

$$f_{n+1} G_n - f_n G_{n+1} = (b - 2a)(-1)^n, \tag{3.17}$$

$$L_{n+1} G_n - L_n G_{n+1} = (b - 3a)(-1)^n. \tag{3.18}$$

(Proof) Let us take

$$f_2 G_1 - f_1 G_2 = 2a - b < 0. \tag{3.19}$$

Similarly we get

$$f_3 G_2 - f_2 G_3 = 3b - 2(a + b) = b - 2a > 0 \tag{3.20}$$

and

$$f_4 G_3 - f_3 G_4 = 5(a + b) - 3(a + 2b) = 2a - b < 0. \tag{3.21}$$

By following these steps we will arrive at (3.17). Since L_n series have a=1 and b=3, (3.16) can also be proved.

Next let us take

$$L_2 G_1 - L_1 G_2 = 3a - b < 0. \tag{3.22}$$

Similarly we get

$$L_3 G_2 - L_2 G_3 = 4b - 3(a+b) = b - 3a > 0 \tag{3.23}$$

and $$L_4 G_3 - L_3 G_4 = 7(a+b) - 4(a+2b) = 3a - b < 0. \tag{3.24}$$

By following these steps we will arrive at (3.18). □

(Geometrical Proof of (3.16)) See Fig. 1 where f_n's and L_n's are, respectively, graduated on the abscissa and ordinate. By noticing the equal areas of the two pairs of hatched rectangles one can obtain the following identities.

$$\begin{aligned} & f_{n+1} L_n - f_n L_{n+1} \\ &= f_{n-1} L_n - f_n L_{n-1} \\ &= -(f_n L_{n-1} - f_{n-1} L_n) \\ &= f_{n-1} L_{n-2} - f_{n-2} L_{n-1} \\ &= \cdots = (-1)^n (f_1 L_0 - f_0 L_1). \end{aligned}$$

Then we have (3.16). □

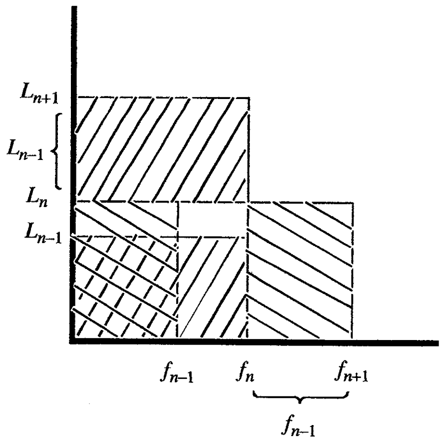

Fig. 1. Geometrical proof of (3.16).

From Theorem 6 interesting identities involving f_n, L_n, and G_n are derived.

[Theorem 7]

$$\frac{L_{n+1}}{f_{n+1}} - \frac{L_n}{f_n} = \frac{(-1)^n}{f_{n+1} f_n} \tag{3.25}$$

$$\frac{f_{n+1}}{L_{n+1}} - \frac{f_n}{L_n} = \frac{(-1)^n}{L_{n+1} L_n} \tag{3.26}$$

$$\frac{f_{n+1}}{f_n} - \frac{L_{n+1}}{L_n} = \frac{(-1)^n}{f_n L_n} \tag{3.27}$$

(Proof) Divide (3.16), respectively, by $f_{n+1} f_n$, $L_{n+1} L_n$, and $f_n L_n$, and one gets (3.25), (3.26), and (3.27). □

[Corollary 2]

$$\frac{f_{n+1}}{f_n} - \frac{G_{n+1}}{G_n} = \frac{(b-2a)(-1)^n}{f_n G_n} \tag{3.28}$$

and

$$\frac{L_{n+1}}{L_n} - \frac{G_{n+1}}{G_n} = \frac{(b-3a)(-1)^n}{L_n G_n} . \tag{3.29}$$

(Proof) Divide (3.17) and (3.18), respectively, by $f_n G_n$ and $L_n G_n$, and one gets (3.28) and (3.29). □

All the Theorems and Corollaries can be ascertained by using the numbers in Table 1.

It is to be remarked here that all the Theorems and Corollaries given in this section are applied to those generalized Fibonacci numbers which obey (2.6) but are not necessarily prime with each other.

4. Comparison with other studies

Although several results in the above section have been reported by other researchers,[8-11] no systematic study seems to have been performed connecting the recursive properties of Fibonacci, Lucas, and generalized Fibonacci numbers, especially on their graph-theoretical interpretation, which will be given in the following section.

Relations (3.1)-(3.3) have sporadically been documented but in different forms as

$$u_{m+n} = f_m u_{n+1} + f_{m-1} u_n \quad (u = f, L, G) \tag{4.1}$$

and the like under the initial condition (1.3). As will be shown later, the forms of relations (3.1)-(3.3) are not only neater than (4.1) but also have clearer graph-theoretical meaning.

In Hoggatt's book[9] the relations corresponding to (3.16) and (3.27) are, respectively, documented as

$$F_{n+1} L_n - F_n L_{n+1} = 2(-1)^n \quad \text{with} \quad F_1 = F_2 = 1, \tag{4.2}$$

and

$$\frac{F_{n+1}}{F_n} - \frac{L_{n+1}}{L_n} = \frac{2(-1)^n}{F_{2n}} \quad \text{with} \quad F_1 = F_2 = 1. \tag{4.3}$$

Both of them are rather neat. However, it is obvious that under the conventional initial condition (1.3), it is rather difficult to derive such systematic relations that connect f, L, and G as obtained in the above section. Actually in hitherto published documents the present author could not find any aesthetic identities as (3.25)-(3.29) involving fractional relations among these series of numbers.

5. Graph-theoretical interpretation of the identities

In this section graph-theoretical interpretation of the identities obtained above will be given. It has already been proved in I that the continuant $K_n(x_1, x_2, \cdots, x_n)$ is identical to the Z-index of the caterpillar graph $C_n(x_1, x_2, \cdots, x_n)$ constructed from the same set of elements $\{x_n\}$. Namely, we have

$$Z_n(x_1, x_2, \cdots, x_n) = \mathrm{K}_n(x_1, x_2, \cdots, x_n), \tag{5.1}$$

where $Z_n(x_1, x_2, \cdots, x_n)$ represents the Z-index of $\mathrm{C}_n(x_1, x_2, \cdots, x_n)$. It is also to be remarked here that the three series of numbers can be represented by the Z-indices of the corresponding series of caterpillar graphs as shown in Fig. 2. Those readers who are not familiar with the recursive properties and manipulation of the Z-index are advised to consult I and Refs. 5, 6, and 14.

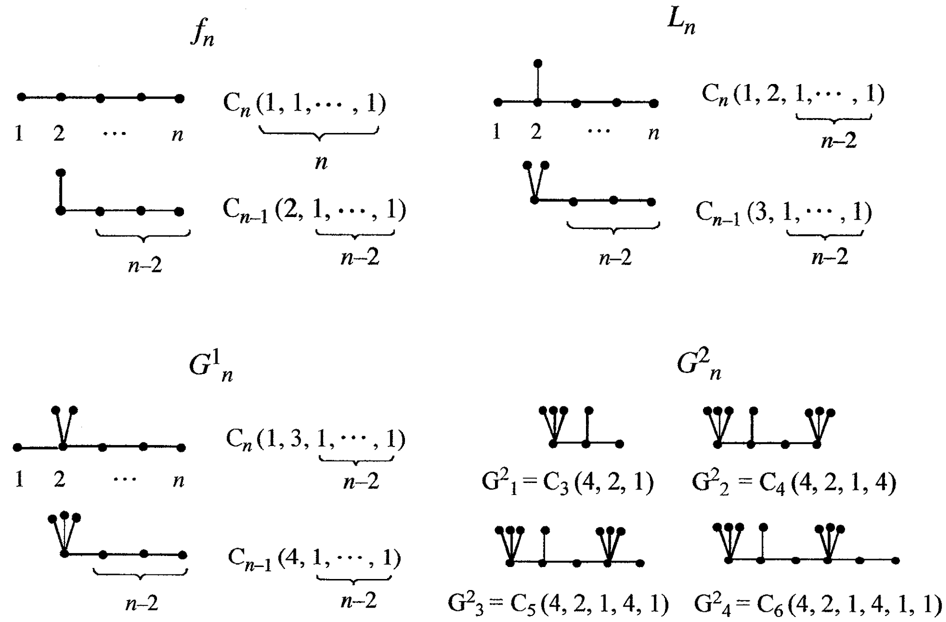

Fig. 2 Caterpillar graphs whose Z-indices correspond to the series of graphs, f_n, L_n, and G_n. See also Table 1.

Parallelism among the caterpillar graphs corresponding to f_n, L_n, and G_n is obvious from the continuant expressions, (2.13), (2.15), and (2.16) for them. However, derivation of (2.17) is straightforward but needs to do some manipulation as in Fig. 3, where procedures for obtaining the continued fraction expression for a given rational number a/b is demonstrated with 13/61 for $G^2{}_n$ as an example. Note that in this case the length of the caterpillar is larger than the arguments for $G^2{}_n$ and Z_n by two. This discrepancy comes from the restriction that the generalized Fibonacci numbers in this paper are in principle defined to satisfy the condition (2.7). Thus, although the continuants, K_n, and caterpillars, C_n, have been defined for $n \geq 0$, the Z-indices and the generalized Fibonacci numbers, G_n's, for $n<0$ do not necesarily obey the recursive relation (2.1).

With these preliminaries it is easy to understand the graph-theoretical interpretation of the set of the

$$\frac{b}{a} = x_1 + \frac{a_2}{a_1} \qquad \frac{61}{13} = 4 + \frac{9}{13}$$

$$\frac{a_1}{a_2} = x_2 + \frac{a_3}{a_2} \qquad \frac{13}{9} = 1 + \frac{4}{9}$$

$$\frac{a_2}{a_3} = x_3 + \frac{a_4}{a_3} \qquad \frac{9}{4} = 2 + \frac{1}{4}$$

$$\cdots\cdots\cdots \qquad \frac{4}{1} = 4 + \frac{0}{1}$$

$$\frac{a_{l-1}}{a_l} = x_l + \frac{a_{l+1}}{a_l} \qquad a_4 \neq 0,\ a_5 = 0$$

if $a_l \neq 0$ and $a_{l+1} = 0$, then $k = l$ $\qquad \therefore\ k = 4$

$$\frac{a}{b} = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\cdots + \cfrac{1}{x_k}}}} = [0; x_1, x_2, \cdots, x_k] \qquad \frac{13}{61} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4}}}} = [0;4,1,2,4]$$

$G_1 = a$	$G_2 = b$	$G_3 = a+b$	$G_4 = a+2b$
$= K_3(1,2,4)$	$= K_4(4,1,2,4)$	$= K_5(1,4,1,2,4)$	$= K_6(1,1,4,1,2,4)$
$= K_3(4,2,1)$	$= K_4(4,2,1,4)$	$= K_5(4,2,1,4,1)$	$= K_6(4,2,1,4,1,1)$

$C_3(4,2,1)$	$C_4(4,2,1,4)$	$C_5(4,2,1,4,1)$	$C_6(4,2,1,4,1,1)$
$G_1 = Z_1 = 13$	$G_2 = Z_2 = 61$	$G_3 = Z_3 = 74$	$G_4 = Z_4 = 135$

ϕ $C_0(\,)$, $K_0(\,)$ $= Z_{-2} = 1$

 $C_1(4)$, $K_1(4)$ $= Z_{-1} = 4$

 $C_2(4,2)$, $K_2(4,2)$ $= Z_0 = 9$

$$[0;4,1,2,4] = \frac{13}{61},\quad [0;1,2,4] = \frac{9}{13},\quad [0;2,4] = \frac{4}{9},\quad [0;4] = \frac{1}{4}$$

Fig. 3. Derivation of the caterpillar graphs corresponding to $G^2{}_n$ in Fig. 2.

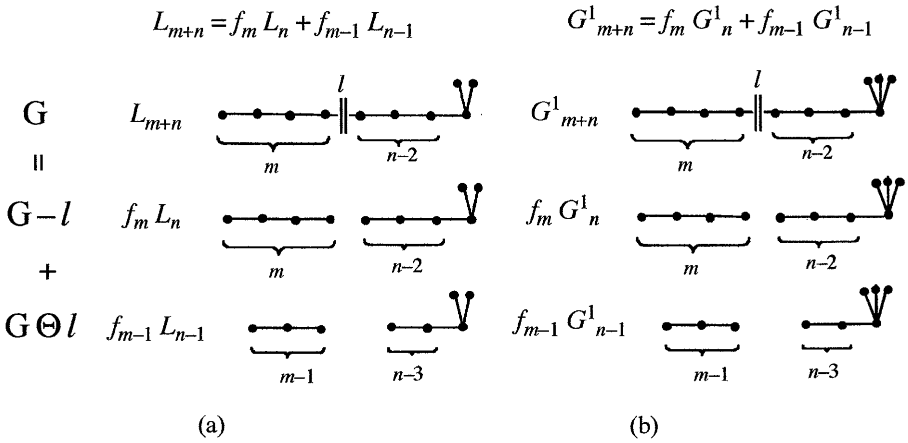

Fig. 4. Graph-theoretical interpretation of the additivity Theorem 4.

additivity Theorem 4.

The fundamental recursive relation for the Z-index of a given graph G is expressed as follows,[5,6]

$$Z(G) = Z(G-l) + Z(G\Theta l), \qquad (5.2)$$

where the subgraph $G-l$ is obtained by deleting an edge l, e.g., marked with the double bar in Fig. 4, and $G\Theta l$ is obtained from $G-l$ by deleting all the edges which *were* incident to l. As shown in Fig. 4a, $G-l$ for L_{m+n} is a pair of f_m and L_n, while $G\Theta l$ is a pair of f_{m-1} and L_{n-1}. From its definition the Z-index of a disjoint pair or group of graphs becomes the product of the Z-indices of all the components. Then (3.2) is directly obtained from Fig. 4a according to (5.2). Similarly (3.3) is obtained from Fig. 4b. Equation (3.1) can be proved by changing L_{m+n} into f_{m+n} by deleting an edge from the right-most moiety of L_{m+n}.

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