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Tangential Boundary Behavior of Solutions to the Dirichlet Problem

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§1. Introduction

Let $\boldsymbol{R}_+^{d+1}$ be the upper half space in $\boldsymbol{R}^{d+1}$. It is well-known that the Poisson integrals of functions f in $L^p(\boldsymbol{R}^d)$ converge to f nontangentially except for a set of d-dimensional Lebesgue measure zero. Moreover it has been also known that the Poisson integrals of functions in a subfamily of $L^p(\boldsymbol{R}^d)$, for example, a family of L^p-potentials, a family of Bessel potentials or the Besov space, have limits within tangential approach regions except for a set of appropriately dimensional Hausdorff measure zero (cf. [4], [5], [1]).

Y. Mizuta investigated tangential boundary behavior of harmonic functions in $\boldsymbol{R}_+^{d+1}$ in [5] and proved that, if $0<\lambda<1$, $p>1$, $p\lambda<d$ and $d-p\lambda<\beta\leqq d$, then the Poisson integral of a function in the Besov space $\Lambda_\lambda^{p,p}(\boldsymbol{R}^d)$ has a limit within a tangential approach region

$$\Omega_{\tau,\eta}(Z):=\{(x,t)\in\boldsymbol{R}_+^{d+1}:\eta|x-z|^\tau<t\}$$

for $\tau=\beta/(d-p\lambda)$ for all $z\in\boldsymbol{R}^d$ except for a set of β-dimensional Hausdorff measure zero.

P. Ahern and A. Nagel also proved in [1] that the above result is still valid even if $\lambda\geqq 1$. Recall that a function f in $L^p(\boldsymbol{R}^d)$ belongs to $\Lambda_\lambda^{p,p}(\boldsymbol{R}^d)$ if the Poisson integral u of f satisfies

$$\iint t^{p(m-\lambda)-1}\left|\frac{\partial^m u}{\partial t^m}(x,t)\right|^p dxdt<\infty\,,$$

where m is the least integer greater than λ.

In this paper we consider a bounded $C^{1,\alpha}$-domain D in $\boldsymbol{R}^d$ $(d\geqq 3)$ instead of the upper half space. We ask what functions f on ∂D allow us to get that the solution to the Dirichlet problem for the Laplacian with boundary data f converges to f through a tangential approach region except for a set of surface measure zero, or that it has a limit within a tangential approach region except for a set of β-dimensional Hausdorff measure zero for $\beta \leqq d-1$.

To answer the questions, we consider an approach region $\Gamma_{\tau,\eta}(Z)$ defined by

$$\Gamma_{\tau,\eta}(Z) := \{X \in D : \langle Z-X, N_Z \rangle > \eta |X-Z|^{\tau}\}$$

for τ, $1<\tau<\alpha+1$ and $0<\eta<1$, where N_Z stands for the outward unit normal to the ∂D at Z.

We will introduce a function space $\Lambda_\lambda^p(\sigma)$, which is a Besov space on ∂D. More precisely, let p, λ be positive real numbers such that $p>1$, $0<\lambda<1$ and σ be the sufface measure of ∂D. We denote by $\Lambda_\lambda^p(\sigma)$ the space of all functions f in $L^p(\sigma)$ such that the functions $f_{p,\lambda}$ defined by

$$f_{p,\lambda}(Z) = \left(\int \frac{|f(Y)-f(Z)|^p}{|Y-Z|^{d-1+\lambda p}} d\sigma(Y) \right)^{1/p}$$

also belong to $L^p(\sigma)$.

The space $\Lambda_\lambda^p(\sigma)$ is a Banach space with norm

$$\|f\|_{p,\lambda} := \|f\|_p + \|f_{p,\lambda}\|_p ,$$

where

$$\|f\|_p = \left(\int |f|^p d\sigma \right)^{1/p}.$$

Using double layer potentials, we will prove the following theorem in § 4.

THEOREM. *Let D be a bounded $C^{1\,\alpha}$-domain in $\boldsymbol{R}^d$* ($0<\alpha\leqq 1$, $d \geqq 3$) *such that $\boldsymbol{R}^d \setminus D$ is connected. Further, let p, β, λ be positive real numbers satisfying $p>1$, $0<\lambda<\alpha$, $p\lambda<d-1$ and $d-1-p\lambda<\beta\leqq d-1$. If $\beta/(d-1-p\lambda) < \alpha+1$, then for*

$$\tau = \frac{\beta}{d-1-p\lambda} \tag{1.1}$$

and for every function $f \in \Lambda_\lambda^p(\sigma)$ there exists a function u on D with the following properties (a)-(c):

(a) $\Delta u = 0$ *in D,*

(b) *The limit of $u(X)$ as $X \to Z$, $X \in \Gamma_{\tau,\eta}(Z)$, exists axcept for a set of β-dimensional Hausdorff measure zero and is equal to $f(Z)$ at σ-almost every point $Z \in \partial D$,*

(c) *There exist positive real numbers c, δ such that*

$$\|u_\delta^*\|_p \leqq c \|f\|_{p,\lambda},$$

where u_δ^* is the function defined by

$$u_\delta^*(Z) = \sup\{|u(X)| : X \in \Gamma_{\tau,\eta}(Z) \cap B(Z,\delta)\}$$

and c, δ do not depend on f.

REMARK. This theorem corresponds to the result obtained by P. Ahern and A. Nagel in [1, § 7] for the upper half domain, although $\lambda<\alpha$. If D is a $C^{m,\alpha}$-domain, $\Lambda_\lambda^p(\sigma)$ will be defined for $\lambda<m$.

§ 2. Local estimates of double layer potentials

In this paper D is a bounded $C^{1,\alpha}$-domain in $\boldsymbol{R}^d$ $(0<\alpha\leqq 1,\ d\geqq 3)$. Recall that a domain D in $\boldsymbol{R}^d$ is called a $C^{1,\alpha}$-domain if to each point $Q\in\partial D$ there correspond a system of coordinates of $\boldsymbol{R}^d$ with origin Q and an open ball $B(Q,\rho)$ with center Q and radius ρ such that with respect to this coordinate system

$$D\cap B(Q,\rho)=\{(x,t): x\in\boldsymbol{R}^{d-1},\ t>\phi(x)\}\cap B(Q,\rho),\tag{2.1}$$
$$\partial D\cap B(Q,\rho)=\{(x,\phi(x)): x\in\boldsymbol{R}^{d-1}\}\cap B(Q,\rho),$$

where $\phi\in C_0^{1,\alpha}(\boldsymbol{R}^{d-1})$ and $\phi(0)=D_j\phi(0)=0$. Note that $C_0^{1,\alpha}(\boldsymbol{R}^{d-1})$ stands for the space of all functions g in $C^1(\boldsymbol{R}^{d-1})$ with compact support satisfying

$$|D_j g(x)-D_j g(y)|\leqq M|x-y|^\alpha\tag{2.2}$$

for all $x, y\in\boldsymbol{R}^{d-1}$ and $1\leqq j\leqq d-1$.

Let us define, for $X\in\boldsymbol{R}^d$, $Y\in\partial D$,

$$k(X,Y)=-\frac{1}{\omega_d(d-2)}\langle\nabla_Y|X-Y|^{2-d}, N_Y\rangle$$

if it is well-defined and $k(X,Y)=0$ otherwise, where ω_d is the area of the surface of the unit ball in $\boldsymbol{R}^d$ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\boldsymbol{R}^d$.

It is well-known that the function k has the following properties.

LEMMA 2.1. *Let $0<\delta\leqq 1$ and $X, Z\in\partial D$. Then*

(a) $|k(X,Z)|\leqq c|X-Z|^{\alpha+1-d}$.

(b) $|k(X,Y)-k(Z,Y)|\leqq c|X-Z|^\delta(|X-Y|^{\alpha-\delta+1-d}+|Z-Y|^{\alpha-\delta+1-d})$

for every $Y\in\partial D$, $Y\neq X$, $Y\neq Z$.

Using Green's formula, we can show the following lemma.

LEMMA 2.2. *The function k has the following properties:*

(a) $\int k(X,Y)d\sigma(Y)=1$ *for* $X\in D$

(b) $\int k(X,Y)d\sigma(Y)=0$ *for* $X\in\boldsymbol{R}^d\backslash\bar{D}$,

(c) $\int k(X, Y)d\sigma(Y)=1/2$ *for* $X\in\partial D$.

Let us now estimate the maximal functions with respect to a tangential approach region. Let $1<\tau<\alpha+1$, $0<\eta<1$ and consider a tangential approach region

$$\Gamma_{\tau,\eta}(P)=\{X\in D: \langle P-X, N_p\rangle>\eta|X-P|^{\tau}\}$$

at $P\in\partial D$. We define, for $f\in L^p(\sigma)$ and $X\in \boldsymbol{R}^d$,

$$u_f(X)=\int k(X, Y)f(Y)d\sigma(Y) \tag{2.3}$$

if it is well-defined and $u_f(X)=0$ otherwise. Then the function u_f is harmonic in $\boldsymbol{R}^d\backslash\partial D$. To study the boundary behavior of u_f, we cover ∂D by finite balls

$$B_j=B(Q_j, \delta_j) \qquad (j=1, \cdots, n) \tag{2.4}$$

which satisfy (2.1) for $Q=Q_j$, $\phi=\phi_j$ and $\rho=40\delta_j$. Furthermore we may assume that

$$\delta_j<1, \quad M\delta_j^{1+\alpha-\tau}<\frac{\eta}{2} \quad \text{and} \quad |\nabla\phi_j|<\eta/4. \tag{2.5}$$

Set

$$\delta_0=\min\{\delta_1, \delta_2, \cdots, \delta_n\}. \tag{2.6}$$

In this paper we fix this covering $\{B_j\}$. To investigate the boundary behavior of u_f, we may suppose that $\operatorname{supp} f\subset B_j$ by using a partition of unity subordinate to $\{B(Q_j, \delta_j)\}$ if necessary.

The following lemma corresponds to Proposition 7 on p. 151 in [6] for the upper half space.

LEMMA 2.3. *Let p, λ be positive real numbers satisfying $p>1$, $\lambda<1$. Suppose that $f\in\Lambda_\lambda^p(\sigma)$ and* $\operatorname{supp} f\subset B(Q_j, \delta_j)$. *Then*

$$\int_0^{4\delta_j}\int_{|x|<40\delta_j} t^{p(1-\lambda)-1}\left|\frac{\partial u_f}{\partial t}(x, \phi_j(x)+t)\right|^p dxdt\leqq\|f\|_{p,\lambda}.$$

PROOF. Put $u=u_f$, $\phi=\phi_j$, $\delta=\delta_j$, $g(x)=f(x, \phi(x))$ and $X=(x, \phi(x)+t)$. From

$$\left|\frac{\partial}{\partial t}k(X, Y)\right|\leqq c_1|X-Y|^{-d}$$

and

$$\frac{\partial}{\partial t}\int k(X, Y)d\sigma(Y)=0$$

it follows that

$$\left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right| \leqq c_1 \int \frac{|g(z)-g(x)|}{(|x-z|^2+|\phi(x)+t-\phi(z)|^2)^{d/2}}dz$$

$$\leqq c_1 \int \frac{|g(x+z)-g(x)|}{(|z|^2+|\phi(x)+t-\phi(x+z)|^2)^{d/2}}dz .$$

If $t>|z|$, then we have

$$|t+\phi(x)-\phi(x+z)|$$

$$\geqq t-|\phi(x+z)-\phi(x)| \geqq t-\frac{\eta}{4}|z| \geqq \frac{3}{4}t ,$$

where

(2.7)
$$\left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right|$$

$$\leqq c_2\Big(\int_{|z|\leq t}\frac{1}{t^d}|g(x+z)-g(x)|dz+\int_{|z|>t}\frac{|g(x+z)-g(x)|}{|z|^d}dz\Big)$$

$$\equiv I_1(x, t)+I_2(x, t) .$$

Set

$$w(z)=w(r\xi)=\Big(\int|g(x+z)-g(x)|^p dx\Big)^{1/p} \quad \text{and} \quad \Omega(r)=\int_S w(r\xi)d\xi$$

where S is the surface of the unit disc. From

$$\Big(\int I_1(x, t)^p dx\Big)^{1/p} \leqq \int_{|z|\leq t}\frac{1}{t^d}\Big(\int|g(x+z)-g(x)|^p dx\Big)^{1/p}dz$$

$$\leqq \frac{1}{t^d}\int_0^t r^{d-2}\Omega(r)dr$$

and

$$\Big(\int I_2(x, t)^p dx\Big)^{1/p} \leqq \int_{|z|>t}\frac{1}{|z|^d}\Big(\int|g(x+z)-g(x)|^p dx\Big)^{1/p}dz$$

$$\leqq \int_t^\infty r^{-2}\Omega(r)dr$$

we deduce

$$J:=\Big(\int_0^{4\delta}t^{p(1-\lambda)-1}dt\int_{|x|<40\delta}\left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right|^p dx\Big)^{1/p}$$

$$\leqq c_3\Big(\int_0^\infty t^{p(1-\lambda)-pd-1}\Big(\int_0^t r^{d-2}\Omega(r)dr\Big)^p dt\Big)^{1/p}$$

$$+c_3\Big(\int_0^\infty t^{p(1-\lambda)-1}\Big(\int_t^\infty r^{-2}\Omega(r)dr\Big)^p dt\Big)^{1/p} .$$

With the aid of Hardy's inequalities we obtain

$$J \leqq c_4\left(\int_0^\infty t^{-p\lambda-1}\Omega(t)^p dt\right)^{1/p}$$

$$\leqq c_5\left(\int_{R^{d-1}} \frac{\|g(x+z)-g(x)\|_p^p}{|z|^{p\lambda+d-1}} dz\right)^{1/p} \leqq c_6\|f\|_{p,\lambda}.$$

This completes the proof. Q. E. D.

We next investigate, for $f \in \Lambda_\lambda^p(\sigma)$ satisfying $\operatorname{supp} f \subset B_j$, how the double layer potential defined by (2.3) behaves near $B_j \cap \partial D$. For the purpose we prepare the following lemma, which can be shown by the same method as in the proof of Lemma 7.1 in [1].

LEMMA 2.4. *Let* $\tau > 1$ *and* $0 < \rho < 1$. *Then there is a positive real number* $\varepsilon = \varepsilon(\tau, z)$ *such that* $|x-z| \geqq t \geqq b|x-z|^\tau$ *implies*

$$\{(y, \phi_j(y)+s) : |x-y| < \rho t, t-s < \rho t\} \subset \{(y, \phi_j(y)+s) : b\varepsilon|y-z|^\tau < s\}.$$

We note that, if $Z = (z, \phi_j(z)) \in B(Q_j, \delta_j)$ and $X = (x, \phi_j(x)+t) \in \Gamma_{\tau,\eta}(Z) \cap B(Z, \delta_0)$, then

$$t > \frac{\eta}{2}|z-x|^\tau. \tag{2.8}$$

In fact, by (2.2) and (2.5) we have

$$t > \eta|z-x|^\tau - \langle z-x, \nabla\phi_j(z)\rangle + \phi_j(z) - \phi_j(x)$$

$$\geqq \eta|z-x|^\tau - M|z-x|^{1+\alpha} \geqq \frac{1}{2}\eta|z-x|^\tau.$$

Let us denote by $b(z, r)$ the ball in $\boldsymbol{R}^{d-1}$ with center z and radius r. The function u_f defined by (2.3) is estimated near $B_j \cap \partial D$ as follows.

LEMMA 2.5. *Let* $p > 1$, $1 < \tau < \alpha + 1$, $0 < \beta \leqq d-1$ *and set*

$$\lambda = \frac{(d-1)\tau - \beta}{p\tau}.$$

Further define

$$\Omega(z) = \left\{(x, \phi_j(x)+t) : t > \frac{1}{2}\eta|x-z|^\tau, |x-z| < \delta_j, t < \delta_j\right\}$$

for $(z, \phi_j(z)) \in B_j$. *Suppose that* ν *is a positive Borel measure on the set* $b(0, 3\delta_j)$ *and*

$$\nu(b(z, r)) \leqq c_0 r^\beta$$

whenever $b(z, r) \subset b(0, 3\delta_j)$. *Then*

$$\left(\int \sup \{|u_f(x, \phi_j(x)+r)|^p : (x, \phi_j(x)+r) \in \Omega(z)\} d\nu(z)\right)^{1/p} \leqq c\|f\|_{p,\lambda}$$

for every $f \in \Lambda_\lambda^p(\sigma)$ *with* $\operatorname{supp} f \subset B_j$.

PROOF. We prove this lemma by the similar method to that in the proof of Theorem 7.1 in [1]. We write simply u, ϕ, δ instead of u_f, ϕ_j, δ_j, respectively and $g(x)=f(x, \phi(x))$. Let $(x, \phi(x)+r) \in \Omega(z)$. Note that

$$u(x, \phi(x)+r)=u(x, \phi(x)+2\delta)-\int_r^{2\delta} \frac{\partial u}{\partial t}(x, \phi(x)+t)dt\,.$$

Since

$$|\phi(x)+2\delta-\phi(y)| \geqq 2\delta-\frac{\eta}{4}|x-y| \geqq 2\delta-\frac{|x-z|+|z-y|}{4} \geqq \frac{3}{4}\delta$$

for $y \in b(0, 3\delta)$, we have

$$|u(x, \phi(x)+2\delta)| \leqq c_1 \int \frac{|g(y)|}{(|x-y|^2+|\phi(x)+2\delta-\phi(y)|^2)^{(d-1)/2}} dy$$

$$\leqq c_1 \int \frac{|g(y)|}{\delta^{d-1}} dy \leqq c_3 \|f\|_p\,.$$

We next consider

$$\int_r^{2\delta} \left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right| dt$$

$$\leqq \int_r^{2^n r} + \int_{|x-z|/2}^{2\delta} \equiv I_1(x, r)+I_2(x, r)\,,$$

where n is the largest integer satisfying $2^n r \leqq |x-z|$.

To estimate $I_2(x, r)$, let $(x, \phi(x)+t) \in \Omega(z)$ and $(1/2)|x-z|<t$. We denote by $J((x, t), \rho)$ the bounded cylinder

$$\{(y, s) : |x-y|<\rho,\ |t-s|<\rho\}$$

for $\rho>0$. Then $J((x, \phi(x)+t),\ (1/2)t) \subset D$. In fact, if $(y, s) \in J((x, \phi(x)+t),\ (1/2)t)$, then

$$\phi(y)=(\phi(y)-\phi(x))+(\phi(x)+t-s)+s-t$$

$$\leqq \frac{t}{8}+\frac{t}{2}+s-t<s\,.$$

By the mean value theorem we obtain

$$\left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right| \leqq c_3 t^{-d} \int_{J((x, \phi(x)+t), t/2)} \left|\frac{\partial u}{\partial s}(y, s)\right| dyds$$

$$\leqq c_4 t^{-d/p} \left(\int_{J((x, \phi(x)+t), t/2)} \left|\frac{\partial u}{\partial s}(y, s)\right|^p dyds\right)^{1/p}\,.$$

If $(y, s)\in J((x, \phi(x)+t), t/2)$ then

$$|\phi(x)-\phi(y)|<\frac{t}{8} \quad \text{and} \quad s<\phi(x)+\frac{3t}{2}.$$

Using this, we easily see that $t>(1/2)(s-\phi(y))$. Therefore we have

$$\text{(2.9)} \qquad \left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right|$$

$$\leqq c_5\Big(\int_{J((x,\phi(x)+t),t/2)}\frac{1}{(s-\phi(y))^d}\left|\frac{\partial u}{\partial s}(y, s)\right|^p dyds\Big)^{1/p}$$

$$\leqq c_5 t^a\Big(\int_{|x-y|<t/2}dy\int_{t/4<s<2t}s^{-pa-d}\left|\frac{\partial u}{\partial s}\Big(y, \phi(y)+s\Big)\right|^p ds\Big)^{1/p},$$

where $a=(\beta(\tau-1)/p\tau)-1$. Putting, for $Z=(z, \phi(z))$,

$$E(Z)=\{(y, \phi(y)+s): |y-z|<10s\}$$

and

$$A=\{(y, s): |y|<3\delta, s<4\delta\},$$

we obtain

$$\left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right|$$

$$\leqq c_5 t^a\Big(\int_{E(Z)\cap A}s^{-d-pa}\left|\frac{\partial u}{\partial s}(y, \phi(y)+s)\right|^p dyds\Big)^{1/p}$$

Since $a>-1$, we have

$$\int_{|x-z|/2}^{2\delta}\left|\frac{\partial u}{\partial t}(x, \phi(x)+t)\right|dt$$

$$\leqq c_5\Big(\int_0^{2\delta}t^a dt\Big)\Big(\int_{E(Z)\cap A}s^{-d-pa}\left|\frac{\partial u}{\partial s}(y, \phi(y)+s)\right|^p dyds\Big)^{1/p}$$

$$\leqq c_6\Big(\int_{E(Z)\cap A}s^{-d-pa}\left|\frac{\partial u}{\partial s}(y, \phi(y)+s)\right|^p dyds\Big)^{1/p}.$$

whence

$$\int \sup\{|I_2(x, r)|^p : (x, \phi(x)+r)\in\Omega(z)\}d\nu(z)$$

$$\leqq c_6\int_A s^{-d-pa}\left|\frac{\partial u}{\partial s}(y, \phi(y)+s)\right|^p\Big(\int\chi_{E(Z)\cap A}(z)d\nu(z)\Big)dyds,$$

$$\leqq c_7\int_A s^{-d-pa+\beta}\left|\frac{\partial u}{\partial s}(y, \phi(y)+s)\right|^p dyds$$

$$\leqq c_7\int_A s^{p(1-\lambda)-1}\left|\frac{\partial u}{\partial s}(y, \phi(y)+s)\right|^p dyds$$

$$\leqq c_8\|f\|_{p,\lambda}^p.$$

The last inequality was deduced from Lemma 2.3.

We next estimate $I_1(x,r)$. We write

$$I_1(x,r)=\sum_k\int_{2^k r}^{2^{k+1}r}\frac{\partial u}{\partial t}(x,t)dt,$$

where the sum is taken over natural numbers k satisfying $2^{k+1}r\leqq|x-z|$. By the same method as in the proof of (2.9) we have

$$\begin{aligned}&\left|\frac{\partial u}{\partial t}(x,\phi(x)+t)\right|\\&\leqq c_9 t^{-d/p}\left(\int_{J((x,\phi(x)+t),t/2)}\left|\frac{\partial u}{\partial s}(y,\phi(y)+s)\right|^p dyds\right)^{1/p}\\&\leqq c_{10}\left(\int_{|x-y|<t/2,\,t/4<s<2t}s^{-d}\left|\frac{\partial u}{\partial s}(y,\phi(y)+s)\right|^p dyds\right)^{1/p},\end{aligned}$$

whence

$$\begin{aligned}&\int_{2^k r}^{2^{k+1}r}\left|\frac{\partial u}{\partial t}(x,\phi(x)+t)\right|dt\\&\leqq c_{11}\left(\int_{|x-y|<2^k r,\,2^{k-2}r<s<2^{k+2}r}s^{-d+p}\left|\frac{\partial u}{\partial s}(y,\phi(y)+s)\right|^p dyds\right)^{1/p}.\end{aligned}$$

Suppose that $|x-y|<2^k r$ and $2^{k-2}r<s<2^{k+2}r$. From $\eta|x-z|^\tau/2<r$ we deduce

$$2^{k+1}r-s<2^{k+1}r-2^{k-2}r\leqq\frac{7}{8}2^{k+1}r,$$

$$|x-y|<2^k r<\frac{7}{8}2^{k+1}r$$

and

$$\frac{\eta}{2}2^{k+1}|x-z|^\tau<2^{k+1}r\leqq|x-z|.$$

On account of Lemma 2.4 we can find $\varepsilon>0$ such that

$$s>\eta\varepsilon 2^k|y-z|^\tau,$$

where ε is independent of r, y, z and k.

Setting

$$E_k(Z)=\{(y,\phi(y)+s):2^k\eta\varepsilon|y-z|^\tau<s<4\delta,\ |y|<3\delta\},$$

we have

$$\begin{aligned}(2.10)\qquad&\int_{2^k r}^{2^{k+1}r}\left|\frac{\partial u}{\partial t}(x,\phi(x)+t)\right|dt\\&\leqq c_{12}\left(\int_{E_k(Z)}s^{-d+p}\left|\frac{\partial u}{\partial s}(y,\phi(y)+s)\right|^p dyds\right)^{1/p}.\end{aligned}$$

Since

$$|y-z|<(2^{-k}\eta^{-1}\varepsilon^{-1}s)^{1/\tau}$$

for every $(y, \phi(y)+s)\in E_k(Z)$, we get

$$\int\Big(\sup_{(x,\phi(x)+r)\in\Omega(x)}\int_{2^k r}^{2^{k+1}r}\Big|\frac{\partial u}{\partial t}(x,\phi(x)+t)\Big|dt\Big)^p d\nu(z)$$

$$\leqq c_{13}(2^{-k/\tau})^{\beta}\int s^{-d+p+\beta/\tau}\Big|\frac{\partial u}{\partial s}(y,\phi(y)+s)\Big|^p dyds$$

$$\leqq c_{13}(2^{-k/\tau})^{\beta}\int s^{p(1-\lambda)-1}\Big|\frac{\partial u}{\partial s}(y,\phi(y)+s)\Big|^p dyds\,.$$

From this and Lemma 2.3 we deduce

$$\Big(\int\sup_{(x,\phi(x)+r)\in\Omega(z)} I_1(x,r)^p d\nu(z)\Big)^{1/p}$$

$$\leqq\sum_k\Big(\int\Big(\sup_{(x,\phi(x)+r)\in\Omega(z)}\int_{2^k r}^{2^{k+1}r}\Big|\frac{\partial u}{\partial t}(x,\phi(x)+t)\Big|dt\Big)^p d\nu(z)\Big)^{1/p}$$

$$\leqq c_{14}\sum_k 2^{-k\beta/(\tau p)}\Big(\int s^{p(1-\lambda)-1}\Big|\frac{\partial u}{\partial s}(y,\phi(y)+s)\Big|^p dyds\Big)^{1/p}$$

$$\leqq c_{12}\|f\|_{p,\lambda}\,.$$

Thus we complete the proof. Q. E. D.

§3. Boundedness of the operator *K*

To study the behavior of double layer potentials on ∂D, we define, for $f\in L^p(\sigma)$ and $Z\in\partial D$,

$$Kf(Z)=\int k(Z,Y)f(Y)d\sigma(Y)$$

if it is well-defined and $Kf(Z)=0$ otherwise. In this section we discuss the boundedness and compactness of the operator K. We begin with the boundedness.

LEMMA 3.1. *Let p,λ and μ be positive real numbers such that $p>1$, $0<\lambda<1$ and $0<\mu<\min\{\lambda+\alpha,1\}$. Then K is a bounded operator from $\Lambda_\lambda^p(\sigma)$ to $\Lambda_\mu^p(\sigma)$.*

PROOF. To prove

$$\|Kf\|_{p,\mu}\leqq c\|f\|_{p,\lambda}$$

for all $f\in\Lambda_\lambda^p(\sigma)$, we may assume that $\operatorname{supp} f\subset B_j=B(Q_j,\delta_j)$ by using a partition of unity if necessary. If $X, Y\in\partial D\backslash B(Q_j, 2\delta_j)$ and $Z\in B_j$, then

$$|k(X,Z)-k(Y,Z)|\leqq c_1|X-Y|.$$

So it is easy to see that the inequality

$$\|(Kf)\chi_{\partial D\setminus B(Q_j,2\delta_j)}\|_{p,\mu}\leqq c_2\|f\|_{p,\lambda}$$

holds.

If $X\in\partial D\cap B(Q_j,2\delta_j)$ and $Y\in\partial D\setminus B(Q_j,3\delta_j)$, then $|X-Y|\geqq\delta_j$, whence

$$\iint\frac{|(Kf)\chi_{B(Q_j,2\delta_j)}(X)-(Kf)\chi_{\partial D\setminus B(Q_j,3\delta_j)}(Y)|}{|X-Y|^{d-1+\mu p}}dxdy\leqq c_3\|f\|_p.$$

We next prove

$$\|(Kf)\chi_{B(Q_j,3\delta_j)}\|_{p,\mu}\leqq c_4\|f\|_{p,\lambda}.$$

Noting that $\phi\equiv\phi_j\in C_0^{1,\alpha}(\boldsymbol{R}^{d-1})$,

$$\partial D\cap B(Q_j,40\delta_j)=\{(x,\phi(x)),x\in\boldsymbol{R}^{d-1}\cap\partial D\},$$

$Q_j=(0,0)$ and $|\nabla\phi|\leqq 1/4$, define

$$h(x,z)=\frac{\phi(x)-\phi(z)-\langle x-z,\nabla\phi(z)\rangle}{\omega_d(|x-z|^2+|\phi(x)-\phi(z)|^2)^{d/2}}$$

and $g(x)=f(x,\phi(x))$ for $f\in\Lambda_\lambda^p(\sigma)$ with $\operatorname{supp}f\subset B_j$.

Then we have

$$|h(x,z)|\leqq c_5|x-z|^{\alpha+1-d}$$

and

$$|h(x,z)-h(y,z)|\leqq c_6|x-y|^{\delta}(|x-z|^{\alpha-\delta+1-d}+|y-z|^{\alpha-\delta+1-d}) \tag{3.1}$$

for δ, $0\leqq\delta\leqq 1$. Moreover define

$$Hg(x)=\int h(x,z)g(z)dz.$$

We note

$$Kf(X)=\int k(X,Z)f(Z)d\sigma(Z)=Hg(x)\qquad\text{for } X=(x,\phi(x))$$

and

$$\begin{aligned}|Hg(x)|&\leqq c_7\int|x-z|^{\alpha+1-d}|g(z)|dz\\&\leqq c_8Mg(x),\end{aligned} \tag{3.2}$$

where

$$Mg(z)=\sup\left\{\frac{1}{|b(z,r)|}\int_{b(z,r)}|g(y)dy:r>0\right\}.$$

Therefore we have $\|Hg\|_p\leqq c_9\|g\|_p$.

We next show

(3.3) $$\left(\iint\frac{|Hg(x)-Hg(y)|^p}{|x-y|^{d-1+\lambda p}}dxdy\right)^{1/p}\leqq c_{10}\|g\|_{p,\lambda},$$

where

$$\|g\|_{p,\lambda}=\left(\int|g(x)|^p dx\right)^{1/p}+\left(\int\frac{|g(x)-g(z)|^p}{|x-z|^{d-1-\lambda p}}dxdz\right)^{1/p}.$$

Since $H1(x)=1/2$ for every x, we have, by (3.1),

$$\begin{aligned}&|Hg(x)-Hg(y)|\\&=\left|\int(h(x,z)-h(y,z))(g(z)-g(x))dz\right|\\&\leqq c_{11}\int_{|x-z|\leqq 2|x-y|}(|x-z|^{\alpha+1-d}+|y-z|^{\alpha+1-d})|g(z)-g(x)|dz\\&\quad+c_{11}\int_{|x-z|<2|x-y|}|x-y|(|x-z|^{\alpha-d}+|y-z|^{\alpha-d})|g(z)-g(x)|dz\\&\equiv I_1(x,y)+I_2(x,y).\end{aligned}$$

We note that $|x-z|\leqq 2|x-y|$ implies $|y-z|\leqq 3|x-y|$. Put $q=p/(p-1)$. From

$$\begin{aligned}&\int_{|x-z|\leqq 2|x-y|}|x-z|^{\alpha+1-d}|g(z)-g(x)|dz\\&\leqq\left(\int_{|x-z|\leqq 2|x-y|}|x-z|^{(\alpha+1-d+(d-1)/p+\lambda)q}dz\right)^{1/q}\left(\int\frac{|g(x)-g(z)|^p}{|x-z|^{d-1-\lambda p}}\right)^{1/p}\\&\leqq c_{12}|x-y|^{\alpha+\lambda}\left(\int\frac{|g(x)-g(z)|^p}{|x-z|^{d-1-\lambda p}}dz\right)^{1/p}\end{aligned}$$

and $\alpha+\lambda>\mu$, we deduce

$$\left(\iint\frac{I_1(x,y)}{|x-y|^{d-1+\mu p}}dxdy\right)^{1/p}\leqq c_{13}\|g\|_{p,\lambda}.$$

Let us estimate $I_2(x,y)$. To do this we pick a positive real number δ satisfying $\delta<\alpha$ and $\mu<\delta+\lambda<1$, and we have, by (3.1),

$$\begin{aligned}&I_2(x,y)\\&\leqq c_{14}|x-y|\int_{|x-z|>2|x-y|}|x-z|^{\delta-d}|g(z)-g(x)|dz\\&\leqq c_{14}|x-y|\int_{|z-x|>2|x-y|}|z-x|^{\delta-d+(d-1)/p+\lambda}\frac{|g(z)-g(x)|}{|z-x|^{(d-1)/p+\lambda}}dz\\&\leqq c_{14}|x-y|\left(\int_{|z-x|>2|x-y|}|z-x|^{(\delta-d+(d-1)/p+\lambda)q}dz\right)^{1/q}\left(\frac{|g(z)-g(x)|^p}{|z-x|^{d-1+\lambda p}}dz\right)^{1/p}\\&\leqq c_{15}|x-y|^{\delta+\lambda}\left(\int\frac{|g(z)-g(x)|^p}{|z-x|^{d-1+\lambda p}}dz\right)^{1/p},\end{aligned}$$

whence

$$\left(\iint \frac{I_2^p}{|x-y|^{d-1+\mu p}} dxdy\right)^{1/p} \leqq c_{16} \|g\|_{p,\lambda}.$$

Thus we have the desired inequality (3.3). Q. E. D.

LEMMA 3.2. *Let p, λ be positive real numbers such that $p>1$ and $\lambda<\alpha$. Then K is a compact operator on $\Lambda_\lambda^p(\sigma)$.*

PROOF. We use the same notations as in the proof of Lemma 3.1. Let $\{f_n\}$ be a sequence of $\Lambda_\lambda^p(\sigma)$ satisfying $\|f_n\|_{p,\lambda} \leqq 1$ and $\operatorname{supp} f_n \cap B_j$. We shall show the existence of a subsequence $\{f_{n_k}\}$ such that $\{Kf_{n_k}\}$ converges in $\Lambda_\lambda^p(\sigma)$. From the consideration of the proof of Lemma 3.1 it suffices to prove that there exist a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and a function g_0 such that

$$\|(Hg_{n_k} - g_0)\chi_{b(0,3\delta_j)}\|_p \longrightarrow 0 \tag{3.4}$$

and

$$\iint_{b(0,3\delta_j)\times b(0,3\delta_j)} \frac{|Hg_{n_k}(x) - g_0(x) - Hg_{n_k}(y) + g_0(y)|^p}{|x-y|^{d-1+\lambda p}} dxdy \longrightarrow 0, \tag{3.5}$$

where $g_n(x) = f_n(x, \phi(x))$. For this purpose we use a mollifier $\{v_\varepsilon\}_{\varepsilon>0}$ on $\boldsymbol{R}^{d-1}$ consisting of functions $v_\varepsilon(x) = \varepsilon^{1-d} v(x/\varepsilon)$, where

$$v(x) = \begin{cases} \gamma \exp\left(\dfrac{-1}{1-|x|^2}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x| \geqq 1, \end{cases}$$

and $\gamma>0$ is so chosen that $\int v(x)dx = 1$.

Let us define

$$h_\varepsilon(x, z) = \int h(x-w, z) v_\varepsilon(w) dw$$
$$= \int h(x - \varepsilon w, z) v(w) dw$$

and

$$H_\varepsilon g(x) = \int h_\varepsilon(x, z) g(z) dz.$$

Then there is a constant c_ε such that

$$h_\varepsilon(x, z) \leqq c_\varepsilon \quad \text{and} \quad |h_\varepsilon(x, z) - h_\varepsilon(y, z)| \leqq c_\varepsilon |x-y|$$

for all $x, y \in \overline{b(0, 3\delta_j)}$ and $z \in \overline{b(0, \delta_j)}$.

Let $\{g_n\}$ be a sequence satisfying $\|g_n\|_{p,\lambda} \leqq 1$. We take a positive real number μ with $\lambda<\mu<1$. Noting that

$$\sup\{|H_\varepsilon g_n(x)| : x\in\overline{b(0, 3\delta_j)}\}$$

and

$$\sup\left\{\frac{|H_\varepsilon g_n(x)-H_\varepsilon g_n(y)|}{|x-y|^\mu} : x, y\in\overline{b(0, 3\delta_j)},\ x\neq y\right\}$$

are uniformly bounded, we can choose a subsequence $\{g_{n_k}\}$ such that $\{H_\varepsilon g_{n_k}\}_k$ converges to g_ε uniformly on $\overline{b(0, 3\delta_j)}$ and

$$\left\{\frac{|H_\varepsilon g_{n_k}(x)-H_\varepsilon g_{n_k}(y)|}{|x-y|^\mu}\right\}_k$$

also converges uniformly on $\overline{b(0, 3\delta_j)}\times\overline{b(0, 3\delta_j)}\backslash\{(x, x) : x\in \boldsymbol{R}^{d-1}\}$. It is easy to see that

$$\|(H_\varepsilon g_{n_k}-g_\varepsilon)\chi_{b(0,3\delta_j)}\|_{p,\lambda}\longrightarrow 0 \qquad \text{as } k\to\infty\,. \tag{3.6}$$

Thus to prove (3.4) and (3.5), it suffices to see the following claim.

Claim. There exist positive real numbers a_ε such that $a_\varepsilon\to 0$ as $\varepsilon\to 0$ and

$$\|(H_\varepsilon g-Hg)\chi_{b(0,3\delta_j)}\|_{p,\lambda}\leqq a_\varepsilon\|g\|_{p,\lambda} \qquad \text{for every } g\in \Lambda_\lambda^p(\sigma)\,. \tag{3.7}$$

Let us prove the claim. We choose $\delta>0$ with $\lambda+2\delta<\alpha$. On account of (3.1) we have

$$\begin{aligned}
&|H_\varepsilon g(x)-Hg(x)|\\
&\quad=\left|\int v_\varepsilon(w)dw\int(h(x-w, z)-h(x, z))g(z)dz\right|\\
&\quad\leqq\int v_\varepsilon(w)|w|^\delta dw\int_{|x-w-z|<5\delta_j}|x-w-z|^{\alpha-\delta+1-d}|g(z)|dz\\
&\qquad+\int v_\varepsilon(w)|w|^\delta dw\int_{|x-z|<4\delta_j}|x-z|^{\alpha-\delta+1-d}|g(z)|dz\,,
\end{aligned}$$

whence

$$\|(H_\varepsilon g-Hg)\chi_{b(0,3\delta_j)}\|_p\leqq c_1\varepsilon^\delta\|g\|_p\,.$$

Further, putting

$$J(x, y\,;\, z, w)=h(x-w, z)-h(x, z)-h(y-w, z)+h(y, z)$$

we write

$$\begin{aligned}
&|H_\varepsilon g(x)-Hg(x)-H_\varepsilon g(y)+Hg(y)|\\
&\quad=\left|\int v_\varepsilon(w)dw\int J(x, y\,;\, z, w)g(z)dz\right|
\end{aligned}$$

$$\leqq \int_{|w|\leqq|x-y|} v_\varepsilon(w)dw \int |J(x,y;z,w)||g(z)|dz$$

$$+\int_{|w|>|x-y|} v_\varepsilon(w)dw \int |J(x,y;z,w)||g(z)|dz \equiv I_1+I_2 .$$

Since

$$I_1 \leqq \int v_\varepsilon(w)dw \int_{|w|\leqq|x-y|} |h(x-w,z)-h(x,z)||g(z)|dz \tag{3.8}$$

$$+\int v_\varepsilon(w)dw \int_{|w|\leqq|x-y|} |h(y-w,z)-h(y,z)||g(z)|dz ,$$

we denote by I_{11} (resp. I_{12}) the first (resp. second) term in the right-hand side. By virtue of (3.1) we have

$$I_{11} \leqq c_2 \int_{|w|\leqq|x-y|} v_\varepsilon(w)|w|^{\lambda+2\delta}dw$$

$$\times \int (|x-w-z|^{\alpha-2\delta-\lambda+1-d}+|x-z|^{\alpha-2\delta-\lambda+1-d})|g(z)|dz$$

$$\leqq c_2|x-y|^{\lambda+\delta}\int v_\varepsilon(w)|w|^\delta dw$$

$$\times \int \{|x-w-z|^{\alpha-2\delta-\lambda+1-d}+|x-z|^{\alpha-2\delta-\lambda+1-d}\}|g(z)|dz .$$

Noting that $\alpha-2\delta-\lambda+1-d>1-d$, we obtain

$$\left(\iint \frac{I_{11}^p}{|x-y|^{d-1+\lambda p}}dxdy\right)^{1/p} \leqq c_3\varepsilon^\delta \|g\|_p \int v(w)|w|^\delta dw \leqq c_4\varepsilon^\delta\|g\|_p .$$

Similarly we have the same estimate for I_{12} and hence

$$\left(\iint \frac{I_1^p}{|x-y|^{d-1+\lambda p}}dxdy\right)^{1/p} \leqq c_5\varepsilon^\delta\|g\|_p .$$

Moreover we also obtain the same estimate for I_2 by using the following inequality, instead of (3.8),

$$I_2 \leqq \int_{|w|>|x-y|} v_\varepsilon(w)dw \int |h(x-w,z)-h(y-w,z)||g(z)|dz$$

$$+\int_{|w|>|x-y|} v_\varepsilon(w)dw \int |h(x,z)-h(y,z)||g(z)|dz .$$

Thus we see that (3.7) holds. This completes the proof. Q. E. D.

§ 4. Estimates of tangential maximal functions

In this section we study the boundary behavior of the double layer potential u_f and prove our theorem. Recall that

$$(u_f)^*_\delta(Z)=\sup\{|u_f(X)| : X\in\Gamma_{\tau,\eta}(Z)\cap B(Z,\delta)\}$$

for $Z\in\partial D$ and $\delta>0$.

The double layer potential u_f is estimated as follows.

LEMMA 4.1. *Let p, β, τ, η be positive real numbers satisfying $p>1$, $0<\beta\leqq d-1$, $0<\eta<1$, $1<\tau<\alpha+1$ and set*

$$\lambda=\frac{(d-1)\tau-\beta}{p\tau}.$$

Furthermore, let ν be a positive Borel measure on ∂D such that

$$\nu(A(Z,r))\leqq r^\beta \tag{4.1}$$

for all surface balls $A(Z,r)=B(Z,r)\cap\partial D$. Then

(a) *There are positive real numbers c and δ such that*

$$\left(\int(u_f)^*_\delta(Z)^p d\nu(Z)\right)^{1/p}\leqq c\|f\|_{p,\lambda}$$

for every $f\in\Lambda^p_\lambda(\sigma)$,

(b) *If $f\in C^1(\partial D)$, then the limit of $u_f(X)$ as $X\to Z$, $X\in\Gamma_{\tau,\eta}(Z)$, exists and is equal to $Kf(Z)+(1/2)f(Z)$ for every $Z\in\partial D$,*

(c) *If $f\in\Lambda^p_\lambda(\sigma)$, then the limit of $u_f(X)$ as $X\to Z$, $X\in\Gamma_{\tau,\eta}(Z)$, exists for every $Z\in\partial D$ except for set of β-dimensional Hausdorff measure zero and is equal to $Kf(Z)+(1/2)f(Z)$ except for a set of surface measure zero.*

PROOF. (a) Let $f\in\Lambda^p_\lambda(\sigma)$. Using a partition of unity subordinate to (2.1), we may suppose that $\mathrm{supp}\, f\subset B_j$. If $Z\notin B(Q_j, 3\delta_j)$, then

$$|u_f(X)|\leqq c_1(3\delta_j)^{1-\delta}\int_{B_j}|f(Y)|d\sigma(Y)\leqq c_2\|f\|_p$$

and hence

$$\left(\int_{\partial D\setminus B(Q_j,3\delta_j)}(u_f)^*_\delta(Y)^p d\nu(Y)\right)^{1/p}\leqq c_3\|f\|_p, \tag{4.2}$$

where δ is the positive real number δ_0 in (2.6).

We next estimate u_f in case $Z\in B(Q_j, 3\delta_j)$. Since ϕ is of class $C^{1,\alpha}$ and the mapping $\Pi : x\mapsto(x,\phi(x))$ is topological, we define, for a positive measure ν satisfying (4.1),

$$\mu(E)=\nu(\Pi(E)\cap B(Q_j,3\delta_j))$$

for a Borel set $E\subset\boldsymbol{R}^{d-1}$. Then

$$\mu(b(z,r))\leqq c_4 r^\beta \qquad \text{for every } b(z,r)\subset\boldsymbol{R}^{d-1}.$$

Applying Lemma 2.5, we obtain

$$\left(\int \sup\{|u_f(x, \phi_j(x)+t)|^p : (x, \phi_j(x)+t) \in \Omega(z)\} d\mu(z)\right)^{1/p} \leqq c_5 \|f\|_{p,\lambda} .$$

From this and (2.8) it follows that

$$\left(\int_{B(Q_j, 3\delta_j)} (u_f)^*_\delta(Z)^p d\nu(Z)\right)^{1/p} \leqq c_6 \|f\|_{p,\lambda} .$$

Combining this with (4.2) we have the estimate (a).

(b) Let $f \in C^1(\partial D)$. We may suppose that $\operatorname{supp} f \subset B_j$. First, assume that $Z \in B(Q_j, 3\delta_j) \cap \partial D$. Using the same notations as in the proof of Lemma 2.5 and noting that the function $X \mapsto \int k(X, Y) d\sigma(Y)$ is constant on D, we can write, for $(x, \phi(x)+r) \in \Omega(z)$,

$$\begin{aligned} &u_{f-f(Z)}(x, \phi(x)+r) \\ &= u_{f-f(Z)}(x, \phi(x)+2\delta) - \int_r^{2\delta} \frac{\partial u_{f-f(Z)}}{\partial t}(x, \phi(x)+t) dt . \end{aligned}$$

If $|x-y|<\delta$, then $|\phi(x)+2\delta-\phi(y)|>\delta$. Therefore we have

$$\lim_{n\to z} u_{f-f(Z)}(x, \phi(x)+2\delta) = u_{f-f(Z)}(z, \phi(z)+2\delta) . \tag{4.3}$$

We next show that

$$\begin{aligned} &\lim_{x\to z, (x,\phi(x)+r)\in\Omega(z)} \int_r^{2\delta} \frac{\partial u_f}{\partial t}(x, \phi(x)+t) dt \\ &= \int_r^{2\delta} \frac{\partial u_f}{\partial t}(z, \phi(z)+t) dt . \end{aligned} \tag{4.4}$$

To see this, we write

$$\begin{aligned} &\int_r^{2\delta} \frac{\partial u_f}{\partial t}(x, \phi(x)+t) dt \\ &= \int_r^{|x-z|/2} \frac{\partial u_f}{\partial t}(x, \phi(x)+t) dt + \int_{|x-z|/2}^{2\delta} \frac{\partial u_f}{\partial t}(x, \phi(x)+t) dt \end{aligned} \tag{4.5}$$

and show that the integrands are dominated by some integrable functions independent of x, respectively.

We begin with estimating the second term in the right-hand side of (4.5). On account of (2.9) we obtain

$$\begin{aligned} I_0 &\equiv \int_{|x-z|/2}^{2\delta} \left|\frac{\partial u_f}{\partial t}(x, \phi(x)+t)\right| dt \\ &\leqq c_1 \int_0^{2\delta} t^{-b} dt \left(\iint_{|x-y|<t/2, t/4<s<2t} s^{-d+pb} \left|\frac{\partial u_f}{\partial s}(x, \phi(x)+s)\right|^p ds dy\right)^{1/p} . \end{aligned}$$

where $0<b<1$. Since $g(w)=f(w, \phi(w))$ is of C^1-class, we can estimate

$$I_0^p \leqq c_2 \iint_{E(Z)\cap A} s^{-d+pb} dyds \Big(\int \frac{|g(w)-g(y)|}{|y-w|^2+|\phi(y)+s-\phi(w)|^2)^{d/2}} dw \Big)^p$$

$$\leqq c_3 \int_{|y-z|<40\delta} \frac{dy}{|y-z|^{d-1-l}} \int_0^{2\delta} s^{pb-1-l-pm} ds$$

$$\times \Big(\int_{|y-w|<4\delta} \frac{dw}{|y-w|^{d-1-m}} \Big)^p$$

from the same consideration as in (2.7). Here we choose small positive real numbers l, m such that $pb-1-l-pm>-1$. Therefore we see that the integrand of the second term in the right-hand side of (4.5) is dominated by an integrable function.

We next estimate the integrand of the first term. Set

$$I_k = \int_{2^k r}^{2^{k+1} r} \Big| \frac{\partial u}{\partial t}(x, \phi(x)+t) \Big| dt .$$

Using (2.10), we obtain

$$I_k^p \leqq c_4 \int_{E_k(Z)} s^{-d+p} \Big| \frac{\partial u}{\partial s}(y, \phi(y)+s) \Big|^p dyds$$

$$\leqq c_5 \int_{E_k(Z)} s^{-d+p} dyds \Big(\int \frac{|g(w)-g(y)|dw}{(|y-w|^2+|\phi(y)+s-\phi(w)|^p)^{d-2}} \Big)^p$$

and hence

$$I_k^p \leqq c_6 \int \frac{1}{(2^k|y-z|^\tau)^u} \int_0^{2\delta} s^{-d+p-bp+a} ds$$

$$\times \Big(\int_{|y-w|<4\delta} \frac{dw}{|y-w|^{d-1-b}} \Big)^p ,$$

where a, b are positive real numbers satisfying

(4.6) $$a\tau < d-1, \quad -d+p-bp+a>-1 \quad b<1 .$$

It is possible to choose a, b satisfying (4.6). Indeed, noting that $\lambda<1$ and $p\lambda<d-1$, we pick a positive real number b with $\lambda<1-b$ and $p(1-b)<d-1$. Since

$$\tau = \frac{\beta}{d-1-p\lambda} < \frac{d-1}{d-1-p(1-b)} ,$$

it suffices to choose a positive real number a satisfying

$$d-1-p(1-b) < a < \frac{d-1}{\tau} .$$

Therefore we have that

$$\int_r^{|x-z|/2}\left|\frac{\partial u_f}{\partial t}(x, \phi(x)+t)\right|dt$$

$$\leqq \sum_{k=1}^{\infty}\int_{2^k r}^{2^{k+1}r}\left|\frac{\partial u_f}{\partial t}(x, \phi(x)+t)\right|dt$$

$$\leqq c_7 \sum_k 2^{-ka}\int_{|y-z|<4\delta}\frac{1}{|y-z|^{\tau a}}dy\,.$$

Thus we see that (4.4) holds, whence

$$\lim_{x\to z,\, r\to 0,\, (x,\phi(x)+r)\in\Omega(z)} u_{f-f(Z)}(x, \phi(x)+r)$$

$$=\lim_{r\to 0}\left\{u_{f-f(Z)}(z, \phi(z)+2\delta)-\int_r^{2\delta}\frac{\partial u_f}{\partial t}(z, \phi(z)+t)dt\right\}$$

$$=u_{f-f(Z)}(z, \phi(z))\,.$$

Noting that $\Gamma_{\tau,\eta}(Z)\subset\Omega(z)$, we have

$$\lim_{X\to Z, X\in\Gamma_{\tau,\eta}(Z)} u_f(X)=\lim_{X\to Z, X\in\Gamma_{\tau,\eta}(Z)}\{u_{f-f(Z)}(X)+f(X)\} \tag{4.7}$$

$$=Kf(Z)+\frac{f(Z)}{2}\,.$$

Finally suppose that $Z\in\partial D$ and $Z\notin B(Q_j, 3\delta_j)$. If $|X-Z|<\delta_j$ and $X\in D$, then we have

$$|X-Y|\geqq\delta\,, \qquad \text{on } \operatorname{supp} f\,.$$

Consequently we also obtain (4.7).

(c) Denote by E_f the set of all boundary points Z at which

$$\lim_{X\to Z, X\in\Gamma_{\tau,\eta}(Z)} u_f(X)$$

do not exist. If E_f is not a set of β-dimensional Hausdorff measure zero, then so is not a compact subset K of E_f. Therefore we can find a positive measure ν with $\operatorname{supp}\nu\subset K$ such that

$$\nu(B(Z, r))\leqq cr^{\beta}$$

for every ball $B(Z, r)$ (cf. [2, Theorem 1 in § II]).

On the other hand, by using a partition of unity and mollifiers, we see that $C^1(\partial D)$ is dense in $\Lambda_\lambda^p(\sigma)$. On account of (a) and (b) we can show that $\nu(E_f)=0$ by the standard argument. This yields a contradiction.

Q. E. D.

Let us now prove our theorem.

PROOF OF THEOREM. In [3, Theorem 2.1] it has been shown that

$K+(1/2)I$ is injective on $L^p(\sigma)$. Therefore $K+(1/2)I$ is also injective on $\Lambda_\lambda^p(\sigma)$. Since K is a compact operator on $\Lambda_\lambda^p(\sigma)$ by Lemma 3.2, the operator $(K+(1/2)I)$ is invertible on $\Lambda_\lambda^p(\sigma)$. Let $f\in\Lambda_\lambda^p(\sigma)$ and choose $g\in\Lambda_\lambda^p(\sigma)$ satisfying $(K+(1/2)I)g=f$. Then Lemma 4.1 shows that $u=u_g$ is the desired function. Q. E. D.

References

[1] P. Ahern and A. Nagel: Strong L^p estimates for maximal functions with respect to singular measures: with applications to exceptional sets, Duke Math. J. **53** (1986), 359-393.

[2] L. Carleson: Selected problems on exceptional sets, Van Nostrand, 1967.

[3] E. B. Fabes, M. Jodeit, JR. and N. M. Rivière: Potential techniques for boundary value problems on C^1-domains, Acta Math. **141** (1978), 165-186.

[4] A. Nagel, W. Rudin and J. Shapiro: Tangential boundary behavior of function in Dirichlet-type spaces, Ann. Math. **116** (1982), 331-360.

[5] Y. Mizuta: On the boundary limits of harmonic functions with gradient in L^p, Ann. Inst. Fourier **34** (1984), 99-109.

[6] E. M. Stein: Singular integrals and differentiability properties of functions, Princeton University Press, Princeton-New Jersey, 1970.