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On Regular Graphs and the Associated Real Algebras Generated by the Adjacency Matrices

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1. Summary

We will observe the relations between regular graphs and the eigenvalues of the adjacencdy matrices. The number of eigenvalues is equal to the dimension of the real algebra generated by the adjacency matrix provided that the multiplicity of the valency is 1 and no eigenvalue is 0. (cf. Th. 1 and Cor. 1).

It is known that a regular graph with 3-demensional associated algebra is strongly regular and this algebra is so-called centralizer algebra which means that a suitable basis of the algebra makes the edge set an association scheme (cf. [2]). We will introduce the concept of quasi strongly regular graph and show its associated algebra is a centraliazer algebra.

2. Notations and terminologies

A graph G consists of a non-void finite set of vertices (denoted by $V(G)$) together with a symmetric irreflexive binary relation (called adjacency) on $V(G)$. 2-point set $\{x, y\}$ is an edge of G if (x, y) belongs to this relation. A graph is complete if every pair of vertices is adjacent, and null if it has no edges at all. The complement of the graph G is the graph $\bar{G}$ with $V(\bar{G})=V(G)$, whose edge set is the complement of the edge set of G. A subgraph of G is a graph whose vertex set is $V(G)$ and edge set is a subset of the one of G. If $x \in V(G)$, the valency of x is the number of vertices adjacent to x. If every vertex has the same valency, the graph G is called regular, and common valency is the valency of the graph G. The adjacency matrix $A(G)$ is the matrix of adjacency relation (with rows and columns indexed by vertices, with (x, y) entry 1 if $\{x, y\}$ is an edge, 0 otherwise). Throughout this paper, I denotes the identity matrix, and J the matrix with every entry 1, of the appropriate size. It follows that $A(\bar{G})=J-I-A(G)$. G is regular if and only if $A(G)J=aJ$ for some constant a (which is then the valency of G). A strongly regular graph is a graph which is regular and has property that the number of vertices adjacent to x and y $(x\neq y)$ depends only on whether x and y are adjacent or not. Its parameters are (n, a, c, d), where n is the number of vertices and a the valency, and the number of vertices adjacent to x and y is c or d according as x and y are adjacent or non-adjacent.

An association scheme $(X;\ R_0, \cdots, R_m)$ consists of a set X together with $R_0, R_1, \cdots, R_m \subset X\times X$ satisfying the conditions

(1) $R_0=\{(x, x)\}$

(2) given x and $y\in X$, there exists one and only one i with $(x, y)\in R_i$

(3) R_i is symmetric

(4) given $(x, y)\in R_k$, the number p_{ij}^k of $z\in X$ with $(x, z)\in R_i$, $(y, z)\in R_j$ depends only on i, j and k.

An association matrix $E_i=(e_{xy}^i)$ of $(X;\ R_0, \cdots, R_m)$ is a matrix whose rows and columns are indexed by X, e_{xy}^i being 1 if $(x, y)\in R_i$, 0 otherwise.

Then, we know that $E_iJ=p_{ii}^0J$, $\sum_{i=0}^m E_i=J$, $E_iE_j=E_jE_i=\sum_{k=0}^m p_{ij}^k E_k$ Hence $\langle E_0, E_1, \cdots, E_m\rangle$ is $m+1$ demensional algebra. This algebra is called centralizer algebra.

3. Regular graphs and eigenvalues of the adjacency matrices

Given a graph G with adjacency matrix A, G is regular if and only if all-1 vector j is an eigenvector of A. The corresponding eigenvalue is the valency of G.

THEOREM 1. *Let A be the adjacency matrix of a regular graph G with the valency a. If A has r different eigenvalues $\rho_1, \cdots, \rho_r$ in $\langle j\rangle^{\perp}$, then*

$$A^r-\sigma_1A^{r-1}+\cdots+(-1)^r\sigma_rI-\frac{\alpha}{n}J=0 \tag{1}$$

where $\sigma_1, \cdots, \sigma_r$ are the elementary symetric polynomials of $\rho_1, \cdots, \rho_r$ and

$$\alpha=a^r-\sigma_1a^{r-1}+\cdots+(-1)^r\sigma_r.$$

PROOF. From $\{\Pi_{i=1}^r(A-\rho_iI)\}j=\{\Pi_{i=1}^r(a-\rho_i)\}j$, $\alpha=\Pi_{i=1}^r(a-\rho_i)$ is an eigen value of $\Pi(A-\rho_i)$ corresponding to j. Let $x\in\langle j\rangle^{\perp}$ be an eigenvector of A corresponding to eigenvalue ρ_i. Then $\{\Pi_{i=1}^r(A-\rho_iI)\}x=0$. Hence there exists an orthogonal matrix $P=((1/\sqrt{n})j, x_1, \cdots, x_{n-1})$ such that

$$
{}^tP\left\{\prod_{i=1}^{r}(A-\rho_i I)\right\}P=\begin{pmatrix} \alpha & 0\cdots 0 \\ 0 & 0\cdots 0 \\ \multicolumn{2}{c}{\cdots\cdots\cdots} \\ 0 & 0\cdots 0 \end{pmatrix}.
$$

$$
\prod_{i=1}^{r}(A-\rho_i I)=P\begin{pmatrix} \alpha & 0\cdots 0 \\ 0 & 0\cdots 0 \\ \multicolumn{2}{c}{\cdots\cdots\cdots} \\ 0 & 0\cdots 0 \end{pmatrix}{}^tP
$$

$$
=\left(\frac{1}{\sqrt{n}}j, x_1, \cdots, x_{n-1}\right)\begin{pmatrix} \alpha & 0\cdots 0 \\ 0 & 0\cdots 0 \\ \multicolumn{2}{c}{\cdots\cdots\cdots} \\ 0 & 0\cdots 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{n}}{}^tj \\ {}^tx_1 \\ \vdots \\ {}^tx_{n-1} \end{pmatrix}
$$

$$
=\left(\frac{1}{\sqrt{n}}j, x_1, \cdots, x_{n-1}\right)\begin{pmatrix} \frac{\alpha}{\sqrt{n}}\cdots\frac{\alpha}{\sqrt{n}} \\ 0 \cdots\cdots 0 \\ \cdots\cdots\cdots \\ 0 \cdots\cdots 0 \end{pmatrix}
$$

$$
=\frac{\alpha}{n}J.
$$

Call the real algebra spanned by I, J and A the associated algebra of G. G and $\bar{G}$ have the same associated algebra since $\langle I, J, A\rangle=\langle I, J, J-I-A\rangle$.

COROLLARY 1. *The adjancy matrix A of a regular graph G has r different eigenvalues in $\langle j\rangle^{\perp}$ if and only if the real algebra generated by I, J, and A is $(r+1)$-dimensional. If a has multiplicity 1 and no eigenvalue is 0, this algeba is generated by A.*

PROOF. Suppose

$$
(A^s+a_1A^{s-1}+\cdots+a_{s-1}A+a_sI+a_{s+1}J)=0.
$$

Let $x\in\langle j\rangle^{\perp}$ be an eigenvector with eigenvalue $\rho_i\,(i=1,\cdots,r)$.

$$
(A^s+a_1A^{s-1}+\cdots+a_{s-1}A+a_sI+a_{s+1}J)x=0
$$

implies $f(\rho_i)=0$ where $f(x)=x^s+a_1x^{s-1}+\cdots+a_{s-1}x+a_s$. It follows that $s\geqq r$. Thus the algebra generated by I, J and A has demension $r+1$.

From (1) we have

$$A^{r+1}-\sigma_1A^r+\cdots+(-1)^r\sigma_rA-\frac{a\alpha}{n}J=0\,.$$

So J and I belong to the algebra generated by A whenever α and σ_r are not 0.

Followings are remarks on strongly regular graphs.

Let A be the adjacency matrix of a strongly regular graph G with parameters (n, a, c, d). Then

$$A^2=(c-d)A+(a-d)I+dJ \tag{2}$$

and

$$a(a-c-1)=(n-a-1)d \tag{3}$$

hold (cf. [2]). $\bar{G}$ is again strongly regular with parameters $(n,\ n-a-1,\ n-2a+d-2,\ n-2a+c)$.

PROP. 1. *Let G and A be as above. If neither G is a disjoint union of a finite copies of the complete graph with $a+1$ vertices nor the complement of a disjoint union of a finite copies of a complete graph (complete multi-partite graph), the algebra generated by I, J and A is generated by A.*

PROOF. Suppose $x\in\langle j\rangle^{\perp}$ be an eigenvector with eigenvalue a. From (2), $a^2x=a(c-d)x+(a-d)x$, and $d=0$ and $a=c+1$. This means that the graph is a union of $n/(a+1)$ copies of a complete graph with $a+1$ vertices. In this case the multiplicity of a is $n/(a+1)$ and the other eigenvalue of A is -1. Next, suppose A has an eigenvalue 0 i.e. $a=d$. Let the parameters of $\bar{G}$ be (n, a', c', d') where $a'=n-a-1$, $c'=n-2a+d-2$, $d'=n-2a+c$. $a'=c'+1$ and (3) imply $d'=0$. Hence G is the complement of one in the first case. A has eigenvalue a of multiplicity 1 and the other eigenvalues are 0 and $c-a$.

4. Quasi strongly regular graphs

Consider the following regular graph G with the valency a and its adjacency matrix A. If x and y are adjacent, (x, y) entry of A^2 takes one of two values c_1 and c_2, and (x, y) entry of A^3 takes c_1' or c_2' according as (x, y) entry of A^2 is c_1 or c_2. If x and y are non-adjacent, (x, y) entry of A^2 is d and (x, y) entry of A^3 is d'. Every diagonal entry of A^3 is a'.

Geometrically speaking, if x and y are adjacent, the number of the walks from x to y of length 3 is c_1' or c_2' according as the number of the triangles which contain edge $\{x, y\}$ is c_1 or c_2. The number of the walks from x to y of length 2 is d and the number of the walks from x to y of length 3 is d' whenever x and y are non-adjacent. (G is disconnected if and only if $d=0$.) For any vertex x, a' is twice the number of triangles with vertex x.

We call such a regular graph and its complement quasi strongly regular.

Comparing corresponding entries in A^3, A^2, A, I and J, we obtain the solution of the equation $A^3=xA^2+yA+zI+wJ$,

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} c_1 & 1 & 0 & 1 \\ c_2 & 1 & 0 & 1 \\ d & 0 & 0 & 1 \\ a & 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} c_1' \\ c_2' \\ d' \\ a' \end{pmatrix} = \begin{pmatrix} (c_2'-c_1')/(c_2-c_1) \\ \{d(c_2'-c_1')+c_2c_1'-c_1c_2'\}/(c_2-c_1)-d' \\ (d-a)(c_2'-c_1')/(c_2-c_1)-d'+a' \\ -d(c_2'-c_1')/(c_2-c_1)+d' \end{pmatrix}$$

provided $c_1 \neq c_2$. Put

$$E_0=I$$

$$E_1=-1/(c_2-c_1)A^2+(c_2-d)/(c_2-c_1)A+(a-d)/(c_2-c_1)I+d/(c_2-c_1)J$$

$$E_2=1/(c_2-c_1)A^2+(d-c_1)/(c_2-c_1)A+(d-a)/(c_2-c_1)I-d/(c_2-c_1)J$$

$$E_3=J-A-I$$

then $\langle E_0, E_1, E_2, E_3\rangle=\langle I, J, A, A^2\rangle$. Hence the associated algebra of quasi strongly regular graph is a centralizer algebra.

The associated algebra of the complement of G has the same centralizer algebra $\langle E_0, E_1, E_2, E_3\rangle$. If x and y are non-adjacent in $\bar{G}$, the number of the vertices adjacent to x and y in $\bar{G}$ is $n-2a+c_1$ or $n-2a+c_2$, and if x and y are adjacent in $\bar{G}$, the number of the vertices adjacent to x and y is $n-2a+d-2$.

THEOREM 2. *The associated algebra of a quasi strongly regular graph is a centralizer algebra of dimension* 4.

EX. 1. A disjoint union of copies of a strongly regular graph is quasi strongly regular, and the multiplicity of the valency is not 1.

EX. 2. The cycle of length 7 is a quasi strongly regular graph of valency 2 with $c=0$, $d_1=1$, $d_2=0$.

EX. 3. The triangular prism is a quasi strongly regular graph.

$$c_1=1, \quad c_2=0, \quad d=2,$$

$$A^3+A^2-2A=5J,$$

$$a=3, \quad \rho_1=1, \quad \rho_2=0, \quad \rho_3=-0,$$

$$E_1=A^2+2A-I-2J, \quad E_2=-A^2+I+2J, \quad E_3=J-A-I.$$

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