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ESTIMATES OF THE BESOV NORMS ON FRACTAL BOUNDARY BY VOLUME INTEGRALS

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ABSTRACT. Consider a bounded domain D with fractal boundary in $\mathbf{R}^d$ such that ∂D is a β-set ($d-1 \leq \beta < d$). Under an additional condition we give the norms defined by the volume integrals "equivalent" to the L^p-norm and the Besov norms on the fractal boundary, respectively.

1. Introduction.

Let D be a bounded domain in $\mathbf{R}^d$ and assume that ∂D is a β-set ($d-1 \leq \beta < d$), i.e., there exist a positive Radon measure μ on ∂D and positive real numbers b_1, b_2, r_0 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_2 r^\beta \tag{1.1}$$

for all points $z \in \partial D$ and all positive real numbers $r \leq r_0$, where $B(z,r)$ stands for the open ball in $\mathbf{R}^d$ with center z and radius r. Such a measure μ is called a β-measure.

We give examples.

1. If D is a bounded Lipschitz domain in $\mathbf{R}^d$, then ∂D is a $(d-1)$-set and the surface measure is a $(d-1)$-measure.

2. If ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set and the β-dimensional Hausdorff measure restricted to ∂D is a β-measure (cf. [H]).

It is well-known that a β-measure on ∂D is equivalent to the β-dimensional Hausdorff measure restricted to ∂D (cf. [JW1], [JW2]).

We fix a β-measure μ on ∂D.

It is natural that we consider the $L^p(\mu)$ as a function space on the boundary of D. But we often need consider spaces with more strong norms than the L^p-norm because of the fractal boundary of D. One of such spaces is a Besov space. In general, let F be a closed β-set in $\mathbf{R}^d$ and μ be a β-measure on F. Let $0 < \alpha \leq 1$. We define a Besov space $\Lambda_\alpha^p(F)$ by the Banach space of all function $f \in L^p(\mu)$ such that

$$\iint \frac{|f(x)-f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x)d\mu(z) < \infty$$

with norm

$$\|f\|_{\alpha,p} = \left(\int |f(x)|^p d\mu(x)\right)^{1/p} + \left(\iint \frac{|f(x)-f(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x)d\mu(z)\right)^{1/p}$$

Especially, since $\mathbf{R}^d$ is a d-set, $\Lambda^p_\alpha(\mathbf{R}^d)$ is the space of all L^p-functions f with respect to the d-dimensional Lebesgue measure, such that

$$\iint \frac{|f(x) - f(z)|^p}{|x - z|^{d+p\alpha}} dxdz < \infty.$$

For such a domain D volume integrals are more easy to deal with than integrals on ∂D, if f is defined on $\overline{D}$. It seems that the existence of a norm defined by a volume integral "equivalent" to the L^p-norm or the Besov norm on the fractal boundary is useful for us to prove that operators are bounded on $L^p(\mu)$ or $\Lambda^p_\alpha(\partial D)$.

A. Jonsson and H. Wallin proved in [JW1] that there exists a continuous extension operator from $\Lambda^p_\alpha(\partial D)$ to $\Lambda^p_\gamma(\mathbf{R}^d)$ and that the restriction operator from $\Lambda^p_\gamma(\mathbf{R}^d)$ to $\Lambda^p_\alpha(\partial D)$ is continuous, where $\gamma = \alpha + (d - \beta)/p$.

But, what type of a volume integral in D is equivalent to the L^p-norm or the Besov norm on the boundary? In this paper we consider this problem.

Hereafter we suppose that $\overline{D} \subset B(0, R/2)$ with $R \geq 1$. We may assume that (1.1) holds for all points $z \in \partial D$ and all positive real numbers $r \leq 3R$.

To consider the above problem, we need add a condition to D. We say that a set G satisfies *the condition (b)* if there exist positive real numbers c and $r_1 > 0$ such that

$$|B(z,r) \cap G| \geq cr^d \tag{1.2}$$

for each point $z \in \partial G$ and each positive real number $r \leq r_1$, where $|A|$ stands for the d-dimensional Lebesgue measure of a set A.

If D satisfies the condition (b) and $r_1 < 3R$, then, for each r satisfying $r_1 < r \leq 3R$ and $z \in \partial D$, we have

$$|B(z,r) \cap D| \geq |B(z, r_1) \cap D| \geq cr_1^d \geq c\Big(\frac{r_1}{3R}\Big)^d r^d.$$

Consequently, if D satisfies the condition (b), then (1.2) holds for every $r \leq 3R$ by replacing with another constant c.

In [W, Lemma 2.1] we proved the following lemma, which is fundamental. Lemma 2.1 in [W] was shown under more strong condition, but, to prove the lemma, it suffices to assume the condition (b).

LEMMA A. *Suppose that D is a bounded domain such that ∂D is a β-set $(d - 1 \leq \beta < d)$ and satisfies the condition* (b). *Let $0 < \epsilon \leq r \leq 3R$ and $z \in \partial D$. Denote by $\delta(y)$ the distance from y to ∂D. Then there exist positive numbers s_1, s_2 such that*

$$s_1 r^\beta \epsilon^{d-\beta} \leq \int_{\{\delta(y)<\epsilon\}\cap B(z,r)\cap D} dy \leq s_2 r^\beta \epsilon^{d-\beta},$$

where s_1 and s_2 are independent of r, ϵ and z.

We fix such numbers s_1 and s_2. We may assume that $s_1 \leq 1$ and $s_2 \geq 1$.

In this paper we shall prove the following two theorems in §3.

THEOREM 1. *Let D be a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set $(d - 1 \leq \beta < d)$ and satisfies the condition* (b). *Let $a > 0$ and put*

$$A_t = \{y \in D; \frac{at}{2}\Big(\frac{s_1}{s_2}\Big)^{1/(d-\beta)} \leq \delta(y) < at\},$$

where s_1, s_2 are constant in Lemma A. If f is a nonnegative continuous function on $\overline{D}$, then

$$c_1 \limsup_{t\to 0} t^{\beta-d} \int_{A_t} f(y)dy \leq \int_{\partial D} f(z)d\mu(z) \leq c_2 \liminf_{t\to 0} t^{\beta-d} \int_{A_t} f(y)dy, \tag{1.3}$$

where c_1 and c_2 are constants independent of f.

THEOREM 2. *Suppose that D satisfies the same conditions as in Theorem 1. Let $1 \le p < \infty$, $p - p\alpha - d + \beta > 0$ and $\alpha + (d-\beta)/p < \lambda < 1$. Further let $a > 0$. If f is λ-Hölder continuous on $\overline{D}$, then*

$$(1.4) \qquad c_1 \limsup_{t\to 0} \int_{A_t}\int_{A_t} \frac{|f(x)-f(y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy \\ \le \int_{\partial D}\int_{\partial D} \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) \le c_2 \liminf_{t\to 0} \int_{A_t}\int_{A_t} \frac{|f(x)-f(y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy,$$

where c_1 and c_2 are constants independent of f.

2. Lemmas

Hereafter we assume that D is a bounded domain in $\mathbf{R}^d$ such that ∂D is a β-set satisfying $d-1 \le \beta < d$ and $\overline{D} \subset B(0, R/2)$ $(R \ge 1)$. Furthermore assume that D satisfies the condition (b).

In this section we prepare several lemmas to prove Theorems 1 and 2.

The following lemma is an easy consequence of the property (1.1).

LEMMA 2.1. *Let $\lambda > 0$ and $z \in \partial D$. Further, let b_1, b_2 are positive real numbers in (1.1).*
(i) *If $\beta < \lambda$ and $a > 0$, then*

$$\int_{\partial D \cap \{a \le |x-z|\}} |x-z|^{-\lambda} d\mu(x) \le b_2 \frac{\lambda}{\lambda - \beta} a^{\beta-\lambda}.$$

(ii) *If $\beta > \lambda > 0$ and $0 < b \le R$, then*

$$\int_{\partial D \cap \{|x-z| \le b\}} |x-z|^{-\lambda} d\mu(x) \le b_2 \frac{\lambda}{\beta - \lambda} b^{\beta-\lambda}.$$

PROOF. (i) By (1.1) we have

$$\int_{\partial D \cap \{a < |z-x|\}} |z-x|^{-\lambda} d\mu(x) \\ \le \int_0^{a^{-\lambda}} \mu(\{x \in \partial D; |z-x|^{-\lambda} > t\}) dt = \int_0^{a^{-\lambda}} \mu(B(z, t^{-1/\lambda}) \cap \partial D) dt \\ \le b_2 \int_0^{a^{-\lambda}} t^{-\beta/\lambda} dt = \frac{b_2 \lambda}{\lambda - \beta} a^{\beta - \lambda},$$

which shows (i).

(ii) This is shown by the same method as (i).

□

LEMMA 2.2. *There exists a positive real number s such that $sr^d \le |B(x,r) \cap D|$ for each $x \in D$ and each positive number $r \le R$.*

PROOF. If $r > (5/4)\delta(x)$, then we take $x' \in \partial D$ satisfying $|x - x'| = \delta(x)$. Since $B(x', (1/4)r) \cap D \subset B(x, r) \cap D$, the condition (b) yields that there is $s > 0$ satisfying

$$s\Big(\frac{1}{4}r\Big)^d \le |B(x', \frac{1}{4}r) \cap D|,$$

whence $s(1/4)^d r^d \le |B(x, r) \cap D|$.

If $\delta(x) \le r < (5/4)\delta(x)$, then $B(x, (4/5)r) \subset D$ and hence $\omega_d (4/5)^d r^d \le |B(x, r) \cap D|$, where ω_d is the surface measure of the unit ball.

Finally if $r < \delta(x)$, then $B(x, r) \subset D$ and hence $\omega_d r^d \le |B(x, r) \cap D|$. Thus we have the conclusion. □

We fix positive real numbers b_3, b_4 satisfying

$$s_3 r^d \le |B(x, r) \cap D| \le s_4 r^d \tag{2.1}$$

for all $x \in \overline{D}$ and for all $0 < r \le R$.

Using Lemma 2.2, we can easily show the following lemma by the same method as in the proof of Lemma 2.1.

LEMMA 2.3. *Let $\lambda > 0$ and $z \in \overline{D}$. Further let s_4 be the positive real number in (2.1).*
(i) *If $\lambda > d$ and $R \ge a > 0$, then*

$$\int_{D \cap \{|x-y| \ge a\}} |x - y|^{-\lambda} dy \le s_4 \frac{\lambda}{\lambda - d} a^{d-\lambda}.$$

(ii) *If $d > \lambda$ and $0 < b \le R$, then*

$$\int_{D \cap \{|x-y| \le b\}} |x - y|^{-\lambda} dy \le s_4 \frac{\lambda}{d - \lambda} b^{d-\lambda}.$$

Fix a C^∞-function ϕ on $\mathbf{R}^d$ such that

$$\phi = 1 \text{ on } \overline{B(0, 1/2)}, \quad 0 \le \phi \le 1, \quad \text{supp}\, \phi \subset B(0, 1), \quad \phi(x) = \phi(-x)$$

and define, for $x \in \mathbf{R}^d$ and $r > 0$,

$$h_{x.r}(y) = \phi\Big(\frac{y - x}{r}\Big).$$

Note that $h_{x,r} \in C^\infty(\mathbf{R}^d)$, $h_{x,r} = 1$ on $\overline{B(x, r/2)}$ and supp $h_{x,r} \subset B(x, r)$. Furthermore, $|\nabla h_{x,r}| \le c/r$, where c is a constant independent of x, r.

LEMMA 2.4. *Let x_0, $y_0 \in \partial D$, $x_0 \ne y_0$, $0 < 4r < |x_0 - y_0|$ and a, $b \in \mathbf{R}$. Assume that $p - p\alpha - d + \beta > 0$. Then*

$$\iint \frac{|a(h_{x_0,r}(x) - h_{x_0,r}(y)) + b(h_{y_0,r}(x) - h_{y_0,r}(y))|^p}{|x - y|^{\beta + p\alpha}} d\mu(x) d\mu(y) \le c(|a|^p + |b|^p) r^{\beta - p\alpha} \tag{2.2}$$

and

$$\int_D \int_D \frac{|a(h_{x_0,r}(x) - h_{x_0,r}(y)) + b(h_{y_0,r}(x) - h_{y_0,r}(y))|^p}{|x - y|^{d + p\alpha + d - \beta}} dx dy \le c(|a|^p + |b|^p) r^{\beta - p\alpha} \tag{2.3}$$

PROOF. To show (2.2) we write

$$\begin{aligned}&\iint \frac{|a(h_{x_0,r}(x)-h_{x_0,r}(y))+b(h_{y_0,r}(x)-h_{y_0,r}(y))|^p}{|x-y|^{\beta+p\alpha}}d\mu(x)d\mu(y)\\ &\le 2^p\iint \frac{|a(h_{x_0,r}(x)-h_{x_0,r}(y))|^p}{|x-y|^{\beta+p\alpha}}d\mu(x)d\mu(y)\\ &+2^p\iint \frac{|b(h_{y_0,r}(x)-h_{y_0,r}(y))|^p}{|x-y|^{\beta+p\alpha}}d\mu(x)d\mu(y)\equiv I_1+I_2.\end{aligned}$$

Then, by Lemma 2.1, (ii) and (i),

$$\begin{aligned}&\iint_{|x-y|<3r} \frac{|a(h_{x_0,r}(x)-h_{x_0,r}(y))|^p}{|x-y|^{\beta+p\alpha}}d\mu(x)d\mu(y)\\ &\le c_1|a|^p r^{-p}\int_{|x-x_0|<4r}d\mu(x)\int_{|x-y|<3r}|x-y|^{-\beta-p\alpha+p}d\mu(y)\\ &\le c_2|a|^p r^{-p}(3r)^{p-p\alpha}(4r)^{\beta}=c_3|a|^p r^{\beta-p\alpha}\end{aligned}$$

and

$$\begin{aligned}&\iint_{|x-y|\ge 3r} \frac{|a(h_{x_0,r}(x)-h_{x_0,r}(y))|^p}{|x-y|^{\beta+p\alpha}}d\mu(x)d\mu(y)\\ &\le c_4|a|^p\int_{|x-x_0|<r}d\mu(x)\int_{|x-y|\ge 3r}\frac{1}{|x-y|^{\beta+p\alpha}}d\mu(y)\\ &+c_4|a|^p\int_{|y-x_0|<r}d\mu(y)\int_{|x-y|\ge 3r}\frac{1}{|x-y|^{\beta+p\alpha}}d\mu(x)\\ &\le c_5|a|^p(3r)^{-p\alpha}r^{\beta}=c_6|a|^p r^{\beta-p\alpha}.\end{aligned}$$

From these we deduce

$$I_1\le c_7|a|^p r^{\beta-p\alpha}.$$

Similarly we also have

$$I_2\le c_8 b|^p r^{\beta-p\alpha}.$$

Thus we have (2.2).

We next prove (2.3). Noting that $p-p\alpha-d+\beta>0$ and using Lemma 2.3, we have

$$\begin{aligned}&\iint_{\{|x-y|<3r\}\cap(D\times D)} \frac{|a(h_{x_0,r}(x)-h_{x_0,r}(y))|^p}{|x-y|^{d+p\alpha+d-\beta}}dxdy\\ &\le c_9|a|^p r^{-p}\int_{\{|x-x_0|<4r\}\cap D}dx\int_{\{|x-y|<3r\}\cap D}|x-y|^{-d-p\alpha-d+\beta+p}dy\\ &\le c_{10}a|^p r^{-p}(3r)^{p-p\alpha-d+\beta}(4r)^d=c_{11}|a|^p r^{\beta-p\alpha}.\end{aligned}$$

We also have

$$\begin{aligned}&\iint_{\{|x-y|\ge 3r\}\cap D} \frac{|a(h_{x_0,r}(x)-h_{x_0,r}(y))|^p}{|x-y|^{d+p\alpha+d-\beta}}dxdy\\ &\le c_{12}|a|^p\int_{\{|x-x_0|<r\}\cap D}dx\int_{\{|x-y|\ge 3r\}\cap D}\frac{1}{|x-y|^{d+p\alpha+d-\beta}}dy\\ &+c_{12}|a|^p\int_{\{|y-x_0|<r\}\cap D}dy\int_{\{|x-y|\ge 3r\}\cap D}\frac{1}{|x-y|^{d+p\alpha+d-\beta}}dx\\ &\le c_{13}|a|^p(3r)^{-p\alpha-d+\beta}r^d=c_{14}|a|^p r^{\beta-p\alpha}.\end{aligned}$$

Thus we have (2.3). □

LEMMA 2.5. *Let* $0 < r \leq a \leq R$ *and* $z \in \partial D$. *Set*

$$F_r = \{y \in D; \frac{r}{2}\Big(\frac{s_1}{s_2}\Big)^{1/(d-\beta)} \leq \delta(y) < r\}.$$

Then

$$|B(z,a) \cap F_r| \geq s_1(1 - 2^{\beta-d})a^\beta r^{d-\beta} \quad \textit{and } |B(z,a) \cap F_r| \leq s_2\big(1 - \big(\frac{s_1}{s_2}\big)^2 2^{\beta-d}\big)a^\beta r^{d-\beta}.$$

PROOF. With the aid of Lemma A we have

$$\begin{aligned}
&|B(z,a) \cap F_r| \\
&= |B(z,a) \cap \{y \in D; \delta(y) < r\}| - |B(z,a) \cap \{y \in D; \delta(y) < \frac{r}{2}\Big(\frac{s_1}{s_2}\Big)^{1/(d-\beta)}\}| \\
&\geq s_1 a^\beta r^{d-\beta} - s_1 a^\beta 2^{\beta-d} r^{d-\beta} = s_1(1 - 2^{\beta-d})a^\beta r^{d-\beta}
\end{aligned}$$

and

$$|B(z,a) \cap F_r| \leq s_2 a^\beta r^{d-\beta} - 2^{\beta-d} s_1^2 s_2^{-1} a^\beta r^{d-\beta} = s_2\big(1 - \big(\frac{s_1}{s_2}\big)^2 2^{\beta-d}\big)a^\beta r^{d-\beta}.$$

Thus we have the conclusion. □

We denote by diam D the diameter of the set D and set

$$t_1 = \max\{1, \frac{2}{3}\Big(\frac{2b_2}{b_1}\Big)^{1/\beta}, \frac{2}{3}\Big(\frac{2(2^{d-\beta}s_2^2 - s_1^2)}{s_1 s_2(2^{d-\beta} - 1)}\Big)^{1/\beta}\}, \tag{2.4}$$

where b_1, b_2 and s_1, s_2 are constants in (1.1) and Lemma A, respectively. Then we have

LEMMA 2.6. *Let* diam $D/(4t_1) \geq r > 0$, $a, b \in \mathbf{R}$, z, $w \in \partial D$ *and* $|z - w| \geq 4rt_1$ *Then*
(i)

$$\int_{F_r}\int_{F_r} \frac{|a\chi_{B(z,r)}(x) - b\chi_{B(w,r)}(y)|^p}{|x - y|^{d+p\alpha+d-\beta}} dxdy \geq cr^{\beta-p\alpha}. \tag{2.5}$$

(ii)

$$\int_{\partial D}\int_{\partial D} \frac{|a\chi_{B(z,r)}(x) - b\chi_{B(w,r)}(y)|^p}{|x - y|^{\beta+p\alpha}} d\mu(x)d\mu(y) \geq cr^{\beta-p\alpha}. \tag{2.6}$$

Here c is constants independent of z and r.

PROOF. (i) Note that

$$\begin{aligned}
&\int_{F_r} dy \int_{F_r} \frac{|a\chi_{B(z,r)}(x) - b\chi_{B(w,r)}(y)|^p}{|x - y|^{d+p\alpha+d-\beta}} dxdy \\
&\geq |a|^p \int_{F_r} \chi_{B(z,r)}(x)dx \int_{F_r \cap \{|w-y|>r\}} \frac{1}{|x - y|^{d+p\alpha+d-\beta}} dy \\
&+ |b|^p \int_{F_r} \chi_{B(w,r)}(y)dy \int_{F_r \cap \{|z-x|>r\}} \frac{1}{|x - y|^{d+p\alpha+d-\beta}} dx \equiv I_{11} + I_{12}.
\end{aligned}$$

Since $x \in B(z,r)$ and $2r < |z-y| < 3rt_1$ imply $|w-y| > rt_1 \geq r$ and $|x-y| < 3/2|z-y|$, we have, by Lemma A,

$$I_{11} \geq |a|^p \int_{F_r} \chi_{B(z,r)}(x)dx \int_{F_r \cap \{2r<|z-y|<3rt_1\}} \left(\frac{2}{3}\right)^{d+p\alpha+d-\beta} \frac{1}{|y-z|^{d+p\alpha+d-\beta}} dy$$
$$\geq \left(\frac{2}{3}\right)^{d+p\alpha+d-\beta} (2r)^{-d-p\alpha-d+\beta} |B(z,r) \cap F_r||F_r \cap (B(z,3rt_1) \setminus B(z,2r))|.$$

Lemma 2.5 yields

$$|B(z,r) \cap F_r| \geq s_1(1-2^{\beta-d})r^\beta r^{d-\beta}.$$

Noting that t_1 is defined by (2.4), we have, by Lemma 2.5,

$$|B(z,3rt_1) \cap F_r| - |B(z,2r) \cap F_r|$$
$$\geq s_1(3rt_1)^\beta r^{d-\beta}(1-2^{\beta-d}) - (2r)^\beta r^{d-\beta}(s_2 - 2^{\beta-d}s_1^2 s_2^{-1}) \geq (s_2 - 2^{\beta-d}s_1^2 s_2^{-1})2^\beta r^d.$$

Therefore we have

$$I_{11} \geq c_1|a|^p r^{\beta-p\alpha}.$$

The same estimate is obtained for I_{12}. Thus we see that (2.5) holds.

(ii) Similarly we write

$$\int_{\partial D}\int_{\partial D} \frac{|a\chi_{B(z,r)}(x) - b\chi_{B(w,r)}(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y)$$
$$\geq |a|^p \int_{\partial D} \chi_{B(z,r)}(x)d\mu(x) \int_{\partial D \cap \{|w-y|>r\}} |x-y|^{-\beta-p\alpha} d\mu(y)$$
$$+ |b|^p \int_{\partial D} \chi_{B(w,r)}(y)d\mu(y) \int_{\partial D \cap \{|z-x|>r\}} |x-y|^{-\beta-p\alpha} d\mu(x) \equiv I_{21} + I_{22}.$$

Using Lemma A and Lemma 2.1, we have

$$I_{21} \geq c_2|a|^p \int_{\partial D} \chi_{B(z,r)}(x)d\mu(x) \int_{\partial D \cap \{2r<|z-y|<3rt_1\}} \left(\frac{2}{3}\right)^{\beta+p\alpha} |z-y|^{-\beta-p\alpha} d\mu(y)$$
$$\geq c_3|a|^p r^\beta (2r)^{-\beta-p\alpha} \mu(B(z,3rt_1) \setminus B(z,2r)).$$

Since

$$\mu(B(z,3rt_1) \setminus B(z,2r)) \geq b_1(3rt_1)^\beta - b_2(2r)^\beta \geq b_2 2^\beta r^\beta,$$

we have

$$I_{21} \geq c_4|a|^p r^{\beta-p\alpha}.$$

We also have the same estimate for I_{22}. Thus we have (2.6). □

3. Proofs of Theorem 1 and Theorem 2

In this section we shall prove Theorem 1 and Theorem 2.

PROOF of THEOREM 1. We first prove the second inequality of (1.3). Suppose that f is nonnegative and continuous on $\overline{D}$. Since f is uniformly continuous on $\overline{D}$, there is, for each $\epsilon > 0$, a positive real number $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. We consider any positive real number t satisfying $t < \delta/(10a)$. Since

$$\partial D \cup \overline{A_t} \subset \cup_{z \in \partial D} B(z, at)$$

and $\partial D \cup \overline{A_t}$ is compact, there is a subfamily of $\{B(z,at)\}_{z\in\partial D}$ which covers $\partial D \cup \overline{A_t}$ and consists of finitely many elements. Using Vitali's covering theorem, we can find, $z_1, z_2, \cdots, z_m \in \partial D$ such that $\{B(z_j,at)\}_{j=1}^m$ is a subfamily of $\{B(z,at)\}_{z\in\partial D}$ and $\{B(z_j,at)\}_{j=1}^m$ are mutually disjoint and

$$\partial D \cup \overline{A_t} \subset \cup_{j=1}^m B(z_j, 5at).$$

Then

$$\begin{aligned}\int_{\partial D} f(z)d\mu(z) &\le \sum_{j=1}^m \int_{B(z_j,5at)\cap\partial D} f(z)d\mu(z)\\ &\le c_1 \sum_{j=1}^m \max\{f(z);\, z \in \overline{B(z_j,5at)} \cap \partial D\}(5at)^\beta\\ &\le c_2 t^{\beta-d} \sum_{j=1}^m \big(\min\{f(y);\, y \in \overline{B(z_j,at)} \cap \overline{A_t}\} + \epsilon\big)t^d.\end{aligned}$$

Since, by Lemma 2.5,

$$|B(z_j,at) \cap A_t| \ge c_3 t^d$$

and $\{B(z_j,at)\}_{j=1}^m$ are mutually disjoint, we have

$$\int_{\partial D} f(z)d\mu(z) \le c_4 t^{\beta-d} \int_{A_t} (f(y)+\epsilon)dy. \tag{3.1}$$

On the other hand we have, by Lemma A,

$$\int_{A_t} dy \le |B(z_0,R) \cap A_t| \le c_5 t^{d-\beta} R^\beta,$$

where z_0 is a fixed point on ∂D. This and (3.1) yield

$$\int_{\partial D} f(z)d\mu(z) \le c_6 (t^{\beta-d} \int_{A_t} f(y)dy + \epsilon).$$

Thus we have the second inequality of (1.3).

We next prove the first inequality of (1.3). Using the above covering, we have, by (1.1),

$$\begin{aligned}t^{\beta-d}\int_{A_t} f(y) &\le t^{\beta-d}\sum_{j=1}^m \int_{B(z_j,5at)\cap A_t} f(y)dy\\ &\le c_7 t^{\beta-d} \sum_{j=1}^m \max\{f(y);\, y \in \overline{B(z_j,5at)} \cap \overline{A_t}\}(5at)^d\\ &\le c_8 \sum_{j=1}^m \big(\min\{f(z);\, z \in \overline{B(z_j,at)} \cap \partial D\} + \epsilon\big)(at)^\beta\\ &\le c_9 \int_{\partial D} (f(z)+\epsilon)d\mu(z) = c_9 \big(\int_{\partial D} f(z)d\mu(z) + \epsilon\mu(\partial D)\big).\end{aligned}$$

This leads the first inequality of (1.3). □

We next prove Theorem 2.

PROOF of THEOREM 2. Choose $\eta > 0$ satisfying $(d-\beta)/p + \alpha < \eta < \lambda$ and $\epsilon > 0$. Since f is λ-Hölder continuous on $\overline{D}$, we can find $t_0 > 0$ such that

$$|x-y| < t_0 \text{ implies } \frac{|f(x)-f(y)|}{|x-y|^\eta} < \epsilon$$

and $t_0 \leq \operatorname{diam} D$.

Consider any positive real number t satisfying $t < t_0/(80at_1)$, where t_1 is the positive real number defined by (2.4). Put $r = at$ and cover

$$\partial D \subset \cup_{z\in\partial D} B(z,r).$$

Using Vitali's covering theorem, we can find a countable subfamily $\{B(z_j,r)\}$ of $\{B(z,r)\}_{z\in\partial D}$ such that $\{B(z_j,r)\}$ are mutually disjoint and

$$\partial D \subset \cup_j B(z_j, 5r).$$

Using the family, we define functions $\{v_{i,j}\}$ on $\mathbf{R}^d \times \mathbf{R}^d$ as follows. If $B(z_i, 20rt_1) \cap B(z_j, 20rt_1) \neq \emptyset$, then $v_{i,j}(x,y) \equiv 0$. If $B(z_i, 20rt_1) \cap B(z_j, 20rt_1) = \emptyset$, then we define

$$v_{i,j}(x,y) = f(z_i)(h_{z_i,10r}(x) - h_{z_i,10r}(y)) + f(z_j)(h_{z_j,10r}(x) - h_{z_j,10r}(y)).$$

Let $(x,y) \in \left(B(z_i,5r)\cap\overline{D}\right) \times \left(B(z_j,5r)\cap\overline{D}\right)$. If $B(z_i, 20rt_1) \cap B(z_j, 20rt_1) = \emptyset$, we have

$$\begin{aligned}|v_{i,j}(x,y) - (f(x)-f(y))| &\leq |f(z_i)h_{z_i,10r}(x) - f(x)| + |f(z_j)h_{z_j,10r}(y) - f(y)| \\ &= |f(z_i) - f(x)| + |f(z_j) - f(y)| \\ &\leq \epsilon|z_i - x|^\eta + \epsilon|z_j - y|^\eta \leq 2\epsilon(5r)^\eta \leq 2\epsilon|x-y|^\eta.\end{aligned}$$

If $B(z_i, 20rt_1) \cap B(z_j, 20rt_1) \neq \emptyset$, then

$$|v_{i,j}(x,y) - (f(x)-f(y))| = |f(x)-f(y)| < \epsilon|x-y|^\eta.$$

We also define functions $\{w_{i,j}\}$ on $\mathbf{R}^d \times \mathbf{R}^d$ as follows. If $B(z_i, 2rt_1) \cap B(z_j, 2rt_1) \neq \emptyset$, then $w_{i,j}(x,y) \equiv 0$. If $B(z_i, 2rt_1) \cap B(z_j, 2rt_1) = \emptyset$, we define

$$w_{i,j}(x,y) = f(z_i)\chi_{B(z_i,r)}(x) - f(z_j)\chi_{B(z_j,r)}(y).$$

Then we can also estimete

$$|w_{i,j}(x,y) - (f(x)-f(y))| < c_1\epsilon|x-y|^\eta$$

for each pair $(x,y) \in \left(B(z_i,r)\cap\overline{D}\right) \times \left(B(z_j,r)\cap\overline{D}\right)$. Note that each $x \in \overline{D}$ belongs to at most N many numbers of $\{B(z_i,5r)\}$, where N is a constant depending only on d. Hence

$$\begin{aligned}I_1 &\equiv \iint \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) \\ &\leq \sum_{i,j} \int_{B(z_i,5r)\cap\partial D} \int_{B(z_j,5r)\cap\partial D} \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) \\ &\leq \sum_{i,j} \iint \frac{|v_{i,j}(x,y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) + c_2\epsilon^p \iint |x-y|^{-\beta-p\alpha+p\eta} d\mu(x)d\mu(y) \\ &\leq \sum_{i,j} \iint \frac{|v_{i,j}(x,y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) + c_3\epsilon^p.\end{aligned}$$

Using Lemma 2.4, we have

$$I_1 \le c_4 \sum_{i,j}{}' (|f(z_i)|^p + |f(z_j)|^p)\, r^{\beta - p\alpha} + c_4 \epsilon^p, \tag{3.2}$$

where $\sum'_{i,j}$ stands for the sum for (i,j) satisfying $|z_i - z_j| \ge 40rt_1$.

On the other hand using Lemma 2.6 and noting that $p\eta - p\alpha - d + \beta > 0$ and $A_t = F_r$, we have

$$\begin{aligned}
&\int_{A_t}\int_{A_t} \frac{|f(x)-f(y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy \\
&\ge \sum_{i,j} \int_{F_r}\int_{F_r} \frac{|w_{i,j}(x,y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy - c_5\epsilon^p \int_D\int_D |x-y|^{-d-p\alpha-d+\beta+p\eta} dxdy \\
&\ge c_6 \sum_{i,j}{}'' (|f(z_i)|^p + |f(z_j)|^p)\, r^{\beta-p\alpha} - c_6\epsilon^p,
\end{aligned} \tag{3.3}$$

where $\sum''_{i,j}$ stands for the sum for (i,j) satisfying $|z_i - z_j| \ge 4rt_1$. Combining (3.2) with (3.3), we have the second inequality of (1.4).

We next show the first inequality of (1.4). Since

$$\begin{aligned}
I_2 &\equiv \int_D\int_D \frac{|f(x)-f(y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy \\
&\le \sum_{i,j} \int_D\int_D \frac{|v_{i,j}(x,y)|^p}{|x-y|^{d+p\alpha+d-\beta}} dxdy + c_7\epsilon^p \int_D\int_D |x-y|^{-d-p\alpha+d+\beta+p\eta} dxdy,
\end{aligned}$$

we have, by Lemma 2.4,

$$I_2 \le c_8 \sum_{i,j}{}' (|f(z_i)|^p + |f(z_j)|^p) r^{\beta-p\alpha} + c_8\epsilon^p.$$

On the other hand Lemma 2.6 yields

$$\begin{aligned}
\int_{\partial D}\int_{\partial D} \frac{|f(x)-f(y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) &\ge c_9 \sum_{i,j} \int_{\partial D}\int_{\partial D} \frac{|w_{i,j}(x,y)|^p}{|x-y|^{\beta+p\alpha}} d\mu(x)d\mu(y) - c_9\epsilon^p \\
&\ge c_{10} \sum_{i,j}{}'' (|f(z_i)|^p + |f(z_j)|^p) r^{\beta-p\alpha} - c_{10}\epsilon^p.
\end{aligned}$$

Thus we also have the first inequality of (1.4).

□

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