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The Analyticity of the Semigroup Generated by the Elliptic Differential Operator in a Lipschitz Domain

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ABSTRACT

We consider the elliptic differential operator in divergence form associated with Dirichlet boundary condition in a Lipschitz domain Ω in $\mathbf{R}^m$. The aim of this paper is to show that the operator is the generator of an analytic semigroup of bounded linear operators in $L^p(\Omega)$. We start from the concrete differential operator with Dirichlet boundary condition in weak sense. Then we show that the smallest closed extension of it is the generator of C_0-semigroup, and that the semigroup satisfies the estimate to be an analytic semigroup.

1 Introduction

This paper is concerned with the elliptic differential operator B of the form:

$$Bu = \sum_{i,j=1}^{m} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x_i}[\sqrt{a(x)} a_{ij}(x) \frac{\partial u}{\partial x_j}], \qquad a(x) = \det[a_{ij}(x)]^{-1} \tag{1.1}$$

considered in a Lipschitz domain Ω in $\mathbf{R}^m$ with Dirichlet boundary condition; the coefficients $a_{ij}(x)$'s are assumed to be bounded and of class C^∞ in the interior of Ω but not necessarily continuous up to the boundary $\partial\Omega$. Furthermore we assume that the partial derivatives of second order of $a_{ij}(x)$ are uniformly Lipschitz continuous in Ω.

Since the Dirichlet boundary condition is considered, it is natural to start from the restriction $\tilde{B}$ of the differential operator B to

$$\tilde{\mathcal{D}} = \{u \mid u \in C^2(\Omega) \cap L^p(\Omega) \cap H_0^1(\Omega), Bu \in L^p(\Omega)\}.$$

We shall show, for any pre-assigned $p(1 \leq p < \infty)$, the smallest closed extension A of the operator $\tilde{B}$ in $L^p(\Omega)$ is the generator of an analytic semigroup of bounded linear operators $T_t(t \geq 0)$ in $L^p(\Omega)$ (Theorem 2 in §4).

One of the most general results for the problem of the same direction is the result of El-Maati Ouhabaz[8], where it is proved that, for any strongly continuous semigroup $\{T_t : t \geq 0\}$ which admits a Gaussian estimate, the semigroup $\{e^{-wt}T_t\}$ (w being a suitable real number) is bounded analytic on the right half-plain, and that the result is applicable to the generator of the semigroup associated with the symmetric elliptic differential operator of the same form as B in (1.1) in the case where Ω possesses the extension property. However, in the present paper, we investigate the problem in entirely different point of view. We start from the concrete differential operator $\tilde{B}$ with the domain $\tilde{\mathcal{D}}$ as mentioned above, and construct the smallest closed extension of $\tilde{B}$ which is the generator of a C_0-semigroup. Then we prove, by means of Moser's mean value theorem [7] and a parabolic version of Caccioppoli's inequality, that the semigroup satisfies the condition to be an analytic semigroup.

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2 Preliminaries

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^m$; the definition of Lipschitz domains is stated as follows [1].

DEFINITION **2. 1** A bounded domain $\Omega \subset \mathbf{R}^m$ is called a *Lipschitz domain* if $\partial\Omega$ is covered by finitely many open right circular cylinders $L_1, L_2, \ldots, L_l$ whose bases have positive distance from $\partial\Omega$ and, corresponding to each cylinder L_k, there exist a coordinate system $(\tilde{x}', \tilde{x}_m) = (\tilde{x}_1, \ldots, \tilde{x}_{m-1}, \tilde{x}_m)$ with the $\tilde{x}_m$-axis parallel to the axis of L_k, a function $f_k(\tilde{x}')$ and a constant K such that $|f_k(\tilde{x}') - f_k(\tilde{y}')| \leq K|\tilde{x}' - \tilde{y}'|$(Lipschitz condition) and that $L_k \cap \partial\Omega$ is represented by $\tilde{x}_m = f_k(\tilde{x}')$.

We consider the parabolic differential equation $\frac{\partial u}{\partial t} = Bu$ with Dirichlet boundary condition on $\partial\Omega$; the differential operator B is expressed in the form:

$$Bu = \sum_{i,j=1}^{m} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x_i} \left(\sqrt{a(x)} a_{ij}(x) \frac{\partial u}{\partial x_j} \right), \qquad \left(a(x) = \det[a_{ij}(x)]^{-1}\right)$$

where $[a_{ij}]$ satisfies the strong ellipticity, i.e. $a_{ij}(x) = a_{ji}(x)$ and there exists a positive number δ_0 independent of x and ξ such that $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \delta_0|\xi|^2$ for any $x \in \Omega$ and any $\xi \in \mathbf{R}^m$. It is well known that the volume element $d_a x = \sqrt{a(x)}dx_1 \ldots dx_m$ is invariant under the transformation of a local coordinate system. Hereafter we always consider the function spaces $L^p(\Omega), H^1(\Omega)$ etc. with respect to the volume element $d_a x$; however, by virture of the strong ellipticity of B, the space $L^p(\Omega)$ e.g. is identical with the usual L^p-space with respect to the Lebesgue measure $dx_1 \ldots dx_m$. The inner product and the norm in $L^2(\Omega)$ are respectively given by

$$(u,v)_{L^2} = (u,v)_a \equiv \int_\Omega u(x)v(x)d_a x$$

and

$$\| u \|_{L^2}^2 = \| u \|_a^2 \equiv \int_\Omega |u(x)|^2 d_a x.$$

We define for any $u, v \in H^1(\Omega)$

$$(\nabla u(x) \cdot \nabla v(x))_a = \sum_{i,j} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}$$

where $\frac{\partial u(x)}{\partial x_i}$ denotes the generalized derivative, and

$$(\nabla u, \nabla v)_a = \int_\Omega (\nabla u(x) \cdot \nabla v(x))_a d_a x;$$

in particular if $u = v$, we use the notations

$$|\nabla u(x)|_a^2 = \sum_{i,j} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} \quad \text{and} \quad \| \nabla u \|_a^2 = \int_\Omega |\nabla u(x)|_a^2 d_a x.$$

Then the norm $\| \cdot \|_{H^1}$ in the Sobolev space $H^1(\Omega)$ is given by

$$\| u \|_{H^1}^2 = \| u \|_a^2 + \| \nabla u \|_a^2 .$$

Let C_0^∞ be the totality of infinitely differentiable functions with compact support in Ω and denote by $H_0^1(\Omega)$ the completion of C_0^∞ with respect to the norm in Sobolev space $H^1(\Omega)$.

Let (T_1, T_2) be any given interval and define $H^1(\Omega \times (T_1, T_2))$ and $H_0^1(\Omega \times (T_1, T_2))$ in the similar way as mentioned above. For the sake of brevity, these function spaces will be denoted by H^1 and H_0^1 respectively if no confusion occurs. For $u = u(x,t) \in H^1$, the notation $u_t = \frac{\partial u}{\partial t}$ is used in the generalized (distributional) sense.

DEFINITION **2. 2** A function $u \in H^1(\Omega \times (T_1, T_2))$ is called *a weak solution* of the parabolic differential equation

$$\frac{\partial u}{\partial t} = Bu \tag{2.1}$$

if it satisfies

$$\iint_{\Omega \times (T_1, T_2)} \{u_t \psi + (\nabla u \cdot \nabla \psi)_a\} d_a x dt = 0 \qquad \text{for any } \psi \in C_0^\infty; \tag{2.2}$$

if furthermore the solution u belongs to $H_0^1(\Omega)$ as a function of x for any fixed t, it is called *a weak solution of* (2.1) *with Dirichlet boundary condition.*

In §4, we have to consider the solution of the parabolic equation in a parabolic ball whose radius is independent of the center of any adopted coordinate neighborhood; so we have to extend the differential operator B and the solution of the equation to some part of the exterior of the domain Ω . For this purpose, we consider the coordinate transformation (such as mentioned in the following paragraphs) in every cylinder stated in Definition 2.1.

Let $L_1, \ldots, L_l$ be the cylinders stated in Definition 2.1. As for the coordinate system $(\tilde{x}_1, \ldots, \tilde{x}_{m-1}, \tilde{x}_m)$ in each cylinder L_k, we may assume that the intersection of the axis of L_k with $\partial\Omega$ is denoted by $(0, \ldots, 0, 0)$. Using the coordinate system in L_k, we define

$$L_k(\alpha) = \{(\tilde{x}_1, \ldots, \tilde{x}_{m-1}, \tilde{x}_m) \mid |\tilde{x}_j| < \alpha \ \ (1 \le j \le m)\}. \tag{2.3}$$

We may assume without loss of generality that

$$L_k(1) \subset L_k \ \ (k = 1, \ldots, l) \quad \text{and} \quad \bigcup_{k=1}^{l} L_k(\frac{1}{2}) \supset \partial\Omega. \tag{2.4}$$

Hereafter we treat an arbitrarily fixed cylinder L_k; the Lipschitz function f_k (mentioned in Definition 2.1) will be denoted simply by f.

We define a new coordinate system in a suitable subdomain of L_k where $|\tilde{x}_m|$ is sufficiently small by the following formulas (2.5); here we denote the new coordinate system also by $(x_1, \ldots, x_m)$ to simplify the notations. Let $\rho(\lambda)$ be a monotone decreasing C^∞ function on $0 \le \lambda < \infty$ satisfying the following conditions:

$$0 \le \rho(\lambda) \le 1 \ \text{ on } 0 \le \lambda < \infty, \ \ \rho(\lambda) = c \ \text{ for } \lambda \le \frac{1}{4K}, \ \ \rho(\lambda) = 0 \ \text{ for } \lambda \ge \frac{1}{2K}$$

(K being the Lipschitz constant of f and c a suitable positive constant)

and

$$\int_{\mathbf{R}^{m-1}} \rho(|\tilde{z}'|^2) d\tilde{z}' = 1 \quad \text{where } d\tilde{z}' = d\tilde{z}'_1 \ldots d\tilde{z}'_{m-1},$$

and define the coordinate transformation by the following formulas (2.5).

$$\begin{cases} \tilde{x}_1 &= x_1 \\ &\vdots \\ \tilde{x}_{m-1} &= x_{m-1} \\ \tilde{x}_m &= x_m + \int_{|z'| \le \frac{1}{2K}} \rho(|z'|^2) f(x' + x_m^2 z') dz' \ ; \end{cases} \tag{2.5}$$

the last expression is equivalent to

$$\tilde{x}_m = x_m + \int_{\mathbf{R}^{m-1}} \frac{1}{x_m^{2(m-1)}} \rho(\frac{|x' - z'|^2}{x_m^4}) f(z') dz' \tag{2.6}$$

as may be shown by a suitable substitution of the variable of integration.

We put $W = \{(x_1, \dots, x_{m-1}, x_m) \mid |x_j| < 1 \ (1 \le j \le m-1), \ \ |x_m| < \frac{1}{2}\}, W^+ = W \setminus \overline{\Omega}$ and $W^- = W \cap \Omega$. Then, by using (2.6) and by simple calculations, we may prove that the transformation $(\tilde{x}_i) \leftrightarrow (x_i)$ is one-to-one and bicontinuous, that $x_m > 0, x_m = 0, x_m < 0$ correspond to $W^+, W \cap \partial\Omega, W^-$ respectively, and that the partial derivetives $\frac{\partial \tilde{x}_i}{\partial \tilde{x}_j}$ $(i, j = 1, \dots, m)$ are continuous and the Jacobian of the transformation (2.5) does not vanish except on $\partial\Omega$. Hence the transformation is a local C^1-diffeomorphism in $W \setminus \partial\Omega$. We may also prove that the transformation is a local C^∞-diffeomorphism in $W \setminus \partial\Omega$. Hence we may consider the partial differentiations $\frac{\partial}{\partial x_j}$ in the inside and outside of Ω, individually.

By using this coordinate system (x_j), we extend the differential operator B and the weak solution u of the differential equation $\frac{\partial u}{\partial t} = Bu$ with Dirichlet boundary condition considered in $(W \cap \Omega) \times (T_1, T_2)$ to the differential operator and the weak solution in $W \times (T_1, T_2)$. The process of such extension is mentioned in the author's previous papers [2],[3] in detail. In the statement above, the *weak solution in* $W \times (T_1, T_2)$ is understood in the similar way as mentioned in Definition 2.2.

3 Construction of a C_0-semigroup

We shall consider $L^p(\Omega)$ where $1 \le p < \infty$. Let $\tilde{\mathcal{D}}$ be $\{u \mid u \in C^2(\Omega) \cap L^p(\Omega) \cap H_0^1(\Omega)$ and $Bu \in L^p(\Omega)\}$. $\tilde{\mathcal{D}}$ is dense in $L^p(\Omega)$ since $C_0^2(\Omega) \subset \tilde{\mathcal{D}}$. Let $\tilde{B}$ denote the restriction of B to $\tilde{\mathcal{D}}$. In this section, we show that the smallest closed extension A of $\tilde{B}$ generates a C_0-semigroup in $L^p(\Omega)$. Hereafter we shall denote by $\overline{C_0(\Omega)}$ the completion of $C_0(\Omega)$ with respect to the supremum-norm.

LEMMA **3.1** *Let λ be an arbitrarily fixed positive number. For any $f \in L^2(\Omega)$, there exists $v \in H_0^1(\Omega)$ such that*

$$\lambda(v, \phi)_a + (\nabla v, \nabla \phi)_a = (f, \phi)_a \quad \text{for any} \quad \phi \in C_0^1(\Omega) \tag{3.1}$$

and it holds that

$$\|v\|_{L^2}^2 \le \|v\|_{L^2}^2 + \frac{1}{\lambda}\|\nabla v\|_{L^2}^2 \le \frac{1}{\lambda^2}\|f\|_{L^2}^2. \tag{3.2}$$

The function $v \in H_0^1(\Omega)$ satisfying (3.1) *is uniquely determined by f. In particular, if $f \in \overline{C_0(\Omega)}$, then v is bounded and continuous in Ω and it holds that*

$$|v(x)| \le \frac{1}{\lambda}\|f\|_{L^\infty} \quad \text{for any} \quad x \in \Omega \tag{3.3}$$

and that

$$f \ge 0 \ \text{implies} \ v \ge 0; \tag{3.4}$$

if further $f \in C_0^1(\Omega)$, then v is of class C^2 and Bv is bounded in Ω.

PROOF The first assertion of this lemma is proved by means of the Lax-Milgram theorem [6]; the estimate (3.2) is readily derived from (3.1). If $f \in C_0^1(\Omega)$, then v is regarded as a function of class C^2 satisfying $\lambda v - Bv = f$ by Weyl's lemma, and accordingly Bv is bounded in Ω. The properties (3.3) and (3.4) may be proved by means of the maximum principle. If $f \in \overline{C_0(\Omega)}$, the boundedness and the continuity of v, (3.3) and (3.4) still hold since $C_0^1(\Omega)$ is dense in $\overline{C_0(\Omega)}$ with respect to the sup-norm $\|\cdot\|_\infty$. ///

We denote by G_λ the operator which maps $f \in \overline{C_0(\Omega)}$ to the function v uniquely determined in the sense of Lemma 3.1. Then G_λ is a bounded linear operator of $\overline{C_0(\Omega)}$ into $L^\infty(\Omega)$ satisfying that $\|G_\lambda f\|_{L^\infty} \le \frac{1}{\lambda}\|f\|_{L^\infty}$ for any $f \in \overline{C_0(\Omega)}$ and that $f \ge 0$ implies $G_\lambda f \ge 0$. Since $(G_\lambda f)(x)$ is continuous in Ω, the value of the function $(G_\lambda f)(x)$ is determined for every $x \in \Omega$. Hence, for any fixed x, $(G_\lambda f)(x)$ is a positive linear functional of $f \in \overline{C_0(\Omega)}$. Therefore, by the Riesz representation theorem, there exists a measure $G_\lambda(x;\cdot)$ in Ω such that $G_\lambda(x;\Omega) \le \frac{1}{\lambda}$ and that

$$(G_\lambda f)(x) = \int_\Omega G_\lambda(x;dy)f(y) \qquad \text{for any } f \in \overline{C_0(\Omega)}. \tag{3.5}$$

By means of this formula G_λ is extended to a bounded linear operator in $L^\infty(\Omega)$ satisfying $\|G_\lambda\| \le \frac{1}{\lambda}$.

We denote by J_λ the operator which maps $f \in L^2(\Omega)$ to the function v uniquely determined in the sense of Lemma 3.1. Then J_λ is a bounded linear operator of $L^2(\Omega)$ into $H_0^1(\Omega)$ and it holds that $\|J_\lambda f\|_{L^2} \le \frac{1}{\lambda}\|f\|_{L^2}$ for any $f \in L^2(\Omega)$.

It is clear from the definition of J_λ that $(J_\lambda f)(x) = (G_\lambda f)(x)$ in Ω for any $f \in \overline{C_0(\Omega)}$. Since $L^\infty(\Omega) \subset L^2(\Omega)$ from the boundedness of Ω, we have the following

LEMMA **3.2**

$$(J_\lambda f)(x) = (G_\lambda f)(x) \quad \textit{a.e.} \qquad \textit{for any } f \in L^\infty(\Omega).$$

PROOF Suppose f_n converges to some f monotonously where f_n and f are bounded and Borel measurable. Then, by means of the bounded convergence theorem, we obtain that $\lim_{n\to\infty}(G_\lambda f_n)(x) = (G_\lambda f)(x)$ for any $x \in \Omega$ and that $\lim_{n\to\infty}\|f_n - f\|_{L^2} = 0$, accordingly $\lim_{n\to\infty}\|J_\lambda f_n - J_\lambda f\|_{L^2} = 0$. Hence the set of all bounded Borel measurable functions f satisfying $(J_\lambda f)(x) = (G_\lambda f)(x)$ a.e. is a monotone class containing $C_0(\Omega)$. Therefore it contains all bounded Borel measurable functions. ///

The following two lemmas may be proved by means of Lemmas 3.1 and 3.2.

LEMMA **3.3** *For any $f \in L^\infty(\Omega)$, the function $u = G_\lambda f$ satisfies*

$$(u, \lambda\phi - B\phi)_a = (f,\phi)_a \quad \textit{for any } \phi \in C_0^2(\Omega).$$

LEMMA **3.4** *For any f and $g \in L^2(\Omega)$, we have $(J_\lambda f, g)_a = (f, J_\lambda g)_a$; in particular if f and $g \in L^\infty(\Omega)$, then $(G_\lambda f, g)_a = (f, G_\lambda g)_a$.*

LEMMA **3.5** *(Resolvent equation)*

$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu$. *In particular* $G_\lambda - G_\mu = (\mu - \lambda) G_\lambda G_\mu$ *in* $L^\infty(\Omega)$.

PROOF Put $u = J_\mu f$ for any $f \in C_0^1(\Omega)$. We get $u \in C^2(\Omega)$ and $(\mu - B)u = f$ by Lemma 3.1. Put $g = (\lambda - B)u$. Then the function $v = J_\lambda g$ satisfies

$$\lambda(v, \phi)_a + (\nabla v, \nabla \phi)_a = (g, \phi)_a \quad \text{for any} \quad \phi \in C_0^1(\Omega). \tag{3.6}$$

On the other hand it holds that

$$\lambda(u, \phi)_a + (\nabla u, \nabla \phi)_a = (\lambda u - Bu, \phi)_a = (g, \phi)_a \quad \text{for any} \quad \phi \in C_0^1(\Omega). \tag{3.7}$$

By (3.6), (3.7) and by the uniqueness of the function v stated in Lemma 3.1, we get $u = v = J_\lambda g = J_\lambda(\lambda - B)u$. Hence

$$\begin{aligned} J_\mu f = u &= J_\lambda(\lambda - \mu + \mu - B)u \\ &= J_\lambda(\lambda - \mu) J_\mu f + J_\lambda(\mu - B) J_\mu f \\ &= (\lambda - \mu) J_\lambda J_\mu f + J_\lambda f. \end{aligned}$$

Since $C_0^1(\Omega)$ is dense in $L^2(\Omega)$, the proof is complete. ///

LEMMA **3.6** *For any fixed* $\lambda > 0$ *and any* $f \in L^\infty(\Omega)$, *we have*

$$\lim_{\mu \to \infty} \|\mu G_\mu G_\lambda f - G_\lambda f\|_{L^\infty} = 0.$$

PROOF By the resolvent equation for $\{G_\mu\}$, we have

$$\mu G_\mu G_\lambda f - G_\lambda f = \lambda G_\lambda G_\mu f - G_\mu f \quad \text{for any } f \in L^\infty(\Omega).$$

Since G_λ satisfies $\|G_\lambda\| \le \frac{1}{\lambda}$, we obtain

$$\begin{aligned} \|\mu G_\mu G_\lambda f - G_\lambda f\|_{L^\infty} &\le \|\lambda G_\lambda G_\mu f\|_{L^\infty} + \|G_\mu f\|_{L^\infty} \\ &= \frac{1}{\mu}\|\lambda G_\lambda \mu G_\mu f\|_{L^\infty} + \frac{1}{\mu}\|\mu G_\mu f\|_{L^\infty} \\ &= \frac{1}{\mu}\{\|\lambda G_\lambda\|\|\mu G_\mu\|\|f\|_{L^\infty} + \|\mu G_\mu\|\|f\|_{L^\infty}\} \\ &\le \frac{2}{\mu}\|f\|_{L^\infty} \end{aligned}$$

Hence $\|\mu G_\mu G_\lambda f - G_\lambda f\|_{L^\infty}$ tends to 0 as $\mu \to \infty$. ///

Now we have the following lemma.

LEMMA **3.7** $(G_\lambda f)(x)$ *is continuous in* Ω *for any* $f \in L^\infty(\Omega)$ *and any* $\lambda > 0$.

PROOF We fix an integer n such that $2n+1 > \frac{m}{2}$ where m denotes the dimension of the space domain Ω, and let $\{\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a monotone increasing sequence.

First of all we consider the equation $(\lambda - B)u = f$ in Ω for $f \in L^\infty(\Omega)$ $(\subset L^2(\Omega))$.By using Lemma 3.1 we may see that the solution of this equation belongs to $H_0^1(\Omega)$, that is, $J_\lambda f \in H_0^1(\Omega)$. Next we consider the equation $(\lambda_1 - B)u = J_\lambda f$. By means of the theorem of regularity of weak solutions of elliptic partial differntial equations([4] Theorem 8.10), we get $u = J_{\lambda_1} J_\lambda f \in H^3_{\text{loc}}(\Omega) \cap L^2(\Omega)$. Repeating the same argument, we may prove successively that

$$J_{\lambda_1} J_\lambda f \in H^3_{\text{loc}}(\Omega) \cap L^2(\Omega), \quad J_{\lambda_2} J_{\lambda_1} J_\lambda f \in H^5_{\text{loc}}(\Omega) \cap L^2(\Omega), \ \ldots, \\ J_{\lambda_n} J_{\lambda_{n-1}} \cdots J_{\lambda_1} J_\lambda f \in H^{2n+1}_{\text{loc}}(\Omega) \cap L^2(\Omega).$$

Since $2n+1 > \frac{m}{2}$, we have $H^{2n+1}_{\text{loc}}(\Omega) \subset C^0(\Omega)$ by virture of Sobolev's lemma. Since operators $J_\lambda, J_{\lambda_1}, J_{\lambda_2}, \ldots, J_{\lambda_n}$ are mutually commutable and $J_\lambda f = G_\lambda f$ a.e. for any $\lambda > 0$ and any $f \in L^\infty(\Omega)$ (Lemma 3.2), we may see that $\lambda_1 G_{\lambda_1} \lambda_2 G_{\lambda_2} \ldots \lambda_n G_{\lambda_n} G_\lambda f$ is continuous in Ω. On the other hand, $\{\mu G_\mu G_\lambda f\}$ converges uniformly to $G_\lambda f$ as $\mu \to \infty$ by Lemma 3.6, and $\|\lambda_\nu G_{\lambda_\nu}\| \leq 1$ for $1 \leq \nu \leq n-1$ (here $\|\cdot\|$ denotes the operator norm in $L^\infty(\Omega)$). Hence, if we rewrite λ_n by μ and let $\mu \to \infty$, then $\lambda_1 G_{\lambda_1} \ldots \lambda_{n-1} G_{\lambda_{n-1}} \mu G_\mu G_\lambda f$ converges to $\lambda_1 G_{\lambda_1} \ldots \lambda_{n-1} G_{\lambda_{n-1}} G_\lambda f$ uniformly; accordingly $\lambda_1 G_{\lambda_1} \ldots \lambda_{n-1} G_{\lambda_{n-1}} G_\lambda f$ is continuous in Ω. Repeating this process we may finally conclude that $G_\lambda f$ is continuous in Ω. ///

LEMMA **3.8** *For any $\lambda > 0$ and any $x \in \Omega$,the measure $G_\lambda(x, E)$ is absolutely continuous, and the density $G_\lambda(x, y)$ is nonnegative for a.a.$y \in \Omega$ and satisfies*

$$\int_\Omega G_\lambda(x, y) d_a y \leq \frac{1}{\lambda}.$$

(Hereafter we use the expression "a.a." as the abbreviation of "almost all".)

PROOF For any Borel set E, the indicator $\chi_E \in L^\infty(\Omega) \subset L^2(\Omega)$. So $G_\lambda(x, E) = (G_\lambda \chi_E)(x)$. Hence $G_\lambda(x, E)$ is continuous in x as mentioned above and it holds that

$$\int_\Omega G_\lambda(x, E)^2 d_a x \leq \frac{1}{\lambda^2} \|\chi_E\|^2_{L^2} = \frac{1}{\lambda^2} |E| \qquad \text{where } |E| = \int_E d_a x.$$

Therefore $|E| = 0$ implies $G_\lambda(x, E) = 0$ for a.a.x and, by the continuity of $G_\lambda(x, E)$ in x, we get $G_\lambda(x, E) = 0$ for all x. Hence $G_\lambda(x, E)$ is absolutely continuous. ///

The following lemma may be proved by means of the standard measure-theoretical argument.

LEMMA **3.9** *Assume that $F(x, y)$ is a function on $\Omega \times \Omega$ satisfying the following two conditions:*

i) for any fixed $x \in \Omega$, $F(x, y)$ is measurable in y and satisfies that $\int_\Omega |F(x, y)| d_a y \leq M$ where M is a constant independent of x;

ii) $\int_\Omega |F(x, y)| \phi(y) d_a y$ is continuous in x for any $\phi \in C_0(\Omega)$.

Then there exists a function $\tilde{F}(x,y)$ measurable on $\Omega \times \Omega$ satifying that $F(x,y) = \tilde{F}(x,y)$ for a.a.$y \in \Omega$ for every $x \in \Omega$.

For any $\lambda > 0$, the density function $G_\lambda(x,y)$ in Lemma 3.8 satisfies the assumption for $F(x,y)$ in Lemma 3.9. Hence this lemma assures the existence of a measurable function $\tilde{G}_\lambda(x,y)$ on $\Omega \times \Omega$ such that $\tilde{G}_\lambda(x,y) = G_\lambda(x,y)$ for a.a.$y \in \Omega$ for every $x \in \Omega$. Therefore we can replace the function $G_\lambda(x,y)$ in Lemma 3.8 by the measurable function $\tilde{G}_\lambda(x,y)$ on $\Omega \times \Omega$. We hereafter consider that the function $G_\lambda(x,y)$ itself is mesurable on $\Omega \times \Omega$.

By virture of the measurability, we may apply Fubini's theorem to the function $G_\lambda(x,y)$; for instance, we may show that Lemma 3.4 implies the following Lemma 3.10.

LEMMA **3.10**

$$G_\lambda(x,y) = G_\lambda(y,x) \qquad \textit{for a.a.}(x,y) \in \Omega \times \Omega.$$

COROLLARY

$$\int_\Omega G_\lambda(x,y) d_a x \leq \frac{1}{\lambda} \qquad \textit{for a.a.y.}$$

From now on, throughout this section, let p $(1 \leq p < \infty)$ be arbitrarily fixed.

For any $\lambda > 0$, the function $u(x)$ defined by

$$u(x) = \int_\Omega G_\lambda(x,y) f(x) d_a y, \quad f \in L^\infty(\Omega)$$

satisfies $\|u\|_{L^p} \leq \frac{1}{\lambda}\|f\|_{L^p}$ as may be seen from the Hölder-Riesz inequality. Hence we can define the operator J_λ in $L^p(\Omega)$ by

$$(J_\lambda f)(x) = \int_\Omega G_\lambda(x,y) f(y) d_a y \qquad \text{for any } f \in L^p(\Omega). \tag{3.8}$$

J_λ is a bounded linear operator in $L^p(\Omega)$ satisfying $\|J_\lambda\| \leq \frac{1}{\lambda}$. In case $p = 2$, the definition of J_λ mentioned above consists with the definition of J_λ in $L^2(\Omega)$ mentioned before.

The following lemma is derived from Lemma 3.5 since $L^\infty(\Omega)$ is dense in $L^p(\Omega)$ with respect to L^p-norm.

LEMMA **3.11** *(Resolvent equation)*

$$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu \qquad \textit{in } L^p(\Omega) \quad \textit{for any } \lambda \textit{ and } \mu.$$

COROLLARY *The range $\mathcal{R}(J_\lambda)$ of J_λ is independent of λ.*

LEMMA **3.12** *$\mathcal{R}(J_\lambda)$ is dense in $L^p(\Omega)$.*

PROOF For any $u \in C_0^2(\Omega)$, $f = (\lambda - B)u$ is in $C_0(\Omega) \subset L^p(\Omega)$. The function $v = J_\lambda f$ is also written as $v = G_\lambda f$. Hence v is the function uniquely determined by f in the sense mentioned in Lemma 3.1. Thus we obtain $u = v \in \mathcal{R}(J_\lambda)$. Since $C_0^2(\Omega)$ is dense in $L^p(\Omega)$, $\mathcal{R}(J_\lambda)$ is also dense in $L^p(\Omega)$. ///

LEMMA **3.13** *The operator J_λ is one-to-one in $L^p(\Omega)$, the inverse operator J_λ^{-1} with the domain $\mathcal{D}(J_\lambda^{-1}) = \mathcal{R}(J_\lambda)$ is a closed operator.*

PROOF For any $f \in L^p(\Omega)$, we put $u = J_\lambda f$. If $f \in C_0^1(\Omega) \subset L^2(\Omega)$, then the function $J_\lambda f$ defined in $L^p(\Omega)$ is identical with that defined in $L^2(\Omega)$. Hence, by Lemma 3.3, we have

$$\langle u, \lambda\phi - B\phi\rangle_a = \langle f, \phi\rangle_a \quad \text{for any } \phi \in C_0^2(\Omega). \tag{3.9}$$

We use the notation $\langle u, v\rangle_a = \int_\Omega u(x)v(x)d_a x$ whenever the right-hand side makes sense. Any $f \in L^p(\Omega)$ is the limit of a sequence $\{f_n\} \subset C_0^1(\Omega)$ in $L^p(\Omega)$, whence $u_n = J_\lambda f_n$ converges to $u = J_\lambda f$ in $L^p(\Omega)$ by the definition (3.8) of J_λ. Therefore (3.9) holds for any $f \in L^p(\Omega)$. Hence $u = 0$ implies $f = 0$. Thus we see that the operator J_λ is one-to-one. Since J_λ is bounded and accordingly closed, the inverse operator J_λ^{-1} is also closed. ///

LEMMA **3.14** *The operator A defined by $A = \lambda I - J_\lambda^{-1}$ is a closed operator independent of λ. The domain $\mathcal{D}(A)$ of A is dense in $L^p(\Omega)$.*

PROOF By Corollary to Lemma 3.11, $\mathcal{D}(A) = \mathcal{R}(J_\lambda)$ is independent of λ, and $\mathcal{D}(A)$ is dense in $L^p(\Omega)$ by Lemma 3.12. It holds that $\lambda J_\lambda J_\mu f - J_\mu f = \mu J_\lambda J_\mu f - J_\lambda f$ for any $f \in L^p(\Omega)$ by the resolvent equation (Lemma 3.11). For any $u \in \mathcal{D}(A)$, we apply the above equation to $f = J_\mu^{-1}u$. Then we get $\lambda J_\lambda u - u = \mu J_\lambda u - J_\lambda J_\mu^{-1}u$, which implies $\lambda u - J_\lambda^{-1}u = \mu u - J_\mu^{-1}u$. Thus we obtain that $A = \lambda I - J_\lambda^{-1}$ is independent of λ. Hence A is a closed operator by virture of Lemma 3.13. ///

LEMMA **3.15** *The operator A is the smallest closed extension of $\tilde{B}$.*

PROOF It may easily be proved that A is a closed extension of $\tilde{B}$. We shall show that it is the smallest one. Let A_1 be an arbitrary closed extension of $\tilde{B}$. For any $u \in \mathcal{D}(A)$ there exists $f \in L^p(\Omega)$ such that $u = J_\lambda f$. We take a sequence $\{f_n\} \subset C_0^2(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$. Then $u_n = J_\lambda f_n$ converges to u in $L^p(\Omega)$. Since u_n is also denoted by $G_\lambda f_n$, u_n is the function uniquly determined by f_n as mentioned in Lemma 3.1. Accordingly $u_n \in \tilde{\mathcal{D}}$ and $\tilde{B}u_n = Bu_n$ by Lemma 3.1. Since A_1 is an extension of $\tilde{B}$, we get $u_n \in \mathcal{D}(A_1)$ and $A_1 u_n = Bu_n = \lambda u_n - f_n$ converges to $\lambda u - f$ in $L^p(\Omega)$. Since A_1 is closed, we obtain $u \in \mathcal{D}(A_1)$ and $A_1 u = \lambda u - f = Au$. Lemma 3.15 is thus proved. ///

We can summarize the discussion mentioned above as follows. The operator $A = \lambda I - J_\lambda^{-1}$ is a natural closed extension of $\tilde{B}$. The domain $\mathcal{D}(A)$ is dense in $L^p(\Omega)$, and $\|(\lambda - A)^{-1}\| = \|J_\lambda\| \leq \frac{1}{\lambda}$ for any $\lambda > 0$. Therefore, by the Hille-Yosida theorem, we obtain the following

THEOREM 1 *The operator A generates a contraction C_0-semigroup $\{T_t\}$ in $L^p(\Omega)$, namely $\frac{d}{dt}T_t = AT_t$ where $\frac{d}{dt}$ denotes the strong derivative in $L^p(\Omega)$.*

4 Analyticity of the C_0-semigroup

We mentioned the main purpose of this paper in §1. In this section we first state our result as the following theoem.

THEOREM 2 *The operator A in §3 is the generator of an analytic semigroup of bounded linear operators $\{T_t\}_{t\geq 0}$ in $L^p(\Omega)$ $(1 \leq p < \infty)$.*

In order to prove Theorem 2, it is sufficient to show the following two propositions i) and ii):

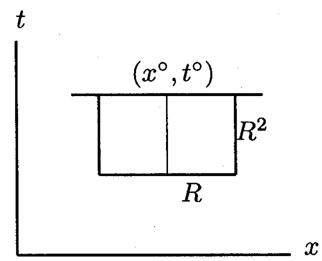
i) The operator A is the infinitesimal generator of a C_0-semigroup $\{T_t\}_{t\geq 0}$ in $L^p(\Omega)$.

ii) For any fixed $M > 0$, there exists a constant $C_M > 0$ such that $u = e^{tA}u_0$ satisfies $\|\frac{\partial u(\cdot,t)}{\partial t}\|_{L^p} \leq \frac{C_M}{t}\|u_0\|_{L^p}$ for any $u_0 \in L^p(\Omega)$ and any $t \in (0, M)$.

We already proved i) in §3. We shall prove ii) in this section.

We define the "ball" $B_R = B_R(x^\circ)$ in $\mathbf{R}^m$ with center x° and radius R and the *parabolic ball $Q_R = Q_R(x^\circ, t^\circ)$ with center (x°, t°) and radius R* as follows:

- $B_R(x^\circ) = \{x \mid |x - x^\circ| < R\}$ $\qquad (|x| = \max_i |x_i|)$
- $Q_R(x^\circ, t^\circ) = \{(x, t) \mid |x - x^\circ| < R,\ 0 \leq t^\circ - t < R^2\}$



Hereafter we always assume $0 < R < 1$.

Let $L_1, \ldots, L_l$ be the cylinders stated in §2. In each L_k, we consider the coordinate system $(x_1, \ldots, x_{m-1}, x_m)$ defined in §2 and the differential operator B extended to W_k defined by

$$W_k = \{(x_1, \ldots, x_{m-1}, x_m) \mid |x_j| < 1 \ \ (1 \leq j \leq m-1), \ \ |x_m| < \frac{1}{2}\}$$

with respect to the coordinate system in L_k (see the last two paragraphs of §2). We put

$$W'_k = \{(x_1, \ldots, x_{m-1}, x_m) \mid |x_j| < \frac{1}{2} \ \ (1 \leq j \leq m-1), \ \ |x_m| < \frac{1}{4}\}.$$

Then we may readily see from (2.3),(2.4) and (2.5) that

$$\bigcup_{k=1}^{l} W'_k \supset \partial\Omega. \tag{4.1}$$

Hence $\delta = \operatorname{dist}(\partial\Omega, \ \Omega \setminus \bigcup_{k=1}^{l} W'_k)$ is positive, where $\operatorname{dist}(\cdot, \cdot)$ denotes the Euclidean distance with respect to the orthogonal coordinate system originally defined in $\mathbf{R}^m$. We put

$$D = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{\delta}{4}\} \ \text{ and } \ D' = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{\delta}{2}\}.$$

Then

$$\overline{\Omega} \subset W'_1 \cup W'_2 \cup \ldots \cup W'_l \cup D'. \tag{4.2}$$

For any point $x^\circ \in W'_k$ $(1 \leq k \leq l)$, we consider the "ball" $B_R(x^\circ)$ and the parabolic ball $Q_R(x^\circ, t^\circ)$ with respect to the coordinate system in W_k; for any point $x^\circ \in D'$, we consider $B_R(x^\circ)$ and $Q_R(x^\circ, t^\circ)$ with respect to the original coordinate system in $\mathbf{R}^m$, where we may assume without loss of generality that D' contains the origin $\mathbf{0} = (0, \ldots, 0)$. We fix a positive number $M \leq \frac{\delta}{8}$. Then, if $0 < R \leq \min\{\frac{1}{8}, \frac{\delta}{8}\}$ and $(2R)^2 < t^\circ < M$, we have

$$B_{2R}(x^\circ) \subset W_k \quad \text{and} \quad Q_{2R}(x^\circ, t^\circ) \subset W_k \times (0, M) \tag{4.3}$$

or

$$B_{2R}(x^\circ) \subset D \quad \text{and} \quad Q_{2R}(x^\circ, t^\circ) \subset D \times (0, M) \tag{4.4}$$

according as $x^\circ \in W'_k$ or $x^\circ \in D'$.

LEMMA **4.1** *If u is a weak solution of $\frac{\partial u}{\partial t} = Bu$ in a domain containing the closure of $Q_R(x^\circ, t^\circ)$, then there exists a constant c_1 such that*

$$|u(x,t)| \leq c_1 \left(\frac{1}{|Q_R(x^\circ, t^\circ)|} \iint\limits_{Q_R(x^\circ, t^\circ)} u(y,s)^2 d_a y ds \right)^{\frac{1}{2}} \quad \text{for any } \ (x,t) \in Q_{\frac{R}{2}}(x^\circ, t^\circ) \tag{4.5}$$

where $|Q_R(x^\circ, t^\circ)|$ denotes the Lebesgue measure of $Q_R(x^\circ, t^\circ)$; the constant c_1 is independent of R and u.

This lemma is essentially due to Theorem 3 in [7; p.113]. Under the assumption of our lemma, the theorem says: the supremum of the left-hand side of (4.5) over $Q_{R'}(x^\circ, t^\circ)$ is estimated by the right-hand side where the constant c_1 depends on R' and R. However, we may see from the proof of the theorem stated in [7] that c_1 depends only on the ratio of R' to R. Hence we obtain Lemma 4.1.

Let $\{T_t\}$ be the contraction C_0-semigroup generated by $A(= \lambda I - J_\lambda^{-1})$ in §3. Then, from the argument in §3, it may readily be seen that

$$\langle -Av, \psi \rangle_a = (\nabla v, \nabla \psi)_a \quad \text{for any } v \in \mathcal{D}(A) \text{ and any } \psi \in C_0^\infty(\Omega). \tag{4.6}$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to H^1-norm, the above relation (4.6) holds for any $\psi \in H_0^1(\Omega)$.

We consider the solution $u \equiv u(x,t)$ defined by $u(\cdot, t) = T_t u_0 = e^{tA}u_0$ for any given $u_0 \in L^p(\Omega)$; see the proposition ii) stated below Theorem 2. Since $\mathcal{D}(A^2)$ is dense in $L^p(\Omega)$, it is sufficient to prove the proposition ii) under the assumption: $u_0 \in \mathcal{D}(A^2)$. Then, for any $s > 0$, we have $u(\cdot, s) \in \mathcal{D}(A)$ and $\dfrac{\partial u(\cdot,s)}{\partial s} = AT_s u_0 = T_s A u_0$. Since $Au_0 \in \mathcal{D}(A)$, we have $\dfrac{\partial u(\cdot,s)}{\partial s} \in \mathcal{D}(A)$. Since $\mathcal{D}(A) = \mathcal{R}(J_\lambda) \subset H_0^1(\Omega)$ as in §3, it follows from (4.6) that

$$\left\langle -Au(\cdot,s), \frac{\partial u(\cdot,s)}{\partial s} \right\rangle_a = \left(\nabla u(\cdot,s), \nabla \frac{\partial u(\cdot,s)}{\partial s} \right)_a \quad \text{for any } s. \tag{4.7}$$

Now we investigate $u(x,t)$ in $(W_k \cap \Omega) \times (0, M)$ and in $D \times (0, M)$ (see (4.2),(4.3) and (4.4)); we may use the fact that the solution $u(x,t)$ considered in $(W_k \cap \Omega) \times (0, M)$ is extended to the solution in $W_k \times (0, M)$ stated in §2. It may readily be seen from the process of the extension that

the extended solution u is a weak solution of the parabolic equation $\dfrac{\partial u}{\partial t} = Bu$ in $W_k \times (0, M)$. (4.8)

Since the operator A is an extension of $\tilde{B}$, it follows from (4.7) and (4.8) that

$$\left\langle -Bu(\cdot,s), \frac{\partial u(\cdot,s)}{\partial s}\varphi(\cdot,s) \right\rangle_a = \left(\nabla u(\cdot,s), \nabla \left(\frac{\partial u(\cdot,s)}{\partial s}\varphi(\cdot,s) \right) \right)_a \tag{4.9}$$

for any function $\varphi(y,s) \in C_0^\infty(\Omega \times (0,\infty))$. Using this fact, we prove the following lemma, which is a parabolic version of Caccioppoli's inequality.

LEMMA **4.2** *Let $u = u(y,s)$ be as mentioned above, and assume that $\dfrac{t}{4\sqrt{2}} < R^2 < \dfrac{t}{4} < M$. Then*

$$\iint_{Q_R(x,t)} \left| \frac{\partial u}{\partial s}(y,s) \right|^2 d_a y ds \le \frac{c_2}{t^2} \iint_{Q_{2R}(x,t)} |u(y,s)|^2 d_a y ds;$$

where c_2 is a constant independent of x, t, R and m.

Proof Throughout the proof of this lemma, $Q_R(x,t)$ will be briefly denoted by Q_R.

Let $\varphi_0(y)$ and $\varphi_1(s)$ be functions of class C^∞ in Ω and in $(0,\infty)$ respectively satisfying that

$$\begin{cases} 0 \le \varphi_0(y) \le 1 \text{ in } \Omega, & \varphi_0(y) = \begin{cases} 1 & \text{if } \ y \in B_R(x) \\ 0 & \text{if } \ y \notin B_{2R}(x), \end{cases} \quad |\nabla\varphi_0(y)|_a \le \frac{C_0}{R} \text{ in } B_{2R}, \\ 0 \le \varphi_1(s) \le 1 \text{ in } (0,\infty), & \varphi_1(s) = \begin{cases} 1 & \text{if } \ 0 \le t-s \le R^2 \\ 0 & \text{if } \ t-s \ge (2R)^2, \end{cases} \quad \left|\frac{\partial\varphi_1(s)}{\partial s}\right| \le \frac{C_0^2}{R^2} \text{ in } (0,\infty) \end{cases}$$

for a suitable constant C_0, and define $\varphi(y,s) = \varphi_0(y)\varphi_1(s)$. Then

$$\begin{aligned} &\iint\limits_{Q_R} \left|\frac{\partial u}{\partial s}(y,s)\right|^2 d_a y ds \le \iint\limits_{Q_{2R}} \left|\frac{\partial u}{\partial s}(y,s)\right|^2 \varphi(y,s)^4 d_a y ds \\ = &\iint\limits_{Q_{2R}} Bu(y,s)\frac{\partial u}{\partial s}(y,s)\varphi(y,s)^4 d_a y ds. \end{aligned}$$

By (4.9), the above integral equals

$$\begin{aligned} & -\iint\limits_{Q_{2R}} \left(\nabla u(y,s)\cdot\nabla\left(\frac{\partial u}{\partial s}(y,s)\varphi(y,s)^4\right)\right)_a d_a y ds \\ = & -\iint\limits_{Q_{2R}} \left(\varphi(y,s)^4\nabla u(y,s)\cdot\nabla\left(\frac{\partial u}{\partial s}(y,s)\right)\right)_a d_a y ds \\ & \qquad -\iint\limits_{Q_{2R}} \left(\frac{\partial u}{\partial s}(y,s)\nabla u(y,s)\cdot\nabla\left(\varphi(y,s)^4\right)\right)_a d_a y ds \\ = & -\frac{1}{2}\iint\limits_{Q_{2R}} \frac{\partial}{\partial s}\left(\nabla u(y,s)\cdot\nabla u(y,s)\right)_a \varphi(y,s)^4 d_a y ds \\ & \quad -\iint\limits_{Q_{2R}} \frac{\partial u}{\partial s}(y,s)\left(\nabla u(y,s)\cdot 4\varphi(y,s)^3\nabla\varphi(y,s)\right)_a d_a y ds. \end{aligned} \tag{4.10}$$

In order to estimate each term of the above expression (4.10), we prepare the following inequalities.

$$\begin{aligned} &\iint\limits_{Q_{2R}} \left(\varphi(y,s)^2\nabla u(y,s)\cdot\nabla u(y,s)\right)_a d_a y ds \\ = & -\iint\limits_{Q_{2R}} \left(\nabla\left(\varphi(y,s)^2\right)\cdot\nabla u(y,s)\right)_a u(y,s) d_a y ds - \iint\limits_{Q_{2R}} \varphi(y,s)^2\left(Bu(y,s)\right)u(y,s) d_a y ds \\ \le & \left|\iint\limits_{Q_{2R}} \left(2\varphi(y,s)\nabla\varphi(y,s)\cdot\nabla u(y,s)\right)_a u(y,s) d_a y ds\right| + \left|\iint\limits_{Q_{2R}} \varphi(y,s)^2\frac{\partial u}{\partial s}(y,s)u(y,s) d_a y ds\right| \\ \le & \iint\limits_{Q_{2R}} \frac{2C_0}{R}|u(y,s)||\nabla u(y,s)|_a|\varphi(y,s)| d_a y ds + \iint\limits_{Q_{2R}} \varphi(y,s)^2\left|\frac{\partial u}{\partial s}(y,s)\right||u(y,s)| d_a y ds\ ; \end{aligned}$$

accordingly

$$\begin{aligned}
&\frac{2C_0^2}{R^2}\iint\limits_{Q_{2R}}|\varphi(y,s)|^2|\nabla u(y,s)|_a^2 d_a y ds\\
\leq\ & \alpha\frac{2C_0^4}{R^4}\iint\limits_{Q_{2R}}|u(y,s)|^2 d_a y ds+\frac{1}{\alpha}\frac{2C_0^2}{R^2}\iint\limits_{Q_{2R}}|\nabla u(y,s)|_a^2|\varphi(y,s)|^2 d_a y ds\\
&+\beta\frac{C_0^4}{R^4}\iint\limits_{Q_{2R}}|u(y,s)|^2 d_a y ds+\frac{1}{\beta}\iint\limits_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds
\end{aligned}$$

where α and β are constants > 1 determined later. Hence

$$\begin{aligned}
&\frac{2C_0^2}{R^2}\left(1-\frac{1}{\alpha}\right)\iint\limits_{Q_{2R}}|\varphi(y,s)|^2|\nabla u(y,s)|_a^2 d_a y ds\\
\leq\ & (2\alpha+\beta)\frac{C_0^4}{R^4}\iint\limits_{Q_{2R}}|u(y,s)|^2 d_a y ds+\frac{1}{\beta}\iint\limits_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds.
\end{aligned}$$

Taking a constant k such that $k(1-\frac{1}{\alpha})\geq 1$, we get

$$\begin{aligned}
&\frac{2C_0^2}{R^2}\iint\limits_{Q_{2R}}|\varphi(y,s)|^2|\nabla u(y,s)|_a^2 d_a y ds\\
\leq\ & k(2\alpha+\beta)\frac{C_0^4}{R^4}\iint\limits_{Q_{2R}}|u(y,s)|^2 d_a y ds+\frac{k}{\beta}\iint\limits_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds \quad (4.11)
\end{aligned}$$

Using the inequality (4.11), we may achieve the following estimates.
The first term of (4.10):

$$\begin{aligned}
&-\frac{1}{2}\iint\limits_{Q_{2R}}\frac{\partial}{\partial s}(\nabla u(y,s)\cdot\nabla u(y,s))_a\varphi(y,s)^4 d_a y ds\\
=\ & -\frac{1}{2}\int_{B_{2R}}|\nabla u(y,t)|_a^2\varphi(y,t)^4 d_a y+\frac{1}{2}\iint\limits_{Q_{2R}}\left(\frac{\partial}{\partial s}\left(\varphi(y,s)^4\right)\nabla u(y,s)\cdot\nabla u(y,s)\right)_a d_a y ds\\
\leq\ & \left|\frac{1}{2}\iint\limits_{Q_{2R}}4\varphi(y,s)^3\frac{\partial\varphi}{\partial s}(y,s)\left(\nabla u(y,s)\cdot\nabla u(y,s)\right)_a d_a y ds\right|\\
\leq\ & \frac{2C_0^2}{R^2}\iint\limits_{Q_{2R}}|\varphi(y,s)|^2|\nabla u(y,s)|_a^2 d_a y ds\\
\leq\ & k(2\alpha+\beta)\frac{C_0^4}{R^4}\iint\limits_{Q_{2R}}|u(y,s)|^2 d_a y ds+\frac{k}{\beta}\iint\limits_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds. \qquad (4.12)
\end{aligned}$$

The second term of (4.10):

$$\left|-\iint\limits_{Q_{2R}}\frac{\partial u}{\partial s}(y,s)\left(\nabla u(y,s)\cdot 4\varphi(y,s)^3\nabla\varphi(y,s)\right)_a d_a y ds\right|$$

$$\leq 2\gamma\frac{C_0^2}{R^2}\iint_{Q_{2R}}\varphi(y,s)^2|\nabla u(y,s)|_a^2 d_a y ds+\frac{2}{\gamma}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds$$

$$\leq k\gamma(2\alpha+\beta)\frac{C_0^4}{R^4}\iint_{Q_{2R}}|u(y,s)|^2 d_a y ds+\frac{k\gamma}{\beta}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds$$

$$+\frac{2}{\gamma}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds \qquad (4.13)$$

where γ is a constant > 1 determined later.

It follows from (4.10),(4.12) and (4.13) that

$$\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds \leq k(2\alpha+\beta)\frac{C_0^4}{R^4}\iint_{Q_{2R}}|u(y,s)|^2 d_a y ds$$

$$+\frac{k}{\beta}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds+k\gamma(2\alpha+\beta)\frac{C_0^4}{R^4}\iint_{Q_{2R}}|u(y,s)|^2 d_a y ds$$

$$+\frac{k\gamma}{\beta}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds+\frac{2}{\gamma}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds,$$

accordingly

$$\{1-\frac{k(1+\gamma)}{\beta}-\frac{2}{\gamma}\}\iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds \leq k(1+\gamma)(2\alpha+\beta)\frac{C_0^4}{R^4}\iint_{Q_{2R}}|u(y,s)|^2 d_a y ds.$$

Here we choose α,β,γ and k in such a way that $1-\dfrac{k(1+\gamma)}{\beta}-\dfrac{2}{\gamma}>0$ and consequently $C_0'\equiv k(1+\gamma)(2\alpha+\beta)/\{1-\frac{k(1+\gamma)}{\beta}-\frac{2}{\gamma}\}>0$; for instance, we may put $\alpha=2,\beta=21,\gamma=4$ and $k=2$. Then we get

$$\iint_{Q_R}\left|\frac{\partial u}{\partial s}(y,s)\right|^2 d_a y ds \leq \iint_{Q_{2R}}\left|\frac{\partial u}{\partial s}(y,s)\right|^2\varphi(y,s)^4 d_a y ds \leq C_0'\frac{C_0^4}{R^4}\iint_{Q_{2R}}|u(y,s)|^2 d_a y ds.$$

Since $t<4\sqrt{2}R^2$, we obtain the following inequlity where $c_2=32C_0'C_0^4$:

$$\iint_{Q_R}\left|\frac{\partial u}{\partial s}(y,s)\right|^2 d_a y ds \leq \frac{c_2}{t^2}\iint_{Q_{2R}}|u(y,s)|^2 d_a y ds.///$$

Now we carry out the estimate of $\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|_{L^p}$ in the proposition ii) stated below Theorem 2. Since $\mathcal{D}(A^2)$ is dense in $L^p(\Omega)$, it suffices to prove the estimate under the assumption: $u_0\in\mathcal{D}(A^2)$. This assumption implies that the strong derivative $\frac{du}{dt}$ in $L^p(\Omega)$ satisfies $\frac{d}{dt}\left(\frac{du}{dt}\right)=A\left(\frac{du}{dt}\right)$. Hence we may see that

the distributional derivative $u_t=\dfrac{\partial u}{\partial t}$ of the above solution
u is a weak solution of the parabolic equation $\dfrac{\partial u_t}{\partial t}=Bu_t$ (4.14)
with Dirichlet boundary condition stated in Definition 2.2.

The original solution $u(x,t)$ in $(\Omega \cap W_k) \times (0,M)$ is extended to the solution in $W_k \times (0,M)$ as stated in §2. By virtue of the property (4.14) of the original solution, the extended solution $u(x,t)$ satisfies that

$$\begin{aligned}&\text{the distributional derivative } u_t = \frac{\partial u}{\partial t} \text{ is a weak solution}\\ &\text{of the parabolic equation } \frac{\partial u_t}{\partial t} = Bu_t \text{ in } W_k \times (0,M).\end{aligned} \tag{4.15}$$

We take an arbitrary point $(x,t) \in W_k' \times (0,M)$ and a positive number $R < \frac{1}{8}$ such that $\frac{t}{\sqrt{2}} < (2R)^2 < t$. Then, by means of (4.3) we have

$$B_{2R}(x) \subset W_k \quad \text{and} \quad Q_{2R}(x,t) \subset W_k \times (0,M).$$

Hence, because of (4.15), we may apply Lemma 4.1 to $u_t = \frac{\partial u}{\partial t}$ and we have

$$\left|\frac{\partial u}{\partial t}(x,t)\right| \le c_1 \left(\frac{1}{|Q_R(x,t)|} \iint_{Q_R(x,t)} |u_s(y,s)|^2 d_a y ds \right)^{\frac{1}{2}}. \tag{4.16}$$

By Lemma 4.2, the right-hand side of the above inequality is less than or equal to

$$\begin{aligned}
& c_1 \left(\frac{c_2}{t^2} \frac{1}{|Q_R(x,t)|} \iint_{Q_{2R}(x,t)} |u(y,s)|^2 d_a y ds \right)^{\frac{1}{2}} \\
= \; & \frac{c_3}{t} \left\{ \frac{1}{|Q_{2R}(x,t)|} \iint_{Q_{2R}(x,t)} |u(y,s)|^2 d_a y ds \right\}^{\frac{1}{2}} \qquad (c_3 = 2^{\frac{m}{2}+1} c_1 c_2^{\frac{1}{2}}) \\
\le \; & \frac{c_3}{t} \left\{ \int_{t-4R^2}^{t} \frac{1}{4R^2 |B_{2R}(x)|} ds \int_{B_{2R}(x)} |u(y,s)|^p d_a y \right\}^{\frac{1}{p}} \qquad (p \ge 2).
\end{aligned} \tag{4.17}$$

The last inequality, which is evident for $p = 2$, may be seen for $p > 2$ from the following relation; by Hölder's inequality, we have

$$\begin{aligned}
& \frac{1}{|Q_{2R}(x,t)|} \iint_{Q_{2R}(x,t)} |u(y,s)|^2 d_a y ds \\
\le \; & \left(\iint_{Q_{2R}(x,t)} |u(y,s)|^p d_a y ds \right)^{\frac{2}{p}} \left(\iint_{Q_{2R}(x,t)} \left(\frac{1}{|Q_{2R}(x,t)|} \right)^r d_a y ds \right)^{\frac{1}{r}} \qquad \left(\frac{2}{p} + \frac{1}{r} = 1 \right) \\
= \; & \frac{1}{|Q_{2R}(x,t)|^{\frac{2}{p}}} \left(\int_{t-4R^2}^{t} ds \int_{B_{2R}(x)} |u(y,s)|^p d_a y \right)^{\frac{2}{p}}.
\end{aligned}$$

The Lebesgue measure $|B_{2R}(x)|$ of $B_{2R}(x)$ with respect to the coordinate system $(x_1, \ldots, x_m)$ in W_k is $(4R)^m$(independent of x), so we denote it simply by $|B_{2R}|$, and define $\phi_{2R}(x) = \frac{1}{|B_{2R}|} \chi_{B_{2R}(0)}(x)$. Then we have

$$\phi_{2R}(x-y) = \frac{1}{|B_{2R}|} \chi_{B_{2R}(0)}(x-y) = \frac{1}{|B_{2R}|} \chi_{B_{2R}(x)}(y) \tag{4.18}$$

and

$$\int_{W_k} \phi_{2R}(x-y)d_a x \le c_0 \tag{4.19}$$

where c_0 is a constant determined by the density $\sqrt{a(x)}$ of the measure $d_a x$; we may assume that $c_0 \ge 1$.

It follows from (4.16), (4.17) and (4.18) that

$$\left|\frac{\partial u}{\partial t}(x,t)\right|^p \le \left(\frac{c_3}{t}\right)^p \int_{t-4R^2}^{t} \frac{1}{4R^2} ds \int_{W_k} \phi_{2R}(x-y)|u(y,s)|^p d_a y. \tag{4.20}$$

Integrating both sides of (4.20) over W'_k with respect to x, changing the order of integrations and using (4.19), we obtain

$$\begin{aligned}\int_{W'_k}\left|\frac{\partial u}{\partial t}(x,t)\right|^p d_a x &\le \left(\frac{c_3}{t}\right)^p \int_{t-4R^2}^{t} \frac{1}{4R^2} ds \int_{W_k} |u(y,s)|^p d_a y \int_{W'_k} \phi_{2R}(x-y) d_a x \\ &\le \left(\frac{c_3}{t}\right)^p c_0 \int_{t-4R^2}^{t} \frac{1}{4R^2} ds \int_{W_k} |u(y,s)|^p d_a y. \end{aligned}\tag{4.21}$$

Since $u(y,s)$ considered in W_k is the extension of the solution $u(y,s)$ originally defined in $W_k \cap \Omega$ as mentioned in §2, we have

$$\int_{W_k} |u(y,s)|^p d_a y \le 2\int_{W_k\cap\Omega} |u(y,s)|^p d_a y \le 2\|u(\cdot,s)\|^p_{L^p(\Omega)} \le 2\|u_0\|^p_{L^p(\Omega)}$$

(the factor 2 comes from the process of the extension mentioned in [2]). Hence (4.21) implies that

$$\int_{W'_k}\left|\frac{\partial u(x,t)}{\partial t}\right|^p d_a x \le \left(\frac{c_3}{t}\right)^p c_0 \int_{t-4R^2}^{t} \frac{1}{4R^2} 2\|u_0\|^p_{L^p(\Omega)} ds = 2\left(\frac{c_3}{t}\right)^p c_0 \|u_0\|^p_{L^p(\Omega)}. \tag{4.22}$$

The above argument is applicable to the solution $u(x,t)$ considered in $D\times(0,M)$ without the process of extending B and u to the outside of Ω, and we get

$$\int_{D'}\left|\frac{\partial u(x,t)}{\partial t}\right|^p d_a x \le \left(\frac{c_3}{t}\right)^p c_0 \int_{t-4R^2}^{t} \frac{1}{4R^2} \|u_0\|^p_{L^p(\Omega)} ds = \left(\frac{c_3}{t}\right)^p c_0 \|u_0\|^p_{L^p(\Omega)}. \tag{4.23}$$

Since a finite number of W_k's and D are concerned, we may choose the same constant c_3 in (4.22) for $W_1, \dots, W_l$ and in (4.23) for D. Hence, taking (4.2) into account, we obtain

$$\int_{\Omega}\left|\frac{\partial u(x,t)}{\partial t}\right|^p d_a x \le \left[\sum_{j=1}^{l}\int_{W'_j} + \int_{D'}\right]\left|\frac{\partial u}{\partial t}(x,t)\right|^p d_a x \le (2l+1)\left(\frac{c_3}{t}\right)^p c_0 \|u_0\|^p_{L^p(\Omega)},$$

namely

$$\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|_{L^p(\Omega)} \le \frac{c_M}{t}\|u_0\|_{L^p(\Omega)} \tag{4.24}$$

where $c_M = c_3 c_0(2l+1) (\ge c_3\{c_0(2l+1)\}^{\frac{1}{p}}$ for any $p \ge 1$);the constant c_M may depend on M. Thus we have proved the desired estimate for $p \ge 2$.

We shall prove the estimate (4.24) in the case $1 \le p < 2$. We first remark that the constant c_M in (4.24) is independent of p (≥ 2).

We shall denote the operators considered in $L^p(\Omega)$ by the symbols with superscript (p); namely, the operators T_t, A in $L^p(\Omega)$ will be denoted by $T_t^{(p)}$, $A^{(p)}$ respectively, and the strong derivative in $L^p(\Omega)$ by $D_t^{(p)}$.

Take an arbitrary function $f \in C_0^2$. Then, from the argument in §3, we may see that $(T_t^{(p)} f)(x)$ as a function of x in Ω is independent of p for every $t > 0$, and accordingly $(D_t^{(p)} T_t^{(p)} f)(x)$ also is independent of p. In fact, Yosida's semigroup theory shows that

$$T_t^{(p)} f = \lim_{\lambda\to\infty} \sum_{n=0}^{\infty} e^{-t\lambda} \frac{(t\lambda)^n}{n!} \left(\lambda J_\lambda^{(p)}\right)^n f \quad \text{(strong convergence in } L^p(\Omega)),$$

while $J_\lambda^{(p)} f$ is given by (3.8) for any p. Since $A^{(2)}$ is the smallest closed extension of $\tilde{B}$ with domain $\tilde{\mathcal{D}}$, we have $C_0^2(\Omega) \subset \tilde{\mathcal{D}} \subset \mathcal{D}(A^{(2)})$. Therefore the following relation holds for any f and $\phi \in C_0^2(\Omega)$ and any $p \ge 1$:

$$\begin{aligned}\langle D_t^{(p)} T_t^{(p)} f, \phi\rangle_a &= \langle D_t^{(2)} T_t^{(2)} f, \phi\rangle_a = \langle A^{(2)} T_t^{(2)} f, \phi\rangle_a \\ &= \langle f, T_t^{(2)} A^{(2)} \phi\rangle_a = \langle f, D_t^{(2)} T_t^{(2)} \phi\rangle_a. \end{aligned} \tag{4.25}$$

For any $M > 0$, c_M denotes the constant which appears in (4.24).

Lemma 4.3 *For any $\phi \in C_0^2$ and any $t \in (0, M)$, it holds that*

$$\operatorname*{ess.sup}_{x\in\Omega} |(D_t^{(2)} T_t^{(2)} \phi)(x)| \le \frac{c_M}{t} \|\phi\|_{L^\infty}. \tag{4.26}$$

Proof Let α be an arbitrary positive number less than

$$\operatorname*{ess.sup}_{x\in\Omega} |(D_t^{(2)} T_t^{(2)} \phi)(x)|,$$

and denote the set $\{x \in \Omega \mid |(D_t^{(2)} T_t^{(2)} \phi)(x)| > \alpha\}$ by $\Omega_{\alpha t}$. Then we have $\Omega_{\alpha t} = \{x \in \Omega \mid |(D_t^{(p)} T_t^{(p)} \phi)(x)| > \alpha\}$ for any $p \ge 1$. Hence

$$\|D_t^{(p)} T_t^{(p)} \phi\|_{L^p} \ge \alpha |\Omega_{\alpha t}|^{\frac{1}{p}}.$$

On the other hand, $\|D_t^{(p)} T_t^{(p)} \phi\|_{L^p} \le \frac{c_M}{t} \|\phi\|_{L^p} \le \frac{c_M}{t} \|\phi\|_{L^\infty} |\Omega|^{\frac{1}{p}}$ for $p \ge 2$ by means of (4.24). Therefore $\alpha |\Omega_{\alpha t}|^{\frac{1}{p}} \le \frac{c_M}{t} \|\phi\|_{L^\infty} |\Omega|^{\frac{1}{p}}$. Let $p \to \infty$, and we get $\alpha \le \frac{c_M}{t} \|\phi\|_{L^\infty}$. Since α can be chosen arbitrarily near to $\operatorname{ess.sup}_{x\in\Omega} |(D_t^{(2)} T_t^{(2)} \phi)(x)|$, we obtain (4.26). ///

Proposition 4. 1 *For any $f \in L^p(\Omega)$ $(1 \le p < 2)$, any $M > 0$ and any $t \in (0, M)$, it holds that $\|D_t^{(p)} T_t^{(p)} f\|_{L^p} \le \frac{c_M}{t} \|f\|_{L^p}$.*

PROOF It suffices to prove this proposition for $f \in C_0^2(\Omega)$. In the case $1 < p < 2$, denote by q the conjugate exponent of p: $\frac{1}{p} + \frac{1}{q} = 1$. For any f and $\phi \in C_0^2(\Omega)(\subset L^q(\Omega))$, we obtain by (4.24) and (4.25) that

$$\begin{aligned} |\langle D_t^{(p)} T_t^{(p)} f, \phi\rangle_a| = |\langle f, D_t^{(2)} T_t^{(2)} \phi\rangle_a| &= |\langle f, D_t^{(q)} T_t^{(q)} \phi\rangle_a| \\ &\le \|f\|_{L^p} \|D_t^{(q)} T_t^{(q)} \phi\|_{L^q} \le \frac{c_M}{t} \|f\|_{L^p} \|\phi\|_{L^q}. \end{aligned}$$

This estimate implies the desired result for $p > 1$.

In the case $p = 1$, we get the following estimate for any $f \in C_0^2(\Omega)$ and any $\phi \in C_0^2(\Omega)$ $(\subset L^\infty(\Omega))$ by means of (4.25) and Lemma 4.3:

$$|\langle D_t^{(1)} T_t^{(1)} f, \phi\rangle_a| = |\langle f, D_t^{(2)} T_t^{(2)} \phi\rangle_a| \le \|f\|_{L^1} \|D_t^{(2)} T_t^{(2)} \phi\|_{L^\infty} \le \frac{c_M}{t} \|f\|_{L^1} \|\phi\|_{L^\infty};$$

this implies the desired result for $p = 1$. ///

We may conclude from (4.24) proved for $p \ge 2$ and Proposition 4.1 that $\{T_t^{(p)}\}$ is an analytic semigroup in $L^p(\Omega)$ for any p $(1 \le p < \infty)$.

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