Natural Science Report, Ochanomizu University, Vol. 45, No. 1, 1994
お茶の水女子大学 自然科学報告 第45巻 第1号

The Central Limit Theorem in Abstract Wiener Spaces

Michie Maeda and Masako Kano

Department of Mathematics, Ochanomizu University, Tokyo
(Received Apr. 8, 1994)

§ 1. Introduction.

The central limit theorem (from now we use an abbreviation "CLT") is one of the oldest and the most important results in probability theory. It is still the main part of classical probability theory and is basic to asymptotic statistical theory.

Recently several mathematicians investigate this theorem in Banach spaces. Many fruitful results are obtained. On the other hand, J. Kawabe proved the CLT in nuclear Fréchet spaces and the strong duals of them. In the Banach space case, the integrability of the square of the norm is used as the hypothesis. Here we analogize Kawabe's method for Hilbert spaces and abstract Wiener spaces, i.e., we consider the CLT with the hypothesis on the weak topology.

§ 2. Preliminaries.

First we introduce the results which were proved by Kawabe ([1]).

Let X be a real locally convex space, $\mathscr{B}(X)$ be the Borel σ-algebra on X and $\mathscr{P}(X)$ be the set of all Borel probability measures on X. Also X' means the topological dual space of X and X'_β is the strong dual of X, i.e., the dual of X endowed with the uniform convergence topology $\beta(X', X)$ on bounded subsets of X. We denote by $\langle \cdot, \cdot \rangle$ the bilinear form on $X \times X'$.

Let $\mathscr{A}$ be a subset of $\mathscr{P}(X)$. Given any $\varepsilon > 0$, there exists a compact subset K_ε of X such that $\mu(K_\varepsilon) \geqq 1 - \varepsilon$ for every $\mu \in \mathscr{A}$. Then we say that $\mathscr{A}$ is uniformly tight.

PROPOSITION 1 ([1]). *Let Φ be a nuclear Fréchet space or the strong dual of a nuclear Fréchet space. For $\mathscr{A} \subset \mathscr{P}(\Phi)$ to be uniformly tight it is sufficient that for each $f \in \Phi'$ the set $\{\mu \circ f^{-1}; \mu \in \mathscr{A}\} \mathscr{P} \subset (\boldsymbol{R})$ is uniformly tight.*

A mapping ξ from a probability measure space (Ω, A, P) into X is called a random variable if it is measurable, i.e., $\xi^{-1}(B)\in A$ for every $B\in\mathcal{B}(X)$. Every random variable ξ with values in X induces a probability measure $\mathcal{L}(\xi)$; $\mathcal{L}(\xi)(B)=P(\{\omega\in\Omega\,;\,\xi(\omega)\in B\})$ for $B\in\mathcal{B}(X)$, which is called its distribution.

Let $\{\xi_n\}$ be a sequence of random variables taking values in X. If $\mathcal{L}(\xi_1)=\mathcal{L}(\xi_2)=\cdots=\mathcal{L}(\xi_n)=\cdots$, then $\{\xi_n\}$ is called to be identically distributed.

PROPOSITION 2 ([1]). *Let Φ be a nuclear Fréchet space or the strong dual of a nuclear Fréchet space. Let $\{\xi_n\}_{n\in N}$ be a sequence of independent identically distributed (we use an abbreviation "i.i.d.") random variables taking values in Φ and satisfying $E\langle\xi_1, f\rangle^2<+\infty$ for every $f\in\Phi'$ and $E\langle\xi_1, f\rangle=0$ for every $f\in\Phi'$. ($E\phi$ means the expectation of ϕ.) Then $\{\mathcal{L}(1/\sqrt{n}\,\sum_{k=1}^{n}\xi_k)_{n\in N}$ converges weakly to a Gaussian probability measure on Φ.*

And we add the next proposition proved by I. Mitoma ([3]).

PROPOSITION 3 ([3]). *Let X be a Fréchet space and $\mathcal{A}$ be a subset of $\mathcal{P}(X'_\beta)$. Suppose that for each $x\in X$ the set $\{\mu\circ x^{-1}\,;\,\mu\in\mathcal{A}\}\subset\mathcal{P}(\boldsymbol{R})$ is uniformly tight. Then the function*

$$M(x)=\sup_{\mu\in\mathcal{A}}\int_{X'_\beta}\frac{|\langle x, f\rangle|}{1+|\langle x, f\rangle|}d\mu(f)$$

for $x\in X$, is continuous on X.

Now we introduce some notations. Let E be a Banach space and E' be the topological dual of E. If $\mu\in\mathcal{P}(E)$, then the characteristic function of μ is a function $\hat{\mu}$ with domain E' and range $\boldsymbol{C}$. It is defined by

$$\hat{\mu}(y)=\int_E \exp(i\langle x, y\rangle)d\mu(x)\,.$$

Let H be a Hilbert space. $L_{(1)}(H)$ denotes the collection of trace class operators of H, i.e., if $u\in L_{(1)}(H)$, then u is a compact operator of H satisfying $\sum_{n=1}^{\infty}\lambda_n<+\infty$, where λ_n's are the eigenvalues of $(u*u)^{1/2}$, where u^* is the adjoint operator of u. An operator is called an S-operator of H if it is in $L_{(1)}(H)$, positive definite and self-adjoint. Let $\mathcal{S}$ denote the family of all S-operators. The class of sets $\{[x\,;\,\langle Sx, x\rangle<1]\,;\,S\in\mathcal{S}\}$ defines a system of neighborhoods at the origin for a certain topology, which is called the S-topology (or Sazonov topology) ([6]).

Let H_i ($i=1$ or 2) be a real separable Hilbert space with norm $|\cdot|_i=\sqrt{\langle\cdot,\cdot\rangle_i}$, where $\langle\cdot,\cdot\rangle_i$ means the inner product on H_i. Let u be a continuous linear operator of H_1 into H_2 satisfying $\sum_{n=1}^{\infty}|ue_n|_2^2<+\infty$ for some orthonormal basis $\{e_n\}_{n\in N}$ of H_1. Then we call it to be of Hilbert-Schmidt type.

We close this section by the next theorem.

PROHOROV'S THEOREM ([5]). *Let H be a real separable Hilbert space, and $\mathcal{A}$ be a subset of $\mathcal{P}(H)$. Suppose that the family $\{\hat{\mu}\,;\,\mu\in\mathcal{A}\}$ is equicontinuous at the origin under the S-topology, then $\mathcal{A}$ is uniformly tight.*

§ 3. The CLT in Hilbert spaces.

In this section we present the following two theorems.

THEOREM 1. *Let H_i $(i=1,2)$ be a real separable Hilbert space with norm $|\cdot|_i=\sqrt{\langle\cdot,\cdot\rangle_i}$ and u be a Hilbert-Schmidt type operator of H_1 into H_2. Let $\mathcal{A}$ be a subset of $\mathcal{P}(H_1)$ satisfying that for each $f\in H_1$, the subset $\{\mu\circ f^{-1}\,;\,\mu\in\mathcal{A}\}$ of $\mathcal{P}(\boldsymbol{R})$ is uniformly tight. Then the subset $\{\mu\circ u^{-1}\,;\,\mu\in\mathcal{A}\}$ of $\mathcal{P}(H_2)$ is uniformly tight.*

REMARK. We denote the image measure $\mu\circ u^{-1}$ under the operator u by $u(\mu)$. In this paper we use both expressions $\mu\circ u^{-1}$ and $u(\mu)$.

PROOF. In this case, Mitoma's function $M(x)$ is as follows:

$$M(x)=\sup_{\mu\in\mathcal{A}}\int_{H_1}|\langle x,y\rangle_1|/\{1+|\langle x,y\rangle_1|\}d\mu(y)$$

for $x\in H_1$. Proposition 3 says that $M(x)$ is continuous on H_1. It is obvious that for any $y\in H_2$ and any $\mu\in\mathcal{A}$

$$\begin{aligned}|1-\widehat{u(\mu)}(y)|&=\left|1-\int_{H_2}e^{i\langle y,t\rangle_2}d(u(\mu))(t)\right|\\&\leqq\int_{H_2}|1-e^{i\langle y,t\rangle_2}|d(u(\mu))(t)\,.\end{aligned}$$

Using the inequality $|1-e^{ix}|\leqq 4|x|/(1+|x|)$ for any $x\in\boldsymbol{R}$, we have

$$\begin{aligned}&\int_{H_2}|1-e^{i\langle y,t\rangle_2}|d(u(\mu))(t)\\&\qquad\leqq 4\int_{H_2}|\langle y,t\rangle_2|/(1+|\langle y,t\rangle_2|)d(u(\mu))(t)\end{aligned}$$

$$=4\int_{H_1} |\langle y, us\rangle_2|/(1+|\langle y, us\rangle_2|)d\mu(s)$$

$$=4\int_{H_1} |\langle u^*y, s\rangle_1|/(1+|\langle u^*y, s\rangle_1|)d\mu(s)\,.$$

Then we conclude that $|1-\widehat{u(\mu)}(y)|\leqq 4M(u^*y)$. Recall that $M(x)$ is continuous on H_1 and nonnegative and $M(0)=0$. It follows that for arbitrary $\varepsilon>0$, there exists a $\delta>0$ such that $|u^*y|_1<\delta$ implies $M(u^*y)<\varepsilon$. Let $S=uu^*/\delta^2$, then S is an S-operator defined on H_2 and satisfying that $y\in\{x\,;\,x\in H_2, \langle Sx, x\rangle_2<1\}$ implies $M(u^*y)<\varepsilon$. Therefore, we have the result that $y\in\{x\,;\,x\in H_2$ and $\langle Sx, x\rangle_2<1\}$ implies $|1-\widehat{u(\mu)}(y)|<4\varepsilon$ for all $\mu\in\mathcal{A}$. Then the set $\{\widehat{u(\mu)}\,;\,\mu\in\mathcal{A}\}$ is equicontinuous at the origin for the Sazonov topology induced by $|\cdot|_2$. It follows from Prohorov's Theorem that $\{u(\mu)\,;\,\mu\in\mathcal{A}\}$ is uniformly tight. $\square$

THEOREM 2. *Let H_1, H_2 and u be the same as them in Theorem 1. Let $\{\xi_n\}_{n=1,2,\cdots}$ be a sequence of i.i.d. random variables taking values in H_1 and satisfying the following conditions* (1) *and* (2).

(1) $E|\langle\xi_1, f\rangle_1|^2<+\infty$ *for any* $f\in H_1$,

(2) $E\langle\xi_1, f\rangle_1=0$ *for any* $f\in H_1$.

Then the sequence of distributions $\{\mathcal{L}(1/\sqrt{n}\sum_{k=1}^n u\circ\xi_k)\}$ converges weakly to a Gaussian probability measure on H_2.

PROOF. Let $\mu_n=\mathcal{L}(1/\sqrt{n}\sum_{k=1}^n \xi_k)$ for all n and $\mathcal{A}=\{\mu_n\,;\,n\in\boldsymbol{N}\}$. It is obvious that $\mathcal{L}(1/\sqrt{n}\sum_{k=1}^n u\circ\xi_k)=u(\mu_n)$. By the CLT in $\boldsymbol{R}$, it follows that the sequence $\{\mu_n\circ f^{-1}\,;\,\mu_n\in\mathcal{A}\}$ converges weakly to a Gauss probability measure on $\boldsymbol{R}$ for each $f\in H_1$. Hence $\{\mu_n\circ f^{-1}\,;\,\mu_n\in\mathcal{A}\}$ is uniformly tight for each $f\in H_1$. Theorem 1 says that $\{u(\mu_n)\,;\,\mu_n\in\mathcal{A}\}$ is uniformly tight. This means that every subsequence of $\{u(\mu_n)\,;\,\mu_n\in\mathcal{A}\}$ has a weakly convergent subsequence. Therefore we only have to show that $\{u(\mu_n)\,;\,\mu_n\in\mathcal{A}\}$ has only one limit point.

Let $\{p_i\}_{i=1,2,\cdots}, \{q_j\}_{j=1,2,\cdots}$ be two subsequences of $\{u(\mu_n)\}_{n=1,2,\cdots}$, i.e., $p_i=u(\mu_{n_i})$ $(i=1,2,\cdots)$, $q_j=u(\mu_{n_j})$ $(j=1,2,\cdots)$. $\{p_i\}_{i=1,2,\cdots}$ has a convergent subsequence $\{p_{i_k}\}_{k=1,2,\cdots}$ and also $\{q_j\}_{j=1,2,\cdots}$ has a convergent subsequence $\{q_{j_l}\}_{l=1,2,\cdots}$. Put $p_0=\lim_{k\to\infty} p_{i_k}$ and $q_0=\lim_{l\to\infty} q_{j_l}$.

Other hand, for each $f\in H_2$, $f\circ u$ is a continuous linear functional on H_1. Then the sequence $\{(f\circ u)(\mu_n)\}_{n=1,2,\cdots}$ converges weakly to some limit, we denote it by ν_f. For each $f\in H_2$, we have

$$\hat{p}_0(f)=\lim_{k\to\infty}\hat{p}_{i_k}(f)$$

$$=\lim_{k\to\infty}\int_{H_2} e^{i\langle t,f\rangle_2}dp_{i_k}(t)$$

$$=\lim_{k\to\infty}\int_R e^{is}d(p_{i_k}\circ f^{-1})(s)$$

$$=\int_R e^{is}d\nu_f(s).$$

Similarly we have $\hat{q}_0(f)=\int_R e^{is}d\nu_f(s)$. Therefore it follows that $\hat{p}_0(f)=\hat{q}_0(f)$ for every $f\in H_2$, and then $p_0=q_0$. This means that the sequence $\{u(\mu_n)\}$ has at most one limit point. We can conclude that the sequence $\{u(\mu_n)\}_{n=1,2,\cdots}$ converges weakly.

Moreover, for every $f\in H_2$, $\{f\circ u\circ\xi_n\}_{n=1,2,\cdots}$ is a sequence of real valued i. i. d. random variables. By hypothesis, we can say that $E|f\circ u\circ\xi_1|^2=E|\langle f, u\xi_1\rangle_2|^2=E|\langle u^*f, \xi_1\rangle_1|^2<+\infty$ and $E(f\circ u\circ\xi_1)=E\langle f, u\xi_1\rangle_2=E\langle u^*f, \xi_1\rangle_1=0$. It follows from the CLT in $\boldsymbol{R}$ that $\mathcal{L}(1/\sqrt{n}\sum_{k=1}^n f\circ u\circ\xi_k)$ weakly converges to a Gauss probability measure on $\boldsymbol{R}$. Hence, the sequence $\{u(\mu_n)\}_{n=1,2,\cdots}$ weakly converges to a Gaussian probability measure on H_2. □

§4. The CLT in abstract Wiener spaces.

We explain the notion of measurable seminorms and abstract Wiener spaces.

Let H be a real separable Hilbert space with norm $|\cdot|=\sqrt{\langle\cdot,\cdot\rangle}$. $\boldsymbol{F}$ will denote the partially ordered set of finite dimensional orthogonal projections P of H. $P>Q$ means $PH\supset QH$ for $P, Q\in\boldsymbol{F}$.

A subset A of H of the following form is called a cylindrical set;

$$A=\{x\in H;\ Px\in D\},$$

where $P\in\boldsymbol{F}$ and D is a Borel subset of PH. C will denote the collection of all cyrindrical sets.

The Gauss cylindrical measure in H is the set function γ from C into $[0,1]$ defined as follows: If $A=\{x\in H;\ Px\in D\}$, then

$$\gamma(A)=(1/\sqrt{2\pi})^n\int_D e^{-|x|^2/2}dx,$$

where $n=\dim PH$ and dx is the Lebesgue measure of PH.

A seminorm $\|\cdot\|$ defined on H is called to be measurable if for every $\varepsilon>0$, there exists $P_0\in\boldsymbol{F}$ such that $\gamma\{\|Px\|>\varepsilon\}<\varepsilon$ for any $P\perp P_0$ and $P\in\boldsymbol{F}$. Let $\|\cdot\|$ be a measurable norm in H, B the completion of H with respect to $\|\cdot\|$ and i will denote the inclusion map of H into B. The triple (i, H, B) is called an abstract Wiener space. If B is a Hilbert space, then i is of Hilbert-Schmidt type.

Now we consider the generalization of Theorems 1 and 2. Recall that every real separable Banach space B can arise in this fashion (i, H, B). First we present the next proposition proved by Y. Okazaki ([4]).

PROPOSITION 4 ([4]). *Let (i, H, B) be an abstract Wiener space and $\mathcal{A}$ be a subset of $\mathcal{P}(H)$. Suppose that the family $\{\hat{\nu}\,;\,\nu\in\mathcal{A}\}$ is equicontinuous at the origin with respect to the B-norm, then $\mathcal{A}$ is uniformly tight.*

REMARK. Since (i, H, B) is the abstract Wiener space, there exists a measurable norm such that B is the completion of H with respect to this norm. We call it to be the B-norm.

We are ready to show the following theorem.

THEOREM 3. *Let (i, H, B) be an abstract Wiener space and B be a real separable Banach space. Let $\mathcal{A}$ be a subset of $\mathcal{P}(B')$ satisfying that for each $f\in B$, the subset $\{\mu\circ f^{-1}\,;\,\mu\in\mathcal{A}\}$ is uniformly tight. Then the subset $\{i^*(\mu)\,;\,\mu\in\mathcal{A}\}$ of $\mathcal{P}(H)$ is uniformly tight, where i^* is the transpose of i.*

PROOF. It is sufficient to show that $\widehat{i^*(\mu)}$ is continuous with respect to the $\boldsymbol{B}$-norm. Let $\langle\cdot,\cdot\rangle_H$ be the inner product on H and $(\cdot,\cdot)$ the bilinear **from** defined on $B\times B'$.

In this case, Mitoma's function $M(x)$ is as follows:

$$M(x)=\sup_{\mu\in\mathcal{A}}\int_{B'}|(x, x')|/\{1+|(x, x')|\}d\mu(x')$$

for $x\in B$. By Proposition 3, it follows that $M(x)$ is continuous with respect to the B-norm. Using the similar method to it in the proof of Theorem 1, we have

$$\begin{aligned}|1-\widehat{i^*(\mu)}(y)| &\leqq \int_H |1-e^{i\langle y,t\rangle_H}|\,d(i^*(\mu))(t)\\ &\leqq 4\int_H |\langle y, t\rangle_H|/\{1+|\langle y, t\rangle_H|\}d(i^*(\mu))(t)\\ &=4\int_{B'} |(iy, s)|/\{1+|(iy, s)|\}d\mu(s)\\ &\leqq 4M(iy)\,.\end{aligned}$$

Then the proof is complete. □

Using Theorem 3, we get the next result. Tne technique of proof is same as it in Theorem 2.

THEOREM 4. *Let (i, H, B) be an abstract Wiener space and B be a real separable Banach space. Let $\{\xi_n\}_{n=1,2,\cdots}$ be a sequence of i.i.d. random variables taking values in B' and satisfying the following conditions* (1) *and* (2).

(1) $E|(f, \xi_1)|^2 < +\infty$ *for any* $f \in B$,

(2) $E(f, \xi_1) = 0$ *for any* $f \in B$.

Then the sequence of distributions $\{\mathcal{L}((1/\sqrt{n}) \sum_{k=1}^{n} i^*(\xi_k))\}$ *converges weakly to a Gaussian probability measure on* H.

References

[1] J. Kawabe: *Uniform tightness of probability measures on nuclear spcces*, Proceedings of the Eleventh Symposium on Applied Functional Analysis (1988), 70-79.

[2] H.H. Kuo: *Gaussian measures in Banach spaces*, Lecture Notes in Math. 463, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[3] I. Mitoma: *Tightness of probability measures on $C([0, 1]; S')$ ond $D([0, 1]; S')$*, Ann. Prob. **11** (1983), 989-999.

[4] Y. Okazaki: *Bochner's theorem on measurable linear functionals of a Gaussian measure*, Ann. Prob. **9** (1981), 663-664.

[5] K.R. Parthasarathy: *Probability measures on metric spaces*, Academic Press, New York and London, 1967.

[6] V.V. Sazonov: *On characteristic functionals*, Theor. Prob. Appl. (SIAM translation) **3** (1958), 201-205.