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Theorems of Fatou Type with Respect to Countably Sublinear Functionals

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§ 1. Introduction and notations

Let U be the unit disc in $\boldsymbol{R}^n$. The classical theorem of Fatou asserts that the nontangential boundary limit of the Poisson integral of a function in $L^p(\partial U)$ exists almost everywhere. Similar results have been obtained for more general domains, for example, Lipschitz domains, NTA-domains and for more general approach regions (cf. [9], [10], [11], [12], [1]). In these papers maximal operators of Hardy-Littlewood type associated to the approach regions and L^p-boundedness of the maximal operators have played important roles.

In this paper, for an open set U we consider a countably sublinear functional γ on the class $J(\partial U)$ of the extended real-valued functions on ∂U, in place of a measure on the boundary ∂U and suppose that for each $x \in U$ a monotone sublinear functional $\Phi_f(x)$ on $\mathcal{L}(\gamma, \mathcal{H})$ is defined, where $\mathcal{L}(\gamma, \mathcal{H})$ is the closure of a family $\mathcal{H}$ of real-valued functions on ∂U relative to γ. We give sufficient conditions for the boundary limits of Φ_f along given filters to exist γ-q.e. on ∂U.

More precisely, let F be a closed subset of a topological space and denote by $J(F)$ the class of all extended real-valued functions on F. A mapping γ from $J(F)$ to $\boldsymbol{R}^+ \cup \{+\infty\}$ is called a countably sublinear functional if it has the following properties:

(c_1) $\gamma(f) = \gamma(|f|)$,

(c_2) $b \in \boldsymbol{R} \Rightarrow \gamma(bf) = |b|\gamma(f)$,

(c_3) $f, f_n \geqq 0,\ f \leqq \sum_n f_n \Rightarrow \gamma(f) \leqq \sum_n \gamma(f_n)$.

To simplify the notations, we use $\gamma(E)$ instead of $\gamma(\chi_E)$ for a subset E of F, where χ_E is the indicator function of E. A subset E of F satisfying $\gamma(E)=0$ is called γ-polar. If a property holds on F except for a γ-polar set, we say that the property holds γ-q.e. on F (simply, it holds γ-q.e.). We note that if $\gamma(f) < +\infty$, then the set $\{x \in F;\ |f(x)| = +\infty\}$ is γ-polar (cf. [7, 1.3 Lemma]).

Put

$$\mathscr{B}(\gamma)=\{f\in J(F)\,;\ \gamma(f)<+\infty\}.$$

We define the algebraic operations with extended real numbers in the usual way. For example,

$$0(\pm\infty)=0, \qquad (+\infty)+(-\infty)=0.$$

Furthermore, fix a linear sublattce $\mathscr{H}$ of Borel measurable functions in $\mathscr{B}(\gamma)$. We denote by $\mathscr{L}(\gamma, \mathscr{H})$ the set of all Borel measurable functions f such that there exists a sequence $\{f_n\}\subset\mathscr{H}$ for which $\gamma(f_n-f)\to 0$.

If $f, g\in\mathscr{L}(\gamma, \mathscr{H})$ and $b, c\in\boldsymbol{R}$, then $bf+cg$ and $|f|$ belong to $\mathscr{L}(\gamma, \mathscr{H})$. For example, let ν be a positive Radon measure on F and p be a real number satisfying $p\geqq 1$. We define

$$\gamma(f)=\left(\int^{*}|f|^{p}d\nu\right)^{1/p} \quad \text{for } f\in J(F),$$

where $\int^{*}$ stands for the upper integral with respect to ν. Then a subset E of F is γ-polar if and only if $\nu^{*}(E)=0$. Moreover if $\mathscr{H}$ is the class $\kappa(F)$ of all continuous real-valued functions on F with compact support, then $\mathscr{L}(\gamma, \kappa(F))$ is equal to $\mathscr{L}^{p}(\nu)$, i.e., the family of all Borel measurable functions f such that $|f|^{p}$ is ν-integrable. Therefore a countably sublinear functional γ on $J(F)$ is regarded as a generalization of an upper integral.

Throughout in this paper, let X be a topological space and U be an open subset of X with boundary B and γ be a countably sublinear functional on $J(B)$. Fix a linear sublattice $\mathscr{H}$ of Borel measurable real-valued functions in $\mathscr{B}(\gamma)$. Suppose that to each $x\in U$ and $f\in\mathscr{L}(\gamma, \mathscr{H})$ there corresponds a real number $\Phi_f(x)$ satisfying the following properties:

(a_1) $f\leqq g$ γ-q.e.$\Rightarrow\Phi_f(x)\leqq\Phi_g(x)$,

(a_2) $\Phi_{f+g}(x)\leqq\Phi_f(x)+\Phi_g(x)$.

Furthermore, let T be a mapping from $\mathscr{L}(\gamma, \mathscr{H})$ to $\mathscr{B}(\gamma)$ satisfying the following properties:

(b_1) $f\leqq g$ γ-q.e.$\Rightarrow Tf\leqq Tg$ γ-q.e.,

(b_2) $T(f+g)\leqq Tf+Tg$ γ-q.e.,

(b_3) T is γ-bounded; there is a positive real number c such that

$$\gamma(T|f|)\leqq c\gamma(f) \quad \text{for all } f\in\mathscr{L}(\gamma, \mathscr{H}).$$

Moreover, we assume that to each $z\in B$ there corresponds a filte $\mathscr{F}_z$ of subsets of U, converging to z.

In §2 we will prove the following theorem.

THEOREM 1. *Suppose that*

$$\mathscr{F}_z-\lim\Phi_f(x)=Tf(z)\ \ \gamma\text{-q.e.} \tag{1.1}$$

for every f in a dense subset $\mathcal{H}_1$ of $\mathcal{H}$ with respect to γ. Set

$$E_{f,b}=\{z\in B;\ \mathcal{F}_z-\limsup \Phi_{|f|}(x)>b\}$$

and suppose that there are positive real numbers c and p such that

$$\gamma(E_{f,b})\leqq cb^{-p}\gamma(f)^p \tag{1.2}$$

for every real number $b>0$ and $f\in\mathcal{L}(\gamma,\mathcal{H})$. Then (1.1) *holds for every $f\in\mathcal{L}(\gamma,\mathcal{H})$.*

We next consider 'S-fine limits'. Let S be a convex cone of non-negative lower semicontinuous functions in X which has the following properties:

(s_1) $\{u_n\}\subset S\Rightarrow\sum_n u_n\in S$,

(s_2) There exists a locally bounded strictly positive function $v\in S$.

A subset E of U is said to be S-thin at $z\in B$, if $z\notin\bar{E}$ or there exists $u\in S$ such that

$$\liminf_{x\to z,\, x\in E} u(x)>u(z)\,.$$

If E_1 and E_2 are S-thin at z, then $E_1\cup E_2$ is also S-thin at z.

We assume that U is not S-thin at each point $z\in\partial D$ and denote by $\mathcal{F}_z$ the filter generated by the family

$$\{F\cup\{z\};\ F\subset U,\ U\backslash F \text{ is } S\text{-thin at } z\}\,.$$

If a function g converges along $\mathcal{F}_z$, we say that g has a S-fine limit at z, which is denoted by $S-\lim_{x\to z,\, x\in U} g(x)$.

Furthermore, fix a positive Radon measure ν on X such that $\int v\,d\nu<+\infty$. For $f\in J(B)$ we define

$$\gamma_\nu^S(f)=\inf\left\{\int u\,d\nu:\ u\in S,\ u\geqq|f| \text{ on } B\right\}.$$

Then γ_ν^S is a countably sublinear functional on $J(B)$ and $\mathcal{B}(\gamma_\nu^S)\supset C_v(B)$, where $C_v(B)$ stands for the space of all continuous real-valued functions f defined on B such that $|f|\leqq\alpha v$ on B for some $\alpha>0$. Therefore we can define $\mathcal{L}(\gamma_\nu^S, C_v(B))$.

In §3 we will show

THEOREM 2. *Let $\mathcal{H}_1$ be a dense subset of $\mathcal{L}(\gamma_\nu^S, C_v(B))$ such that*

$$S-\lim_{x\to z,\, x\in U}\Phi_g(x)=Tg(z)\ \gamma_\nu^S\text{-q.e.} \tag{1.3}$$

for every $g\in\mathcal{H}_1$. Suppose that for some real number $c>0$ the following condition $(*)$ is satisfied:

$$u\in S,\quad g\in\mathcal{L}(\gamma_\nu^S, C_v(B)),\quad 0\leqq g\leqq u \quad \text{on } B \tag{$*$}$$

$$\Longrightarrow \ \Phi_g(x) \leqq cu(x) \quad \text{for all } x \in U.$$

Then (1.3) *holds* γ_v^S-q.e. *for every* $g \in \mathcal{L}(\gamma_v^S, C_v(B))$.

In §4 the above results will be applied to potential theory. Let U be an open set in the strong harmonic space X in the sense of Bauer. We intend to construct a function space such that for each function f in the space the generalized solution

$$x \longmapsto \int f d\varepsilon_x^{CU}$$

has the fine limits on ∂U except for a polar set. To do this we consider the convex cone S of nonnegative hyperharmonic functions in X and a strictly positive continuous potential v, and define a countably sublinear functional γ_v^S on $J(\partial U)$ such that a subset E of B is γ_v^S-polar if and only if E is polar. Further it will be proved that, under the assumption that there is no point of ∂U at which U is thin, the harmonic function: $x \mapsto \int g d\varepsilon_x^{CU}$ for every $g \in \mathcal{L}(\gamma_v^S, C_v(\partial U))$ has the fine limit $\int g d\varepsilon_z^{CU}$ at every $z \in \partial U$ except for a polar set.

§2. Proof of Theorem 1

Before the proof of the theorem, recall that, a sequence $\{f_n\}$ of functions in $\mathcal{B}(\gamma)$ converges γ-quasi uniformly to a function $f \in \mathcal{B}(\gamma)$ if there exists, for every $\varepsilon > 0$, a set $E \subset B$ such that $\gamma(E) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on $B \backslash E$ as $n \to \infty$.

It is well-known that $\mathcal{B}(\gamma)$ has the following property (cf. [7, 1.4. Theorem]).

LEMMA 2.1. *Let* $\{f_n\}$ *be a sequence in* $\mathcal{B}(\gamma)$. *If* $\gamma(f_n) \to 0$, *then* $\{f_n\}$ *has a subsequence which converges to* 0 γ*-quasi uniformly.*

Furthermore, let λ be a nonnegative extended real-valued functions defined on a family of subsets of B. An extended real-valued function on B is said to be λ-quasicontinuous if for each $\varepsilon > 0$ there exists an open set V such that $\lambda(V) < \varepsilon$ and the restriction of f to $B \backslash V$ is finite and continuous.

In view of the proof of [7, 1.4. Theorem] we obtain

LEMMA 2.2. *Let* $\{f_n\}$ *be a sequence of real-valued continuous functions in* $\mathcal{B}(\gamma)$ *which is a Cauchy one with respect to* γ. *Then there exists a* γ*-quasicontinuous function* f *such that* $\gamma(f - f_n) \to 0$. *In particular, if* $\mathcal{H} \subset C(B)$, *then for every* $g \in \mathcal{L}(\gamma, \mathcal{H})$ *there exists a* γ*-quasicontinuous* h

such that $g=h$ γ-q.e., *where* $C(B)$ *is the space all of continuous real-valued functions.*

PROOF OF THEOREM 1. Let $f\in\mathcal{L}(\gamma,\mathcal{H})$ and ε be a positive real number. Further, set

$$B_\varepsilon=\{z\in B\,;\ \mathcal{F}_z-\limsup|\Phi_f(x)-Tf(z)|>\varepsilon\}\,.$$

It suffice to see that $\gamma(B_\varepsilon)=0$. By the definition of $\mathcal{L}(\gamma,\mathcal{H})$, we can choose a sequence $\{f_n\}\subset\mathcal{H}_1$ such that $\gamma(f-f_n)\to 0$ and hence $\gamma(T|f-f_n|)\to 0$ by the assumptions of T. By the aid of Lemma 2.1 there is a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{T|f-g_n|\}$ converges γ-quasi uniformly to 0. For each $\delta>0$ satisfying $\delta<\varepsilon$ there exist a set F_δ and a natural number m such that

$$\gamma(B\backslash F_\delta)<\delta\,,\quad T|f-g_m|<\delta\ \text{ on } F_\delta \ \text{ and }\ \gamma(f-g_m)<\delta\,. \tag{2.1}$$

We set

$$E_0=\{z\in B\,;\ |Tf(z)|=+\infty\}\cup(\cup_n\{z\in B;\ |Tg_n(z)|=+\infty\})$$

and denote by E_n the set of points z such that $\mathcal{F}_z-\lim\Phi_{g_n}(x)$ do not exist or $\mathcal{F}_z-\lim\Phi_{g_n}(x)\neq Tg_n(z)$. Furthermore, denote by G_n the set

$$\{z\in B\,;\ |Tf(z)-Tg_n(z)|>T|f-g_n|(z)\}\,.$$

Then $E:=\cup_{n=0}^{\infty}(E_n\cup G_n)$ is a γ-polar set by the assumptions for $\mathcal{H}_1$ and T. Let $z\in F_\delta\backslash E$. Then we have

$$\begin{aligned}
&\mathcal{F}_z-\limsup|\Phi_f(x)-Tf(z)|\\
&\quad\leqq\mathcal{F}_z-\limsup|\Phi_f(x)-\Phi_{g_m}(x)|\\
&\qquad+\mathcal{F}_z-\limsup|\Phi_{g_m}(x)-Tg_m(z)|+|Tg_m(z)-Tf(z)|\\
&\quad\leqq\mathcal{F}_z-\limsup\Phi_{|f-g_m|}(x)+\delta\,.
\end{aligned}$$

Therefore we see that

$$B_{\varepsilon,\delta}\subset A_{\varepsilon,\delta}\,, \tag{2.2}$$

where

$$A_{\varepsilon,\delta}=\{z\in F_\delta\backslash E\,;\ \mathcal{F}_z-\limsup\Phi_{|f-g_m|}(x)>\varepsilon-\delta\}$$

and

$$B_{\varepsilon,\delta}=\{z\in F_\delta\backslash E\,;\ \mathcal{F}_z-\limsup\Phi_f(x)-Tf(z)|>\varepsilon\}\,.$$

By (2.1), (2.2) and (1.2) we obtain

$$\begin{aligned}
\gamma(B_\varepsilon)&\leqq\gamma(B\backslash F_\delta)+\gamma(E)+\gamma(B_{\varepsilon,\delta})\\
&\leqq\delta+\gamma(A_{\varepsilon,\delta})\leqq\delta+c(\varepsilon-\delta)^{-p}\gamma(f-g_m)^p
\end{aligned}$$

$$<\delta+c(\varepsilon-\delta)^{-p}\delta^{p}.$$

As $\delta\to 0$, we see that $\gamma(B_\varepsilon)=0$, which completes the proof.

REMARK. We see easily that Lemmas 2.1, 2.2 and Theorem 1 remain valid even if we replace the property (c_3) of γ by the following one:

(c_3') There is a positive real number c such that

$$f, f_n\geqq 0, \quad f\leqq \Sigma_n f_n \Longrightarrow \gamma(f)\leqq c\Sigma_n \gamma(f_n).$$

§ 3. Proof of Theorem 2

We prepare the following lemma.

LEMMA 3.1. *Assume that condition* (*) *in Theorem* 2 *is satisfied. For* $g\in\mathcal{L}(\gamma_\nu^S, C_v(B))$ *and* $b\in \boldsymbol{R}^+$, *set*

$$F_{g,b}=\{z\in B;\ S-\limsup_{x\to z,\, x\in U}\Phi_{|g|}(x)>b\},$$

where $S-\limsup_{x\to z,\, x\in U}\Phi_{|g|}(x)=\lim_{E\in\mathcal{F}_z}\sup_{x\in E}\Phi_{|g|}(x)$. *Then there exists a real number* $c>0$ *such that*

$$\gamma_\nu^S(F_{g,b})\leqq cb^{-1}\gamma_\nu^S(g) \tag{3.1}$$

for every $g\in\mathcal{L}(\gamma_\nu^S, C_v(B))$ *and every real number* $b>0$.

PROOF. For $g\in\mathcal{L}(\gamma_\nu^S, C_v(B))$ and a positive real number b, put $F=F_{g,b}$. To show (3.1), let u be an arbitrary function in S such that $u\geqq|g|$ on B. By assumption (*) we have $u(x)\geqq c^{-1}\Phi_{|g|}(x)$ for every $x\in U$. Let $z\in F$. Then $\{x\in U;\ \Phi_{|g|}(x)\leqq b\}\notin\mathcal{F}_z$. Therefore the set $E=\{x\in U;\ \Phi_{|g|}(x)>b\}$ is not S-thin at z. Consequently we have

$$u(z)=\liminf_{x\to z,\, x\in E}u(x)$$
$$\geqq\liminf_{x\to z,\, x\in E}c^{-1}\Phi_{|g|}(x)\geqq c^{-1}b,$$

whence $cb^{-1}u\geqq 1$ on F. From the definition of γ_ν^S it follows that

$$cb^{-1}\int u\,d\nu\geqq\gamma_\nu^S(F),$$

whence

$$cb^{-1}\gamma_\nu^S(g)\geqq\gamma_\nu^S(F).$$

This completes the proof.

PROOF OF THEOREM 2. This is an easy consequence of Theorem 1 and Lemma 3.1.

§ 4. Harmonic spaces

In this section, let X be a strong harmonic space in the sense of Bauer [2] and U be an open set with boundary B. Further, let S be the convex cone of all nonnegative hyperharmonic functions in X. Evidently S has the property (s_1) in § 1. Since there exists a strictly positive continuous potential v, S also has the property (s_2). Denote by P the family of all strictly positive continuous potentials.

In this section we also assume that there is no point in B at which U is thin, i.e.,

$\liminf_{x\to z,\, x\in U} u(x)=u(z)$ for every $u\in S$ and every $z\in B$.

Fix a countable regular open base $\{V_p\}$ of X and points $\{x_p\}$ such that $x_p\in V_p$. We shall define a measure ν, depending on a function $v\in P$, by

$$\nu=\sum_{p=1}^{\infty}2^{-p}v(x_p)^{-1}\mu_p\,, \tag{4.1}$$

where μ_p is the harmonic measure of V_p at x_p.

We shall define for $g\in J(B)$

$$\gamma_v^S(g)=\inf\left\{\int u\,d\nu\,;\ u\in S, u\geqq|g| \text{ on } B\right\}.$$

Then γ_v^S is a countably sublinear functional such that $\mathscr{B}(\gamma_v^S)\supset C_v(B)$.

A set E is said to be polar if there is a nonnegative superharmonic function u such that $u=+\infty$ on E.

LEMMA 4.1. *A subset E of B is γ_v^S-polar if and only if E is polar.*

PROOF. Suppose that $\gamma_v^S(E)=0$ and choose a sequence $\{u_n\}\subset S$ such that $u_n\geqq 1$ on E and $\int u_n d\nu<1/2^n$. If we set

$$u=\sum_n u_n\,,$$

then $u\in S$ and $u=+\infty$ on E. Moreover we have for every p

$$\int u\,d\mu_p=\sum_n\int u_n d\mu_p\leqq 2^p v(x_p)\sum_n\int u_n d\nu\leqq 2^p v(x_p)<+\infty\,.$$

Therefore we can find $z_p\in\partial V_p$ such that $u(z_p)<+\infty$. Since $\{V_p\}$ is a base, we see that the set $\{x\in X;\ u(x)<+\infty\}$ is dense. Thus E is polar.

Conversely, suppose that E is polar. Choose a nonnegative superharmonic function u such that $u=+\infty$ on E. Theh we have $\int u\,d\nu<+\infty$ (cf. [2, Satz 2.3.1]). Let $\varepsilon>0$. Since $\varepsilon u\geqq 1$ on E, we have $\gamma_v^S(E)\leqq\varepsilon\int u\,d\nu$, which leads to $\gamma_v^S(E)=0$.

H. Bauer proved in [3] also in [4] for harmonic spaces in the sense of Constantinescu-Cornea, that the generalized sulution H_f has a fine limit at each irregular point $z\in B$ for every bounded resolutive boundary function f and this limit is identified with the integral $\int f d\varepsilon_z^{CU}$, where ε_z^{CU} is the balayaged measure of ε_z to CU. W. Hansen also obtained the similar results concerning finely harmonic functions and finely open sets (cf. [8]).

The following theorem gives the boundary behavior of the harmonic function: $x\mapsto\int g d\varepsilon_z^{CU}$ for $g\in\mathcal{L}(\gamma_v^S, C_v(B))$.

THEOREM 3. *Let U be an open set with boundary B in a strong harmonic space in the sense of Bauer. Furthermore assume that there in no point in B at which U is thin. Then every $g\in\bigcup_{v\in P}\mathcal{L}(\gamma_v^S, C_v(B))$ is ε_x^{CU}-integrable for all $x\in U$ and*

(4.2) $$\mathrm{f}-\lim_{x\to z,\, x\in U}\int g d\varepsilon_x^{CU}=\int g d\varepsilon_z^{CU}$$

at every $z\in B$ except for a polar set, where f$-$lim *stands for the fine limit.*

PROOF. First we show that every $g\in\mathcal{L}(\gamma_v^S, C_v(B))$ is ε_x^{CU}-integrable for all $x\in U$. Since $\gamma_v^S(g)<+\infty$, there is $u\in S$ such that

$$u\geqq|g| \quad\text{on } B \quad\text{and}\quad \int u d\nu<+\infty\,,$$

where $\nu=\sum_{p=1}^{\infty}2^{-p}v(x_p)^{-1}\mu_p$. Assume that $\bar{V}_p\subset U$. Then we have

$$\int|g|d\varepsilon_{x_p}^{CU}\leqq\int u d\mu_p\leqq 2^p v(x_p)\int u d\nu<+\infty\,.$$

Since $\{x_p\}$ is dense, the function $x\mapsto\int|g|d\varepsilon_x^{CU}$ is harmonic in U and hence $\int|g|d\varepsilon_x^{CU}<+\infty$ for all $x\in U$.

We next define

$$\Phi_g(x)=\int g d\varepsilon_x^{CU} \quad\text{for } g\in\mathcal{L}(\gamma_v^S, C_v(B)) \quad\text{and}\quad x\in U.$$

It is easy to see that Φ_g has the properties (a_1) and (a_2) in § 1. Noting that the support of ε_z^{CU} is contained in B for every $z\in B$, we also define

$$Tg(z)=\int g d\varepsilon_z^{CU}$$

for $g\in\mathcal{L}(\gamma_v^S, C_v(B))$ and $z\in B$. To see that T has also the property (b_3) in § 1, let $g\in\mathcal{L}(\gamma_v^S, C_v(B))$ and u be a function in S such that $|g|\leqq u$ on B. Then

$$T|g|(z)\leqq\int u d\varepsilon_z^{CU}\leqq u(z) \quad\text{for every } z\in B\,,$$

which leads to $\gamma_v^S(T|g|)\leqq\int u d\nu$. From the definition of γ_v^S it follows that $\gamma_v^S(T|g|)\leqq\gamma_v^S(g)$. Thus we see that T has property (b_3) and $|Tg|<+\infty$ γ_v^S-q.e. Furthermore it is obvious that T has the properties (b_1) and (b_2) in § 1.

To see that the assumptions of Theorem 2 are satisfied, let $g\in C_v(B)$. If CU is not thin at $z\in B$, then we have

$$\lim_{x\to z,\, x\in U}\int g d\varepsilon_x^{CU}=g(z)=\int g d\varepsilon_z^{CU} \tag{4.3}$$

(cf. [5, VII 3.1. Proposition]).

Next, assume that CU is thin at $z\in B$. Since $U\cup\{z\}$ is a fine neighborhood, we obtain

$$\mathrm{f}-\lim_{x\to z,\, x\in U}\hat{R}_w^{CU}(x)=\hat{R}_w^{CU}(z)$$

for all nonnegative continuous hyperharmonic functions w, whence

$$\mathrm{f}-\lim_{x\to z,\, x\in U}\int g d\varepsilon_x^{CU}=\int g d\varepsilon_z^{CU} \tag{4.4}$$

(cf. [5, VI 11.4. Proposition]). Noting that a set in $\mathcal{F}_z$ is also a fine neighborhood, we have

$$S-\lim_{x\to z,\, x\in U}\int g d\varepsilon_x^{CU}=\mathrm{f}-\lim_{x\to z,\, x\in U}\int g d\varepsilon_x^{CU}\,. \tag{4.5}$$

By combining this with (4.3), (4.4), we conclude that

$$S-\lim_{x\to z,\, x\in U}\int g d\varepsilon_x^{CU}=\int g d\varepsilon_z^{CU}$$

for all $g\in C_v(B)$ and all $z\in B$.

Moreover the condition (*) is satisfied. Indeed, if $u\geqq g\geqq 0$ on B, then

$$u(x)\geqq\int u d\varepsilon_x^{CU}\geqq\int g d\varepsilon_x^{CU}\quad\text{for every } x\in U.$$

Thus we see that all assumptions of Theorem 2 are satisfied. Therefore, by the aid of Theorem 2, Lemma 4.1 and (4.5), we have the conclusion.

REMARK. In particular if g is the sum of a series of continuous potentials g_n such that for some $v\in P$ the function g_n/v is bounded for every n and $\sum_{n=1}^{\infty}\int g_n d\nu<+\infty$ for the measure ν defined by (4.1), then the restriction g^* of g to B belongs to $\mathcal{L}(\gamma_v^S, C_v(B))$. In fact, this follows from $\sum_{n=1}^{m} g_n^*\in C_v(B)$ and

$$\gamma_v^S(g^*-\textstyle\sum_{n=1}^{m} g_n^*)\leqq\int(\sum_{n=m+1}^{\infty} g_n)d\nu \longrightarrow 0\,,$$

where g_n^* is the restriction of g_n to B.

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