Natural Science Report, Ochanomizu University, Vol. 41, No. 2, 1990
お茶の水女子大学 自然科学報告 第41巻 第2号

On Some Cylindrical Vector-Valued Measures

Michie Maeda

Department of Mathematics, Ochanomizu University, Tokyo

§ 1. Introduction

I. Kluvánek has introduced an operator-valued measure M_t which is called SPt-measure in [4, 5] as follows:

Let E be a Banach space and $L(E)$ the space of bounded linear operators on E. Let $S: \{(t, s): 0 \leqq s \leqq t < \infty\} \to L(E)$ be a map such that

(i) $S(t, t) = I$, the identity operator, for every $t \geqq 0$;

(ii) $S(t, r) = S(t, s) \cdot S(s, r)$ for any r, s and t such that $0 \leqq r \leqq s \leqq t < \infty$;

(iii) S is continuous in the strong operator topology of $L(E)$.

Such a map is called a propagator in the space E. If $S(t, s) = S(t-s, 0)$, for any $0 \leqq s \leqq t < \infty$, then we write without ambiguity $S(t) = S(t, 0)$, for every $t \geqq 0$. Then we call it a semigroup.

Let Λ be a locally compact Hausdorff space, $\mathscr{B}(\Lambda)$ the σ-algebra of Baire sets in Λ. Let $P: \mathscr{B}(\Lambda) \to L(E)$ be a spectral measure. That is, P is σ-additive in the strong operator topology, $P(\Lambda) = I$ and $P(B \cap C) = P(B)P(C)$ for any $B \in \mathscr{B}(\Lambda)$ and $C \in \mathscr{B}(\Lambda)$.

For every $t \geqq 0$, let Γ_t be a set of maps $v: [0, t] \to \Lambda$ to be called paths. Let P_t be the family of all sets

$$\Gamma = \{v \in \Gamma_t : v(t_j) \in B_j,\ j = 1, 2, \cdots, k\}$$

corresponding to arbitrary $k = 1, 2, \cdots$, numbers $0 \leqq t_1 < t_2 < \cdots < t_{k-1} < t_k \leqq t$ and sets $B_j \in \mathscr{B}(\Lambda)$, $j = 1, 2, \cdots, k$. Let $M_t: P_t \to L(E)$ be a map such that $M_t(\Gamma) = S(t, t_k)P(B_k)S(t_k, t_{k-1})P(B_{k-1}) \cdots P(B_2)S(t_2, t_1)P(B_1)S(t_1, 0)$ for every set $\Gamma \in P_t$.

We consider the case that $\Lambda = R^n$, where n is a positive integer and $\Gamma_t = Y_t$ consists of all continuous paths $v: [0, t] \to R^n$. M_t is a cylindrical operator-valued measure. I. Kluvánek has considered in [4] the case that M_t is extensible to a σ-additive measure on $\sigma(P_t)$ which is the σ-algebra generated by P_t. However, it is very rare cases except the Wiener measure. So, we investigate the special case of M_t which is available to get some kind of extension.

§ 2. An operator S_t

In this section we investigate a special operator which relates to M_t. We follow Nelson's method of construction of the Wiener measure ([6]) (see also Ichinose [3]).

Let $\dot{R}^n = R^n \cup \{\infty\}$ be the one-point compactification of R^n. In section 1 we have defined Y_t. We introduce the infinite path $v_\infty : [0, t] \to \dot{R}^n$ defined by $v_\infty(s) = \infty$, for $0 \leqq s \leqq t$. Then we understand that Y_t contains the infinite path.

Let $C(\dot{R}^n)$ be the Banach space of the C-valued, where C is the complex number field, continuous functions on $\dot{R}^n$, denote by X. Let $\tilde{X}$ be the Banach space of all C-valued bounded Borel measurable functions defined on $\dot{R}^n$. Let $Z = C(\Pi_{[0,t]}\dot{R}^n ; C)$ denote the Banach space of the C-valued continuous functions on $\Pi_{[0,t]}\dot{R}^n$, where $\Pi_{[0,t]}\dot{R}^n$ is the product of the uncountably many $\dot{R}^n$; $Z_{\text{fin}} = C_{\text{fin}}(\Pi_{[0,t]}\dot{R}^n ; C)$ the subspace of those Φ in Z for which there exist a finite partition $0 = t_0 < t_1 < \cdots < t_m = t$ of the interval $[0, t]$ and a C-valued bounded continuous function $F(x^{(0)}, x^{(1)}, \cdots, x^{(m)})$ on $(\dot{R}^n)^{m+1}$ such that

$$(*) \qquad \Phi(v) = F(v(t_0), v(t_1), \cdots, v(t_m)) .$$

We want to introduce a linear operator S_t mapping Z_{fin} into X using S which is a propagator in X.

Take Φ from Z_{fin}, so that there exist a finite partition $0 = t_0 < t_1 < \cdots < t_m = t$ of $[0, t]$ and a C-valued bounded continuous function $F(x^{(0)}, x^{(1)}, \cdots, x^{(m)})$ on $(\dot{R}^n)^{m+1}$ such that $(*)$ holds. We may suppose F is defined everywhere and continuous in $(\dot{R}^n)^{m+1}$ so that $\|\Phi\|_\infty = \|F\|_\infty$, where $\|\cdot\|_\infty$ means the sup norm. Suppose that S defines a kernel function K such that

$$(**) \qquad (S(t, s)f)(x) = \int K(t, x ; s, y) f(y) dy , \qquad \text{for} \quad f \in C(\dot{R}^n) .$$

Define $S_t(\Phi)$ by

$$(S_t\Phi)(x) = \overbrace{\int_{\dot{R}^n} \cdots \int_{\dot{R}^n}}^{m} K(t_m, x^{(m)} ; t_{m-1}, x^{(m-1)}) \cdots K(t_2, x^{(2)} ; t_1, x^{(1)}) K(t_1, x^{(1)} ; t_0, x^{(0)}) F(x^{(0)}, x^{(1)}, \cdots, x^{(m)}) dx^{(0)} \cdots dx^{(m-1)} ,$$

where $x^{(m)} = x \in \dot{R}^n$. $S_t\Phi$ is independent of the choice of F, then S_t is well-defined. We have the following proposition.

PROPOSITION 1. *If $\{S(t) ; 0 \leqq t < +\infty\}$ is a contraction semigroup and also defines a kernel function K satisfying $(**)$, then S_t is uniquely extended to* *a continuous operator of* $C(\Pi_{[0,t]}(\dot{R}^n))$ $(=Z)$ *into* $C(\dot{R}^n)$ $(=X)$.

We also denote the extension S_t.

The following theorem is well known. All notations are the same as above.

THEOREM 1. *If S_t is a continuous linear operator of Z into X, then there exists a unique set function μ, defined on the Borel sets in $\Pi_{[0,t]}\dot{R}^n$ and having values in X'', where X'' is the second dual of X, such that*

(a) *$\mu(\cdot)x'$ is in* $\mathrm{rca}(\Pi_{[0,t]}\dot{R}^n)$ *for each x' in X', where* $\mathrm{rca}(A)$ *is the space of all regular countably additive measures on A;*

(b) *the mapping $x'\to\mu(\cdot)x'$ of X' into* $\mathrm{rca}(\Pi_{[0,t]}\dot{R}^n)$ *is continuous with the X and Z topologies in these spaces respectively;*

(c) $x'S_tf=\int_{\Pi_{[0,t]}\dot{R}^n} f(u)\mu(du)x'$, *for $f\in Z$ and $x'\in X'$;*

(d) $\|S_t\|=\|\mu\|(\Pi_{[0,t]}\dot{R}^n)$, *$\|\mu\|(A)$ means the total variation of μ on A.*

If S_t is weakly compact, we have the following one instead of Theorem 1.

THEOREM 2. *If S_t is a weakly compact operator of Z into X, then there exists a vector-valued measure μ defined on the Borel sets in $\Pi_{[0,t]}\dot{R}^n$ and having values in X such that*

(a) *$x'\mu$ is in* $\mathrm{rca}(\Pi_{[0,t]}\dot{R}^n)$, *for $x'\in X'$;*

(b) $S_tf=\int_{\Pi_{[0,t]}\dot{R}^n} f(u)\mu(du)$, *for $f\in Z$;*

(c) $\|S_t\|=\|\mu\|(\Pi_{[0,t]}\dot{R}^n)$;

(d) $S_t'x'=x'\mu$, *for $x'\in X'$, where S_t' is the dual operator of S_t.*

§ 3. Relations between S_t and M_t

Let $\{S(t)\,;\,0\leqq t<+\infty\}$ be a contraction semigroup in $\tilde{X}$ and also having a kernel function K satisfying $(**)$, and P be defined as follows:

$$P(B)f=1_B\cdot f \qquad \text{where} \quad f\in\tilde{X} \quad \text{and} \quad B\in\mathscr{B}(\dot{R}^n).$$

P is a spectral measure. Then we have an SPt-measure M_t defined by S and P.

Here we consider the relation between M_t and the operator S_t defined by S.

First we suppose that S_t is weakly compact. In this case we have the X-valued measure μ. If $f\in X$ and $\Gamma=\{v\in Y_t\,;\,v(t_j)\in B_j,\ j=1,2,\cdots,m\}$, where $B_j\in\mathscr{B}(R^n)$ for every j, then $M_t(\Gamma)f=\int_\Gamma F\,d\mu$, where F is in Z_{fin} and

$F(v)=f(v(t_0))$. Therefore $M_t(\Gamma)\in L(X)$ and M_t is countably additive on $\sigma(P_t)$ in the strong topology.

Second we consider the general case. In this case we have the X''-valued measure μ such that

$$x'S_tF=\int F(u)\mu(du)x' \qquad \text{for} \quad F\in Z \quad \text{and} \quad x'\in X'.$$

If $f\in X$ and $\Gamma=\{v\in Y_t\,;\, v(t_j)\in B_j,\, j=1, 2, \cdots, m\}$, then $x'(M_t(\Gamma)f)=\int_\Gamma F(u)\mu(du)x'$, where F is in Z_{fin} and $F(v)=f(v(t_0))$. Therefore M_t is countably additive on $\sigma(P_t)$ in the weak topology.

§ 4. Examples

In this section we treat some examples.

EXAMPLE 1. Let $K(t, x\,;\, s, y)=\{4\pi D(t-s)\}^{-1/2}\exp(-|x-y|^2/4D(t-s))$, where D is a constant and $|\cdot|$ means the norm of R^n. It is clear that S is a contraction semigroup. The following results are known.

Let $(\tilde{S}_t\Phi)(x)=\overbrace{\int_{R^n}\cdots\int_{R^n}}^{m+1}K(t_m, x^{(m)}\,;\, t_{m-1}, x^{(m-1)})\cdots K(t_1, x^{(1)}\,;\, t_0, x^{(0)})F(x^{(0)}, \cdots x^{(m)})dx^{(0)}dx^{(1)}\cdots dx^{(m)}$, then $\tilde{S}_t$ defines the Wiener measure γ on $\Pi_{[0,t]}R^n$. Also let $(\tilde{\tilde{S}}_t\Phi)(x)=\overbrace{\int_{R^n}\cdots\int_{R^n}}^{m+1}K(t_m, x^{(m)}\,;\, t_{m-1}, x^{(m-1)})\cdots K(t_1, x^{(1)}\,;\, t_0, x^{(0)})F(x^{(0)}, \cdots x^{(m)})dx^{(0)}\cdots dx^{(m-1)}d\mu(x^{(m)})$, where $\mu\in\text{rca}(R^n)$, then $\tilde{\tilde{S}}_t$ defines the measure γ_μ on $\Pi_{[0,t]}R^n$. Consider the dual operator S_t', then we have $S_t'\mu=\gamma_\mu$ for every $\mu\in\text{rca}(R^n)$. Therefore S_t' is weakly compact ([2]). Then S_t is also weakly compact, so that we have $C(R^n)$-valued measure defined on $\Pi_{[0,t]}R^n$.

EXAMPLE 2. ([3])

Consider the homogeneous hyperbolic system of the first order

$$(***) \qquad \partial_t\phi(t, x)=\left[\sum_{i=1}^{n}P_1\partial_1+iQ(t, x)\right]\phi(t, x), \qquad 0<t<T,\ x\in R^n,$$

where $0<T<\infty$. We assume, for the $N\times N$-matrices P_1 and $Q(t, x)$, (P) and one of $(Q)_i$, $(Q_b)_i$ and $(Q_c)_i$, $i=0, 1$.

(P) The P_l, $1\leqq l\leqq n$, are mutually commuting, constant matrices having only real eigenvalues $\{\lambda_j\}_{j=1,\cdots,N}$, so that they are simultaneously diagonalizable. $(Q)_i$ $Q:[0, T)\ni t\mapsto Q(t, \cdot)\in E^i(R^n\,;\, C^{N^2})$ is continuous; $(Q_b)_i$ $Q:[0, T)\ni t\mapsto Q(t, \cdot)\in B^i(R^n\,;\, C^{N^2})$ is continuous; $(Q_c)_i$ $Q:[0, T)\ni t\mapsto Q(t, \cdot)\in C^i(R^n\,;\, C^{N^2})$ is continuous; E^i is the Fréchet space of the C^{N^2}-valued C^i (i-times continuously differentiable) function in R^n, B^i the Banach space of those functions in E^i which together with their derivatives up to the ith order are bounded and C^i is the Banach space of those functions in B^i which together with their derivatives up to the ith order have the finite limits as $|x|\to\infty$.

By $(S(t, s)g)(x)$ we denote the solution $\phi(t, x)$ of the Cauchy problem for $(***)$ with datum $\phi(s, x)=g(x)$ at time s:

$$(S(t, s)g)(x)=\int_{R^n} K(t, x; s, y)g(y)dy$$

with the fundamental solution $K(t, x; s, y)$ for $(***)$.

This case S is not necessarily a contraction semigroup, however S_t is a continuous linear operator. Then we have $(C(R^n))''$-valued measure on $\Pi_{[0,t]}R^n$.

EXAMPLE 3. ([1])

Assume that $K(t, x; s, y)$ satisfies the following conditions:

(i) The real-valued function $K(t, x; s, y)$ is continuous with respect to (x, y) for $s<t$.

(ii) $\int_{R^n} K(t, x; s, y)K(r, z; t, x)dx=K(r, z; s, y)$ $(r<t<s)$.

(iii) For each x and y, $|K(t, x; s, y)|^2$ is integrable with respect to the Lebesgue measure.

(iv) $\int_{R^n} K(t, x; s, y)dy=1$ for $s<t$, $x\in R^n$.

(v) $\sup_{x,t,s}\frac{1}{t-s}\left\{\int_{R^n}|K(t, x; s, y)|dy-1\right\}<+\infty$.

Consider $\tilde{S}_t$ as the same as in Example 1, then we have the signed measure which is of bounded variation on $\Pi_{[0,t]}R^n$ ([1]). Using the same method of Example 1, we have $C(R^n)$-valued measure on $\Pi_{[0,t]}R^n$.

References

[1] Yu. L. Daletskii: Functional integrals connected with operator evolution equations, Uspehi Mat. Nauk., **17**(5) (1962), 3–115; English transl., Russ. Math. Surveys, **17** (1962), 1–107.

[2] N. Dunford and J.T. Schwartz: Linear operators part I, Interscience publishers, INC., New York, 1957.

[3] T. Ichinose: Path integral for a hyperbolic system of the first order, Duke Math. J., **51** (1984), 1–36.

[4] I. Kluvánek: Operator-valued measures and perturbations of semi-groups, preprint.

[5] ———: Integration and the Feynman-Kac formula, Studia Math., **86** (1987), 35–57.

[6] E. Nelson: Feynman integrals and the Schrödinger equation, J. Math. Phys., **5** (1964), 332–343.