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Equilibrium Measures of the Recurrent Markov Processes in Duality

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§ 1. Introduction.

We are concerned in this paper with the potential theory for recurrent Markov processes introduced by T. Ueno [**6**]. He studied a pair of measures μ_L^K and μ_K^L satisfying

(1.1) $$\mu_L^K(\cdot)=\mu_K^L H_K(\cdot)\,,\quad \mu_K^L(\cdot)=\mu_L^K H_L(\cdot)\,,$$

where $H_K(x,\cdot)$ is the hitting measure to the set K. In the case of the Markov processes with Brownian hitting measures on R^2, Ueno [**6**] proved that μ_L^K coincides with the equilibrium measure on K with respect to the classical Green kernel of the component. Under the assumption of symmetry of the Green kernel, Kitamura [**5**] showed that the measure ν_L^K multiplied μ_L^K by the Ueno capacity is the equilibrium measure on $K\subset L^c$.

From these arguments we anticipated that such a fine relation is valid in broader recurrent Markov processes, namely, a pair of measures μ_L^K and μ_K^L satisfying (1.1) is a new probabilistic characterization of the equilibrium measure. However, it now turns out to fall short of our expectations. In fact when the underlying Ueno process X is in duality with another Ueno process $\hat{X}$, the corresponding measure $\hat{\nu}_L^K$ for the dual process $\hat{X}$ is the equilibrium measure on K with respect to the original Green kernel. Our aim here is to prove this and the fact that the potential induced by $\hat{\nu}_L^K$ equals to the hitting probability for K before attaining to L.

On the other hand, Chung [**2**, p. 139] ensures the existence of dual processes for the spatially homogeneous Markov processes. In particular, the spatially homogeneous Markov processes with the temporally homogeneity constitute a large and important class which are called additive processes. Thus our situation treats rather broader Markov processes and our results are better than before.

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§ 2. Preliminaries.

We refer the reader to [**4**] and [**5**] for all terminology and notation not explicitly defined here. Let R be a separable Hausdorff locally compact space containing at least two points and satisfying

(R. 1) For each point $x \in R$, we can take a countable base of neighborhoods of x consisting of arcwise connected open sets,

(R. 2) R is connected.

We denote by $\boldsymbol{B}$ the topological Borel field of subsets of R. For a set $A \in \boldsymbol{B}$ and a path function $X(t)$ from $[0, \infty)$ to R, the entry time D_A to the set A is defined by

$$D_A = \inf\{t \geqq 0 \mid X(t) \in A\}, \qquad \text{if such } t \text{ exists},$$
$$= \infty, \qquad \text{otherwise}.$$

First, we define Ueno processes [**6**]. We will say that a standard process $X = \{X(t)\}$ with state space R is an Ueno process provided:

(X. 1) Recurrence: $P_x(X(t) \in A$ for some $0 \leqq t < \infty) = 1$ for any $x \in R$, $A \in \boldsymbol{B}$,

(X. 2) For any continuous function f on A, $H_A f$ is continuous in A^c, where the operator H_A is defined by $H_A f = E.(f(X(D_A))\,;\, D_A < \infty)$ and A is a closed set in R containing an inner point,

(X. 3) Maximum principle: For non-negative continuous function f in A, $H_A f(x)$ is either strictly positive or 0 for all points x of any one component of A^c, where A is a closed set in R containing an inner point,

(X. 4) For any continuous function f on R, the resolvent $U^\alpha f$ is continuous on R,

(X. 5) There is no point of positive holding time.

Now, we introduce the Green measure

$$U_L(x, A) = E_x\left(\int_0^{D_L} I_A(X(t))\,dt\right), \qquad x \in R, \quad A \in \boldsymbol{B}$$

for any closed set L containing an inner point, where I_A takes 1 on A, 0 on A^c respectively. Let $\mathfrak{F}$ be the family of all $\{K, L\}$, where K and L are mutually disjoint closed sets in R and in particular K is compact. Ueno [**6**] proves that for each $\{K, L\} \in \mathfrak{F}$ there is a unique pair of measures μ_L^K and μ_K^L with total mass 1 on K and L respectively, satisfying

$$\mu_L^K(\cdot)=\mu_K^L H_K(\cdot)=\int_L \mu_K^L(dx)H_K(x,\cdot),$$

$$\mu_K^L(\cdot)=\mu_L^K H_L(\cdot)=\int_K \mu_L^K(dx)H_L(x,\cdot).$$

Making use of these μ_L^K, μ_K^L, Ueno defines his own Green capacity denoted by $C(K,L)$, which is the Green capacity of K with respect to L. Set

$$\nu_L^K(\cdot)=C(K,L)\,\mu_L^K(\cdot).$$

Then we have the following lemma concerning the Ueno capacity.

LEMMA 2.1. *Suppose that* $\{K',L\}$, $\{K,L\}\in\mathfrak{F}$ *and* $K'\subset K$. *Then we get*

$$C(K',L)=\int \nu_L^K(dx)P_x(D_{K'}<D_L).$$

(Theorem 2 of [**5**]).

Moreover we introduce the measure

$$m(\cdot)=\int_K \nu_L^K(dx)\,U_L(x,\cdot)+\int_L \nu_K^L(dx)\,U_K(x,\cdot). \tag{2.1}$$

It is known that this measure is independent of the choice of $\{K,L\}\in\mathfrak{F}$ and takes positive value for each Borel set with inner points. Then Ueno [**6**] proved that the Green measure $U_L(x,\cdot)$ has a density function $u_L(x,y)$ relative to m, that is,

$$U_L(x,A)=\int_A u_L(x,y)m(dy)$$

holds for a set $A\in\boldsymbol{B}$.

Next, we say that the Ueno processes X and $\hat{X}$ are in duality relative to the measure m provided that for each $\alpha>0$ there is non-negative function u^α with the following properties:

(i) the function $x\to u^\alpha(x,y)$ is α-excessive relative to the resolvent $\{U^\alpha\}$ for each $y\in R$ and $\alpha>0$,

(ii) the function $y\to u^\alpha(x,y)$ is α-excessive relative to the resolvent $\{\hat{U}^\alpha\}$ for each $x\in R$ and $\alpha>0$,

(iii) $U^\alpha f(x)=\int u^\alpha(x,y)f(y)\,m(dy)$ and $f\hat{U}^\alpha(y)=\int f(x)\,u^\alpha(x,y)\,m(dx)$ for $\alpha>0$ and a bounded function f and $x,y\in R$, where $\hat{U}^\alpha$ is the resolvent for the dual process $\hat{X}$.

This definition comes from Blumenthal-Getoor [**1**]. In this paper we consider the Ueno processes whose dual processes exist. The next two lemmas will be useful to proof of the main results later on.

LEMMA 2.2. *Suppose that standard processes* X *and* $\hat{X}$ *are in duality relative to the measure* m. *Then for each* $\alpha\geqq 0$ *and a Borel set* A

$$P_A^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_A^\alpha(x, y), \qquad x, y \in R,$$

where for a Borel set A

$$P_A^\alpha f(x) = E_x(e^{-\alpha T_A} f(X(T_A))\,;\ T_A < \infty),$$

$$T_A = \inf\{t > 0 \mid X(t) \in A\}, \qquad \text{if such } t \text{ exists,}$$

$$= \infty, \qquad \text{otherwise}$$

(Theorem 1.16 in Chapter VI of Blumenthal-Getoor [**1**]).

LEMMA 2.3. *Let X and $\hat{X}$ be a pair of standard processes in duality relative to m. Then for each $\alpha \geqq 0$ and a Borel set A there exists a function $u_A^\alpha(x, y)$ such that*

(i) $U_A^\alpha f(x) = \int u_A^\alpha(x, y) f(y) m(dy)$

(ii) $u^\alpha(x, y) = u_A^\alpha(x, y) + P_A^\alpha u^\alpha(x, y)$.

Set $f\hat{U}_A^\alpha(y) = \int f(x) u_A^\alpha(x, y) m(dx)$. Then $x \to u_A^\alpha(x, y)$ is α-excessive relative to the resolvent $\{U_A^\alpha\}$ for each y, and $y \to u_A^\alpha(x, y)$ is α-excessive relative to the resolvent $\{\hat{U}_A^\alpha\}$ for each x, where for a Borel set A

$$U_A^\alpha f(x) = E_x\left(\int_0^{T_A} e^{-\alpha t} f(X(t)) dt\right)$$

(Theorem 2.5 of Getoor [**3**]).

§ 3. Equilibrium measures of Ueno processes.

In this section we prove two theorems. In order to prove them the following lemma is essential.

LEMMA 3.1. *Let $\{K, L\} \in \mathfrak{F}$. For $x, y \in K$ we have*

$$S_K u_L(x, y) = u_L \hat{S}_K(x, y).$$

Here we denote

$$S_K(x, \cdot) = H_L H_K(x, \cdot) = \int_R H_L(x, dy) H_K(y, \cdot).$$

PROOF. Choose $\alpha > 0$. Let G be an open neighborhood of K such that $\bar{G} \cap L = \emptyset$. Set

$$S_G^\alpha(x, \cdot) = P_L^\alpha P_G^\alpha(x, \cdot) = \int P_L^\alpha(x, dy) P_G^\alpha(y, \cdot).$$

First of all we show that $S_G^\alpha u_L^\alpha(x, y) = u_L^\alpha \hat{S}_G^\alpha(x, y)$ for $x, y \in K$. By Lemma 2.3 we get

$$(3.1) \qquad S_G^\alpha u_L^\alpha(x, y) = P_L^\alpha P_G^\alpha u_L^\alpha(x, y)$$

$$= P_L^\alpha P_G^\alpha u^\alpha(x, y) - P_L^\alpha P_G^\alpha P_L^\alpha u^\alpha(x, y)$$

for $x, y \in K$.

Consider the first term of the right side in (3.1). It follows from Lemma 2.3 that

$$P_L^\alpha P_G^\alpha u^\alpha(x, y) = \int P_L^\alpha(x, dz)(u^\alpha(z, y) - u_G^\alpha(z, y)). \tag{3.2}$$

Now for $z \in L$ and $A \subset G$

$$U_G^\alpha(z, A) = E_z\left(\int_0^{T_G} e^{-\alpha t} I_A(X(t))dt\right) = \int_A u_G^\alpha(z, y)\, m(dy).$$

The middle term of the above identity vanishes and so $U_G^\alpha(z, y)$ equals zero for almost every $y \in G$ relative to the measure m. Moreover $u_G^\alpha(z, y) = 0$ is true for all $y \in G$ because G is open and $u_G^\alpha(z, y)$ is α-coexcessive with respect to y. Therefore by applying Lemma 2.2 to (3.2),

$$\begin{aligned} P_L^\alpha P_G^\alpha u^\alpha(x, y) &= \int P_L^\alpha(x, dz) u^\alpha(z, y) \\ &= \int u^\alpha(x, z) \hat{P}_L^\alpha(dz, y). \end{aligned}$$

Besides $u_G^\alpha(x, z)$ is equal to zero for $x \in K$ and $z \in L$ since $U_G^\alpha(x, A) = 0$ for $A \subset R$ and $u_G^\alpha(x, z)$ is α-coexcessive with respect to z. Consequently by the another aide of Lemma 2.2 and Lemma 2.3 it follows that

$$\begin{aligned} P_L^\alpha P_G^\alpha u^\alpha(x, y) &= \int (u^\alpha(x, z) - u_G^\alpha(x, z)) \hat{P}_L^\alpha(dz, y) \\ &= \int P_G^\alpha u^\alpha(x, z) \hat{P}_L^\alpha(dz, y) \\ &= \int u^\alpha(x, w) \hat{P}_G^\alpha(dw, z) \hat{P}_L^\alpha(dz, y) \\ &= u^\alpha \hat{S}_G^\alpha(x, y) \end{aligned} \tag{3.3}$$

for $x, y \in K$.

We turn now to the second part of the right hand in (3.1). Applying Lemma 2.2, we get for $x, y \in K$

$$\begin{aligned} P_L^\alpha P_G^\alpha P_L^\alpha u^\alpha(x, y) &= P_L^\alpha P_G^\alpha u^\alpha \hat{P}_L^\alpha(x, y) \\ &= P_L^\alpha u^\alpha \hat{P}_G^\alpha \hat{P}_L^\alpha(x, y) \\ &= P_L^\alpha u^\alpha \hat{S}_G^\alpha(x, y). \end{aligned} \tag{3.4}$$

Therefore we substitute (3.3) and (3.4) for (3.1) to get

$$\begin{aligned} S_G^\alpha u_L^\alpha(x, y) &= u^\alpha \hat{S}_G^\alpha(x, y) - P_L^\alpha u^\alpha \hat{S}_G^\alpha(x, y) \\ &= \int (u^\alpha(x, z) - P_L^\alpha u^\alpha(x, z)) \hat{S}_G^\alpha(dz, y) \\ &= u_L^\alpha \hat{S}_G^\alpha(x, y) \end{aligned}$$

for $x, y \in K$ as desired.

For typographical convenience, we write D_n for the entry time to the set G_n and suppress the letter G in our notations of D_n, S_n^α, P_n^α and H_n^α. We can find a decreasing sequence of open neighborhoods $\{G_n\}$ containing K such that $D_n \uparrow D_K$ almost surely P_x and $\hat{D}_n \uparrow \hat{D}_K$ almost surely $\hat{P}_x$. See Lemma (10.10) of Blumenthal-Getoor [**1**]. Then making use of the quasi-left continuity of paths we obtain

$$\begin{aligned}\lim_{n\to\infty} S_n^\alpha(x, A) &= \int P_L^\alpha(x, dy)\Big(\lim_{n\to\infty} P_n^\alpha(y, A)\Big) \\ &= \int H_L^\alpha(x, dy)\Big(\lim_{n\to\infty} H_n^\alpha(y, A)\Big) \\ &= \int H_L^\alpha(x, dy) H_K^\alpha(y, A)\end{aligned}$$

for $x \in K$ and $A \in \boldsymbol{B}$, where

$$H_A^\alpha f(x) = E_x(e^{-\alpha D_A} f(X(D_A))\,;\ D_A < \infty)\,.$$

Thus by setting $S_K^\alpha = H_L^\alpha H_K^\alpha$ this yields

$$\lim_{n\to\infty} S_n^\alpha(x, A) = S_K^\alpha(x, A)\,. \tag{3.5}$$

By the same argument as (3.5)

$$\lim_{n\to\infty} \hat{S}_n^\alpha(A, y) = \hat{S}_K^\alpha(A, y)$$

is true for $y \in K$. Therefore we have $\lim_{n\to\infty} S_n^\alpha u_L^\alpha(x, y) = S_K^\alpha u_L^\alpha(x, y)$ and $\lim_{n\to\infty} u_L^\alpha \hat{S}_n^\alpha(x, y) = u_L^\alpha \hat{S}_K^\alpha(x, y)$. And so

$$S_K^\alpha u_L^\alpha(x, y) = u_L^\alpha \hat{S}_K^\alpha(x, y)\,, \qquad x, y \in K\,. \tag{3.6}$$

Consequently the desired conclusion follows by letting $\alpha \downarrow 0$ in the equation (3.6).

Now we add the following two assumptions.

(A. 1) If a bounded function f on L^c vanishes outside a compact subset of L^c, then $f\hat{U}_L(y) = \int f(x) u_L(x, y) m(dx)$ is a continuous function of y on L^c.

(A. 2) If $\{f_\lambda, \lambda \in \Lambda\}$ is a uniformly bounded class of bounded functions on K, $\{H_K f_\lambda, \lambda \in \Lambda\}$ is equicontinuous on each fixed compact subset of K^c, where K is a closed set $(\neq R)$ with an inner point. The assumptions (A. 1) and (A. 2) go back to Blumenthal-Getoor [**1**, p. 265] and to Ueno [**6**], respectively.

LEMMA 3.2. *Assume that* (A. 2) *holds. For any measure μ on K and ν on L with total mass* 1, *$\mu(S_K)^n$, $\nu(S_L)^n$, $n = 1, 2, \cdots$, converge to the unique limit μ_L^K and μ_K^L respectively, with respect to the norm of total* *variation, and exponentially fast* (Proposition 2.2 of Ueno [**6**]).

We are now ready to prove the main theorems on equilibrium problems of Ueno processes.

THEOREM 3.1. *Suppose that* (A. 1) *and* (A. 2) *hold. Then for* $\{K, L\} \in \mathfrak{F}$

$$u_L \hat{\nu}_L^K = 1 \qquad \text{on} \quad K.$$

PROOF. Under the assumption (A. 1) a finite measure concentrated on K may be chosen so that

$$u_L \mu = H_{K,L} 1 = P_{\cdot}(D_K < D_L) \qquad \text{on} \quad L^c \tag{3.7}$$

is true, where

$$H_{K,L}(x, A) = P_x(X(D_K) \in A,\ D_K < D_L), \qquad A \in \boldsymbol{B}.$$

This is proved by the similar way as page 271 of [**1**]. Then it follows from Lemma 3.1 that

$$u_L \hat{S}_K \mu(x) = S_K u_L \mu(x) = S_K(x, K)$$

for $x \in K$. Since recurrence of our process yields $S_K(x, K) = 1$, we get

$$u_L \hat{S}_K \mu = 1 \qquad \text{on} \quad K.$$

Hence by induction on n

$$u_L (\hat{S}_K)^n \mu = 1 \qquad \text{on} \quad K. \tag{3.8}$$

Combining Lemma 3.2 with (3.8) we obtain

$$\begin{aligned} \frac{1}{\mu(K)} &= \frac{1}{\mu(K)} \lim_{n \to \infty} u_L (\hat{S}_K)^n \mu(x) \\ &= u_L \hat{\mu}_L^K(x) \end{aligned}$$

for $x \in K$. That is,

$$\mu(K) u_L \hat{\mu}_L^K = 1 \qquad \text{on} \quad K.$$

In order to complete the proof of Theorem 3.1 it remains to show that $\mu(K) = C(K, L) = \hat{C}(K, L)$. Choose a compact set K' such that $K \subset K'^{\circ}$ and $K' \cap L = \emptyset$, where K'° denotes the interior of the set K'. Then it follows from Lemma 2.1 and (3.7)

$$\begin{aligned} C(K, L) &= \int \nu_L^{K'}(dx) P_x(D_K < D_L) \\ &= \int \nu_L^{K'}(dx) u_L \mu(x) \\ &= \int \left(\int \nu_L^{K'}(dx) u_L(x, y) \right) \mu(dy). \end{aligned} \tag{3.9}$$

We will prove that $\int \nu_L^{K'}(dx)u_L(x,y)=1$ for $y \in K$. In fact for any set A contained in K'

$$m(A)=\int \nu_L^{K'}(dx)U_L(x,A)$$
$$=\int_A\Big(\int \nu_L^{K'}(dx)u_L(x,y)\Big)m(dy)$$

by the definition (2.1) of the measure m. Therefore $\int \nu_L^{K'}(dx)u_L(x,y)=1$ for almost every $y \in K'$ relative to the measure m. Also since $\int \nu_L^{K'}(dx)u_L(x,y)$ is coexcessive with respect to y, we have $\int \nu_L^{K'}(dx)u_L(x,y)=1$ for y in K'° and in particular $\int \nu_L^{K'}(dx)u_L(x,y)=1$ for y in K as desired. Thus it follows from (3.9) that $C(K,L)=\mu(K)$. On the other hand it can be shown similarly as on page 285 of [**1**] that $\mu(K)=\hat{\mu}(K)$. Also $\hat{\mu}(K)=\hat{C}(K,L)$ is true by duality. Consequently we have

$$\mu(K)=C(K,L)=\hat{\mu}(K)=\hat{C}(K,L).$$

Theorem 3.1 tells us that $\hat{\nu}_L^K$ is the equilibrium measure for the kernel $u_L(x,y)$.

Finally, under the same conditions as the previous theorem we prove that the potential of $\hat{\nu}_L^K$ is the hitting probability of K before reaching L. Since Kakutani this beautiful theorem is fundamental in the probabilistic potential theory.

THEOREM 3.2. *Suppose the assumption* (A. 1) *and* (A. 2). *Then for* $\{K, L\} \in \mathfrak{F}$

$$u_L\hat{\nu}_L^K = P_\cdot(D_K < D_L) \qquad \text{on} \quad L^c.$$

PROOF. Choose $x \in L^c$. Let $\{G_n\}$ be a decreasing sequence of open sets containing K such that $D_n \uparrow D_K$ almost surely P_x. As before, we suppress the letter G in the notation of D_n and $H_{n,L}$. Then for a set $A \subset G_n$, we have

$$U_L(x,A)=E_x\Big(\int_0^{D_L} I_A(X(t))dt\Big)$$
$$=E_x\Big(\int_0^{D_L} I_A(X(t))dt\,;\ D_L<D_n\Big)$$
$$+E_x\Big(\int_0^{D_n} \text{〃}\,;\ D_L>D_n\Big)+E_x\Big(\int_{D_n}^{D_L} \text{〃}\,;\ D_L>D_n\Big)$$
$$=E_x\Big(E_{X(D_n)}\Big(\int_0^{D_L} I_A(X(t))dt\,;\ D_L>D_n\Big)$$
$$=E_x(U_L(X(D_n),A)\,;\ D_L>D_n)$$
$$=\int H_{n,L}(x,dy)U_L(y,A)$$

by the strong Markov property. Thus

$$\int_A u_L(x, y)m(dy) = \int_A \left(\int H_{n,L}(x, dz) u_L(z, y) \right) m(dy)$$

for $A \subset G_n$, namely we have for y

$$u_L(x, y) = \int H_{n,L}(x, dz) u_L(z, y) \qquad \text{a. e. } (m) \quad \text{on} \quad G_n .$$

Since $\int H_{n,L}(x, dz) u_L(z, y)$ is coexcessive with respect to y, we get for all $y \in G_n$

$$u_L(x, y) = \int H_{n,L}(x, dz) u_L(z, y) . \tag{3.10}$$

Particularly (3.10) is true for all y in K. Letting $n \uparrow \infty$ in (3.10), we have

$$u_L(x, y) = \int H_{K,L}(x, dz) u_L(z, y) \tag{3.11}$$

for $y \in K$. Consequently it follows by combining (3.11) with Theorem 3.1

$$\begin{aligned} u_L \hat{\nu}_L^K(x) &= \int H_{K,L}(x, dz) u_L \hat{\nu}_L^K(z) \\ &= \int_K H_{K,L}(x, dz) \cdot 1 \\ &= P_x(D_K < D_L) , \end{aligned}$$

which completes the proof of the theorem.

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